Chapter 3

Shared Inverse Gaussian Frailty Models

3.1 Introduction

The gamma distribution is most commonly used frailty distribution because of its mathematical convenience. Another choice is the inverse Gaussian distribution. The inverse Gaussian makes the population homogeneous with time, whereas for gamma the relative heterogeneity is constant (Hougaard, 1984). Duchateau and Janssen (2008) fit the inverse Gaussian (IG) frailty model with Weibull hazard to the udder quarter infection data. The IG distribution has a unimodal density and is a member of the exponential family. While its shape resembles that of other skewed density functions, such as lognormal and gamma, it provides much flexibility in modeling. Furthermore, there are many striking similarities between the statistics derived from this distribution and those of the normal; see Chhikara and Folks (1986). These properties make it potentially attractive for modeling purposes with survival data. The models derived above are bases on the assumption that a common random effect acts multiplicatively on the hazard rate function.
Alternative to the gamma distribution, Hougaard (1984) introduced the inverse Gaussian as a frailty distribution. It provides much flexibility in modeling, when early occurrences of failures are dominant in a life time distribution and its failure rate is expected to be non-monotonic. In such situations, the inverse Gaussian distribution might provide a suitable choice for the lifetime model. Also inverse Gaussian is almost an increasing failure rate distribution when it is slightly skewed and hence is also applicable to describe lifetime distribution which is not dominated by early failures. Secondly, for the inverse Gaussian distribution, the surviving population becomes more homogeneous with respect to time, where as for gamma distribution the relative heterogeneity is constant. The inverse Gaussian distribution has shape resembles the other skewed density functions, such as log-normal and gamma. These properties of inverse Gaussian distribution motivate us to use inverse Gaussian as frailty distribution. The inverse Gaussian distribution has a history dating back to 1915 when Schrodinger and Smoluchowski presented independent derivations of the density of the first passage time distribution of Brownian motion with positive drift. Villman et al., (1990) have studied the histomorphometrical analysis of the influence of soft diet on masticatory muscle development in the muscular dystrophic mouse. The muscle fibre size distributions were fitted by an inverse Gaussian law. Barndorff-Nielsen (1994) considers a finite tree whose edges are endowed with random resistances, and shows that, subject to suitable restrictions on the parameters, if the resistances are either inverse Gaussian or reciprocal inverse Gaussian random variables, then the overall resistance of the tree follows a reciprocal inverse Gaussian law. Gacula and Kubala (1975) have analyzed shelf life of several products using the IG law and found to be a good fit. For more real life applications (see Seshadri, 1999).
3.2 Shared Inverse Gaussian Frailty Model

Let a continuous random variable $Z$ follow an inverse Gaussian distribution with parameters $\mu$ and $\theta$, then the density function of $Z$ is,

\[
f_Z(z) = \begin{cases} 
  \left[ \frac{1}{2\pi\theta} \right]^{\frac{1}{2}} z^{-\frac{3}{2}} e^{-\frac{(z-\mu)^2}{2z\theta^2}} & ; z > 0, \mu > 0, \theta > 0 \\
  0 & ; \text{otherwise},
\end{cases}
\]  

(3.1)

and the Laplace transform is,

\[
L_Z(s) = \exp \left[ \frac{1}{\mu\theta} - \left( \frac{1}{\theta^2\mu^2} + \frac{2s\theta}{\theta} \right)^{\frac{1}{2}} \right].
\]  

(3.2)

The mean and variance of the frailty variable are $E(Z) = \mu$ and $V(Z) = \mu^3\theta$. For identifiability, we assume $Z$ has expected value equal to one i.e. $\mu = 1$. Under this restriction, the density function and the Laplace transformation of the inverse Gaussian distribution reduces to,

\[
f_Z(z) = \begin{cases} 
  \left[ \frac{1}{2\pi\theta} \right]^{\frac{1}{2}} z^{-\frac{3}{2}} e^{-\frac{(z-1)^2}{2z\theta^2}} & ; z > 0, \theta > 0 \\
  0 & ; \text{otherwise},
\end{cases}
\]  

(3.3)

and the Laplace transform is,

\[
L_Z(s) = \exp \left[ \frac{1 - (1 + 2\theta s)^{\frac{1}{2}}}{\theta} \right],
\]  

(3.4)

with variance of $Z$ as $\theta$. The frailty variable $Z$ is degenerate at $Z = 1$ when $\theta$ tends to zero. Replacing the Laplace transform in equation (1.8), we get the unconditional bivariate distribution function for the $j^{th}$ individual as,

\[
S(t_{1j}, t_{2j}) = \exp \left[ 1 - (1 + 2\theta \eta_j(H_{01}(t_{1j}) + H_{02}(t_{2j})))^{\frac{1}{2}} \right],
\]  

(3.5)

where $H_{01}(t_{1j})$ and $H_{02}(t_{2j})$ are the cumulative baseline hazard functions of the lifetime $T_{1j}$ and $T_{2j}$ respectively. The bivariate distribution in the presence
of covariates, when the frailty variable is degenerate is given by (2.3). Clayton (1978) define cross-ratio function as,

\[ \zeta(t_1, t_2) = \frac{\partial^2 S(t_1, t_2)}{\partial t_1 \partial t_2} S(t_1, t_2) \frac{\partial S(t_1, t_2)}{\partial t_1} \frac{\partial S(t_1, t_2)}{\partial t_2} \]  

(3.6)

The cross ratio function of inverse Gaussian frailty is,

\[ \zeta(t_1, t_2) = 1 + \frac{1}{\theta - \ln(S(t_1, t_2))} \]  

(3.7)

The highest value is obtained at the start and equals \( 1 + \theta \), and goes to one as the survival function goes to zero.

### 3.3 Proposed Models

Substituting cumulative hazard function for the generalized log logistic type II and the generalized Weibull baseline distribution in equation (3.5), we get the unconditional bivariate survival functions at time \( t_{1j} > 0 \) and \( t_{2j} > 0 \) as,

\[ S(t_{1j}, t_{2j}) = \exp \left[ 1 - \left[ 1 + 2\theta \eta_j \left( \alpha_1 \ln \left( 1 + \lambda_1 t_{1j}^{\gamma_1} \right) + \alpha_2 \ln \left( 1 + \lambda_2 t_{2j}^{\gamma_2} \right) \right) \right]^{\frac{1}{\theta}} \right] \]  

(3.8)

\[ S(t_{1j}, t_{2j}) = \exp \left[ 1 - \left[ 1 + 2\theta \eta_j \left( \ln \left( 1 - \left( 1 - e^{-\lambda_1 t_{1j}^{\gamma_1}} \right)^{\alpha_1} \right) + \ln \left( 1 - \left( 1 - e^{-\lambda_2 t_{2j}^{\gamma_2}} \right)^{\alpha_2} \right) \right) \right]^{\frac{1}{\theta}} \right] \]  

(3.9)

Here onwards we call equation (3.8) and (3.9) as Model V and Model VI respectively. Model V is for the generalized log-logistic distribution with frailty and Model VI is for the generalized Weibull distribution with frailty. The joint
bivariate survival function (3.8) and (3.9) can be expressed in terms of survival copula as,

\[
C(u, v) = \exp \left\{ \frac{1 - [(1 - \theta \ln(u))^2 + (1 - \theta \ln(v))^2 - 1]^\frac{1}{2}}{\theta} \right\} \quad (3.10)
\]

where \( u = S_{T_1}(\cdot) \) and \( v = S_{T_2}(\cdot) \)

For Model V

\[
S_{T_i}(t_i) = \exp \left[ \left[ 1 - \frac{1 + 2\theta \eta \alpha_i \ln (1 + \lambda_i t_i^\gamma_i)}{\theta} \right]^{\frac{1}{2}} \right], \quad i = 1, 2
\]

For Model VI

\[
S_{T_i}(t_i) = \exp \left[ \left[ 1 - \frac{1 - 2\theta \eta \ln \left( 1 - e^{\left( \frac{t_i^\gamma_i}{\theta} \right)} \right)^{\alpha_i}}{\theta} \right]^{\frac{1}{2}} \right], \quad i = 1, 2
\]

### 3.4 Likelihood Specification and Bayesian Estimation of Parameters

Suppose there are \( n \) individuals under study, whose first and second observed failure times are represented by \( (t_{1j}, t_{2j}) \). Let \( c_{1j} \) and \( c_{2j} \) be the observed censoring times for the \( j^{th} \) individual \((j = 1, 2, ..., n)\) for the first and the second recurrence times respectively. We also assume the independence between censoring scheme and life times of individuals.

The contribution of bivariate life time random variable of the \( j^{th} \) individual in likelihood function is given by,

\[
L_j(t_{1j}, t_{2j}) = \begin{cases} 
  f_1(t_{1j}, t_{2j}), & t_{1j} < c_{1j}, t_{2j} < c_{2j}, \\
  f_2(t_{1j}, c_{2j}), & t_{1j} < c_{1j}, t_{2j} > c_{2j}, \\
  f_3(c_{1j}, t_{2j}), & t_{1j} > c_{1j}, t_{2j} < c_{2j}, \\
  f_4(c_{1j}, c_{2j}), & t_{1j} > c_{1j}, t_{2j} > c_{2j}.
\end{cases}
\]
and the likelihood function is,

\[ L(\psi, \beta, \theta) = \prod_{j=1}^{n_1} f_1(t_{1j}, t_{2j}) \prod_{j=1}^{n_2} f_2(t_{1j}, c_{2j}) \prod_{j=1}^{n_3} f_3(c_{1j}, t_{2j}) \prod_{j=1}^{n_4} f_4(c_{1j}, c_{2j}) \]

(3.11)

where \( \theta, \psi \) and \( \beta \) are respectively the frailty parameter, the vector of baseline parameters and the vector of regression coefficients. For without frailty model likelihood function is,

\[ L(\psi, \beta) = \prod_{j=1}^{n_1} f_1(t_{1j}, t_{2j}) \prod_{j=1}^{n_2} f_2(t_{1j}, c_{2j}) \prod_{j=1}^{n_3} f_3(c_{1j}, t_{2j}) \prod_{j=1}^{n_4} f_4(c_{1j}, c_{2j}) \]

(3.12)

The counts \( n_1, n_2, n_3 \) and \( n_4 \) are the number of individuals for which the first and the second failure times \( (t_{1j}, t_{2j}) \) lie in the ranges \( t_{1j} < c_{1j}, t_{2j} < c_{2j}; t_{1j} < c_{1j}, t_{2j} > c_{2j}; t_{1j} > c_{1j}, t_{2j} < c_{2j} \) and \( t_{1j} > c_{1j}, t_{2j} > c_{2j} \) respectively and

\[ f_1(t_{1j}, t_{2j}) = \frac{\partial^2 S(t_{1j}, t_{2j})}{\partial t_{1j} \partial t_{2j}} = \frac{h_{01}(t_{1j})h_{02}(t_{2j})S(t_{1j}, t_{2j})\phi_1(t_{1j}, t_{2j})\eta_j^2}{[\phi_2(t_{1j}, t_{2j})]^\frac{3}{2}} \]

\[ f_2(t_{1j}, c_{2j}) = \frac{\partial S(t_{1j}, c_{2j})}{\partial t_{1j}} = \frac{h_{01}(t_{1j})S(t_{1j}, c_{2j})\phi_1(t_{1j}, c_{2j})\eta_j}{[\phi_2(t_{1j}, c_{2j})]^\frac{3}{2}} \]

\[ f_3(c_{1j}, t_{2j}) = \frac{\partial S(c_{1j}, t_{2j})}{\partial t_{2j}} = \frac{h_{02}(t_{2j})S(c_{1j}, t_{2j})\phi_1(c_{1j}, t_{2j})\eta_j}{[\phi_2(c_{1j}, t_{2j})]^\frac{3}{2}} \]

and \( f_4(c_{1j}, c_{2j}) = S(c_{1j}, c_{2j}) \)

(3.13)

where \( \phi_1(a_j, b_j) = 1 + \theta[1 - ln(S(a_j, b_j))] \) and \( \phi_2(a_j, b_j) = 1 + 2\theta\eta_j(H_{01}(a_j) + H_{02}(b_j)) \). Substituting the hazard function \( h_{01}(t_{1j}), h_{02}(t_{2j}) \), the distribution function \( S(t_{1j}, t_{2j}) \) and the cumulative hazard function \( H_{01}(t_{1j}) \) and \( H_{02}(t_{2j}) \), we get the likelihood function given by equation (3.11).

For Model V...
\[ f_1(t_{ij}, t_{2j}) = (\alpha_1 \frac{\lambda_1 \gamma_1 t_{ij}^{\gamma_1-1}}{1 + \lambda_1 t_{ij}^{\gamma_1}}) \alpha_2 \frac{\lambda_2 \gamma_2 t_{2j}^{\gamma_2-1}}{1 + \lambda_2 t_{2j}^{\gamma_2}} \eta_j^2 \\
\quad \quad \quad \exp \left[ \frac{1 - \left[ 1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 t_{ij}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 t_{2j}^{\gamma_2}) \right) \right]}{\theta} \right] \\
\quad \quad \quad \left(1 + \theta \left[ 1 - \left( \frac{1 - \left[ 1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 t_{ij}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 t_{2j}^{\gamma_2}) \right) \right]}{\theta} \right) \right] \\
\quad \quad \quad \left(1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 t_{ij}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 t_{2j}^{\gamma_2}) \right) \right)^{\frac{1}{2}} \]

\[ f_2(t_{ij}, c_{2j}) = (\alpha_1 \frac{\lambda_1 \gamma_1 t_{ij}^{\gamma_1-1}}{1 + \lambda_1 t_{ij}^{\gamma_1}}) \eta_j \\
\quad \quad \quad \exp \left[ \frac{1 - \left[ 1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 t_{ij}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 c_{2j}^{\gamma_2}) \right) \right]}{\theta} \right] \\
\quad \quad \quad \left(1 + \theta \left[ 1 - \left( \frac{1 - \left[ 1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 t_{ij}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 c_{2j}^{\gamma_2}) \right) \right]}{\theta} \right) \right] \\
\quad \quad \quad \left(1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 t_{ij}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 c_{2j}^{\gamma_2}) \right) \right)^{\frac{1}{2}} \]

\[ f_3(c_{1j}, t_{2j}) = (\alpha_2 \frac{\lambda_2 \gamma_2 t_{2j}^{\gamma_2-1}}{1 + \lambda_2 t_{2j}^{\gamma_2}}) \eta_j \\
\quad \quad \quad \exp \left[ \frac{1 - \left[ 1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 c_{1j}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 t_{2j}^{\gamma_2}) \right) \right]}{\theta} \right] \\
\quad \quad \quad \left(1 + \theta \left[ 1 - \left( \frac{1 - \left[ 1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 c_{1j}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 t_{2j}^{\gamma_2}) \right) \right]}{\theta} \right) \right] \\
\quad \quad \quad \left(1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 c_{1j}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 t_{2j}^{\gamma_2}) \right) \right)^{\frac{1}{2}} \]

and

\[ f_4(c_{1j}, c_{2j}) \]

\[ = \exp \left[ 1 - \frac{\left[ 1 + 2\theta \eta_j \left( \alpha_1 \ln (1 + \lambda_1 c_{1j}^{\gamma_1}) + \alpha_2 \ln (1 + \lambda_2 c_{2j}^{\gamma_2}) \right) \right]}{\theta} \right] \]  

(3.14)
For Model VI

\[
f_1(t_{ij}, t_{2j}) = \frac{\alpha_1 \lambda_1 \gamma_1 t_{ij}^{\gamma_1-1} e^{-\lambda_1 t_{ij}^{\gamma_1}} (1 - e^{-\lambda_2 t_{ij}^{\gamma_1}})^{-1}}{1 - (1 - e^{-\lambda_1 t_{ij}^{\gamma_1}}) \alpha_1} \\
\exp \left[ \frac{1 - [1 - 2\theta \eta_j (ln(1 - (1 - e^{-\lambda_1 t_{ij}^{\gamma_1}}) \alpha_1) + ln(1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_1}}) \alpha_2))]^{1/2}}{\theta} \right] \\
(1 + \theta[1 - (\frac{1 - [1 - 2\theta \eta_j (ln((1 - (1 - e^{-\lambda_1 t_{ij}^{\gamma_1}}) \alpha_1) \times (1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_1}}) \alpha_2))]^{1/2})]}]) / \\
(1 - 2\theta \eta_j (ln(1 - (1 - e^{-\lambda_1 t_{ij}^{\gamma_1}}) \alpha_1) + ln(1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_1}}) \alpha_2))]^{1/2})^{1/2}
\]

\[
f_2(t_{1j}, c_{2j}) = \frac{\alpha_1 \lambda_1 \gamma_1 t_{ij}^{\gamma_1-1} e^{-\lambda_1 t_{ij}^{\gamma_1}} (1 - e^{-\lambda_2 t_{ij}^{\gamma_1}})^{-1}}{1 - (1 - e^{-\lambda_1 t_{ij}^{\gamma_1}}) \alpha_1} \\
\exp \left[ \frac{1 - [1 - 2\theta \eta_j (ln(1 - (1 - e^{-\lambda_1 t_{ij}^{\gamma_1}}) \alpha_1) + ln(1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_1}}) \alpha_2))]^{1/2}}{\theta} \right] \\
(1 + \theta[1 - (\frac{1 - [1 - 2\theta \eta_j (ln((1 - (1 - e^{-\lambda_1 t_{ij}^{\gamma_1}}) \alpha_1) \times (1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_1}}) \alpha_2))]^{1/2})]}]) / \\
(1 - 2\theta \eta_j (ln(1 - (1 - e^{-\lambda_1 t_{ij}^{\gamma_1}}) \alpha_1) + ln(1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_1}}) \alpha_2))]^{1/2})^{1/2}
\]

\[
f_3(c_{1j}, t_{2j}) = \frac{\alpha_2 \lambda_2 \gamma_2 t_{ij}^{\gamma_2-1} e^{-\lambda_2 t_{ij}^{\gamma_2}} (1 - e^{-\lambda_2 t_{ij}^{\gamma_2}})^{-1}}{1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_2}}) \alpha_2} \eta_j^2 \\
\exp \left[ \frac{1 - [1 - 2\theta \eta_j (ln(1 - (1 - e^{-\lambda_1 c_{ij}^{\gamma_1}}) \alpha_1) + ln(1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_2}}) \alpha_2))]^{1/2}}{\theta} \right] \\
(1 + \theta[1 - (\frac{1 - [1 - 2\theta \eta_j (ln((1 - (1 - e^{-\lambda_1 c_{ij}^{\gamma_1}}) \alpha_1) \times (1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_2}}) \alpha_2))]^{1/2})]}]) / \\
(1 - 2\theta \eta_j (ln(1 - (1 - e^{-\lambda_1 c_{ij}^{\gamma_1}}) \alpha_1) + ln(1 - (1 - e^{-\lambda_2 t_{ij}^{\gamma_2}}) \alpha_2))]^{1/2})^{1/2}
\]
and

\[ f_4(c_{1j}, c_{2j}) = \exp \left[ 1 - \left[ 1 - 2\theta \eta_4 (\ln(1 - (1 - e^{-\lambda_1 c_{1j}^1})^{\alpha_1}) + \ln(1 - (1 - e^{-\lambda_2 c_{2j}^2})^{\alpha_2})) \right] \theta \right] \]

\[ (3.15) \]

Computing the maximum likelihood estimators (MLEs) involves solving a twelve dimensional optimization problem for Model V and Model VI eleven dimensional optimization problem for Model II and Model IV. As the method of maximum likelihood fails to estimate the several parameters due to convergence problem in the iterative procedure, so we use the Bayesian approach. The traditional maximum likelihood approach to estimation is commonly used in survival analysis, but it can encounter difficulties with frailty models. Moreover, standard maximum likelihood based inference methods may not be suitable for small sample sizes or situations in which there is heavy censoring see Kheiri et al. (2007). Thus, in our problem a Bayesian approach, which does not suffer from these difficulties, is a natural one, even though it is relatively computationally intensive.

The joint posterior density function of the parameters for given failure times in the proposed frailty models is obtained as,

\[ \pi(\alpha_1, \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \theta, \beta) \propto L(\alpha_1, \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \theta, \beta) \]

\[ \times g_1(\alpha_1)g_2(\lambda_1)g_3(\gamma_1)g_4(\alpha_2)g_5(\lambda_2)g_6(\gamma_2)g_7(\theta) \prod_{i=1}^{5} p_i(\beta_i) \]

where \( g_i(.) \) \( (i = 1, 2, \cdots, 7) \) indicates the prior density function with known hyper parameters of corresponding arguments for baseline parameters and frailty variance; \( p_i(.) \) is prior density function for regression coefficient \( \beta_i; \overline{\beta} \) represents a vector of regression coefficients except \( \beta_i, i = 1, 2, \ldots, k \) and likelihood function \( L(.) \) is given by equation (3.11) or (3.12). Here we assume
that all the parameters are independently distributed. In the same way one
can write the joint posterior density function of the parameters in the models
without frailty.

To estimate the parameters of the model, we used Metropolis-Hastings
algorithm and Gibbs sampler. We monitored the convergence of a Markov
chain to a stationary distribution by the Geweke test (Geweke, 1992) and the
Gelman-Rubin Statistics (Gelman et al., 1992). The trace plots, the coupling
from the past plots and the sample autocorrelation plots are used to check the
behaviour of the chain, to decide the burn-in period and the autocorrelation
lag respectively.

Algorithm consists in successively obtaining a sample from the conditional
distribution of each of the parameter given all other parameters of the model.
These distributions are known as full conditional distributions. In our case
full conditional distributions are not easy to integrate out. So full conditional
distributions are obtained by considering that they are proportional to the
joint distribution of the parameters of the model.

We have full conditional distribution of the parameter $\alpha_1$ with frailty as,

$$
\pi_1(\alpha_1 \mid \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \theta, \beta) \propto L(\alpha_1, \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \theta, \beta) \cdot g_1(\alpha_1)
$$

(3.16)

We have full conditional distribution of the parameter $\alpha_1$ without frailty as,

$$
\pi_1(\alpha_1 \mid \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \beta) \propto L(\alpha_1, \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \beta) \cdot g_1(\alpha_1)
$$

(3.17)

Similarly full conditional distributions for other parameters can be obtained.
3.5 Simulation Study

To evaluate the performance of the Bayesian estimation procedure for inverse Gaussian frailty models we carried out a simulation study. For the simulation purpose we have considered only one covariate \( X_0 \) which we assume to follow binomial distribution. The frailty variable \( Z \) is assumed to have inverse Gaussian distribution with known variance. Life times \((T_{1j}, T_{2j})\) for \( j^{th} \) individual are conditionally independent for given frailty \( Z_j = z_j \). We assume that \( T_{ij} \) \((i = 1, 2; j = 1, 2, \cdots, n)\) follows one of the baseline distributions; the generalized log logistic type II (Model V) or the generalized Weibull (Model VI) distribution. Simulation details of model without frailty are not discussed here.

As the Bayesian methods are time consuming, we generate only 50 pairs of life times. According to the assumption, for given frailty \((Z)\), life times of individuals are independent. So the conditional survival function for an individual for given frailty, \( Z = z \) and a covariate \( X_0 \) at time \( t > 0 \) is,

\[
S(t | z, X_0) = e^{-zH_0(t)\eta}
\]

where \( \eta = e^{X_0\beta_0} \). Equating \( S(t | z, X_0) \) to a random number, say \( r \) \((0 < r < 1)\), over \( t > 0 \) we get,

for Model V

\[
t = \left[ \frac{r^{-\frac{1}{\alpha}} - 1}{\lambda} \right]^\frac{1}{\gamma}
\]

(3.18)

for Model VI

\[
t = \left[ -\frac{\ln(1-(1-r^{\frac{1}{\alpha}})^{1/\alpha})}{\lambda} \right]^{1/\gamma}
\]

(3.19)

Equations (3.18) and (3.19) are generators to generate life times for model V and model VI respectively. Samples are generated using the following procedure;
1. Generate 50 covariate values for $X_0$ from the binomial distribution with $p = 0.45$ and $n = 1$.

2. Compute $\eta = e^{X_0\beta}$ with the regression coefficient (known).

3. Generate 50 pairs of lifetimes $(t_{1j}, t_{2j})$ for a given frailty ($z_j$) using the following generators,

$$t_{ij} = \left[ \frac{1 - \frac{1}{r_{ij} \eta_i \alpha_i}}{\lambda_i} - 1 \right]^{\frac{1}{\gamma_i}}, \quad i = 1, 2; j = 1, 2, \ldots, n$$

$$t_{ij} = \left[ -\ln \left( 1 - (1 - \frac{1}{r_{ij} \eta_i \alpha_i})^{1/\alpha_i} \right) \right]^{1/\lambda_i}, \quad i = 1, 2; j = 1, 2, \ldots, n$$

for Model V and Model VI, $r_{ij}$ is the random variables having $U(0,1)$ distribution and $\alpha_1, \gamma_1, \lambda_1$ are the parameters of the baseline distribution for the first survival time and $\alpha_2, \gamma_2, \lambda_2$ are that of the second survival time.

4. Generate the censoring times ($c_{1j}$ and $c_{2j}$) from the exponential distribution with the failure rates 0.01 and 0.005 for Model V and VI respectively.

5. Observe the $i^{th}$ survival time $t_{ij}^* = \min(t_{ij}, c_{ij})$ and the censoring indicator $\delta_{ij}$ for the $j^{th}$ individual ($i = 1, 2$ and $j = 1, 2, \ldots, 50$), where

$$\delta_{ij} = \begin{cases} 
1, & t_{ij} < c_{ij} \\
0, & t_{ij} > c_{ij} 
\end{cases}$$

Thus we have data consisting of 50 pairs of survival times $(t_{1j}^*, t_{2j}^*)$ and the censoring indicators $\delta_{ij}$.

A widely used prior for frailty parameter $\theta$ is the gamma distribution $G(0.0001, 0.0001)$. In addition, we assume that the regression coefficients are normal with mean zero and large variance say 1000. In our study we also
use non informative prior for frailty parameter $\theta$ and regression coefficient $\beta_0$. Since we do not have any prior information about baseline parameters, $\lambda_1, \gamma_1, \alpha_1, \lambda_2, \gamma_2$ and $\alpha_2$, prior distributions are assumed to be flat. We consider two different non-informative prior distributions for baseline parameters, one is $G(a_1, a_2)$ and another is $U(b_1, b_2)$. All the hyper-parameters $a_1, a_2, b_1$ and $b_2$ are known. Here $G(a, b)$ is the gamma distribution with the shape parameter $a$ and the scale parameter $b$ and $U(b_1, b_2)$ represents uniform distribution over the interval $(b_1, b_2)$.

Table 3.1: Baseline Distribution Generalized Log-Logistic Type II Distribution Model V with inverse Gaussian frailty (Simulation for Model V).

<table>
<thead>
<tr>
<th>Parameter (True value)</th>
<th>Estimate</th>
<th>Standard error</th>
<th>Lower Credible Limit</th>
<th>Upper Credible Limit</th>
<th>Geweke p values</th>
<th>p values</th>
<th>Gelman &amp; Rubin values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$ (2.0)</td>
<td>1.9925</td>
<td>0.1715</td>
<td>1.6418</td>
<td>2.3448</td>
<td>0.00192</td>
<td>0.500766</td>
<td>1.0007</td>
</tr>
<tr>
<td>$\alpha_2$ (2.2)</td>
<td>2.0238</td>
<td>0.2623</td>
<td>1.5386</td>
<td>2.4683</td>
<td>-0.01467</td>
<td>0.49414</td>
<td>1.0013</td>
</tr>
<tr>
<td>$\lambda_1$ (0.005)</td>
<td>0.0051</td>
<td>0.0011</td>
<td>0.0031</td>
<td>0.0068</td>
<td>-0.00302</td>
<td>0.49879</td>
<td>1.0000</td>
</tr>
<tr>
<td>$\lambda_2$ (0.005)</td>
<td>0.0049</td>
<td>0.0009</td>
<td>0.0033</td>
<td>0.0067</td>
<td>-0.00100</td>
<td>0.49960</td>
<td>1.0001</td>
</tr>
<tr>
<td>$\gamma_1$ (1.9)</td>
<td>1.8607</td>
<td>0.1223</td>
<td>1.6602</td>
<td>2.1207</td>
<td>-0.01260</td>
<td>0.49497</td>
<td>1.0834</td>
</tr>
<tr>
<td>$\gamma_2$ (2.9)</td>
<td>1.8568</td>
<td>0.1169</td>
<td>1.6451</td>
<td>2.1119</td>
<td>-0.00256</td>
<td>0.49898</td>
<td>1.0027</td>
</tr>
<tr>
<td>$\theta$ (0.02)</td>
<td>0.0198</td>
<td>0.0051</td>
<td>0.0107</td>
<td>0.0288</td>
<td>-0.00209</td>
<td>0.49916</td>
<td>1.0002</td>
</tr>
<tr>
<td>$\beta_0$ (0.56)</td>
<td>0.0543</td>
<td>0.0289</td>
<td>-0.0265</td>
<td>0.1404</td>
<td>0.00989</td>
<td>0.50395</td>
<td>1.0011</td>
</tr>
</tbody>
</table>

We use different value of baseline parameters for Model V and Model VI, details are given in Table (3.1) and (3.2). We assume the value of the hyper-parameters as $a_1 = 1, a_2 = 0.0001, b_1 = 0$ and $b_2 = 100$. We run two parallel chains for model one using two sets of prior distributions with the different starting points using Metropolis-Hastings algorithm and Gibbs sampler based on normal transition kernels. We iterate both the chains for 100000 times. There is no effect of prior distribution on posterior summaries because the estimates of parameters are nearly the same and the convergence rate of Gibbs sampler for both the prior sets is almost the same. Also for both the chains the
Table 3.2: Baseline Distribution Generalized Weibull Distribution Model-VI with inverse Gaussian frailty (Simulation for Model VI).

<table>
<thead>
<tr>
<th>Parameter (value)</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Lower Credible Limit</th>
<th>Upper Credible Limit</th>
<th>Geweke values</th>
<th>p values</th>
<th>Gelman &amp; Rubin values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$ (2.00)</td>
<td>2.0581</td>
<td>0.1539</td>
<td>1.7696</td>
<td>2.3792</td>
<td>-0.00587</td>
<td>0.4976</td>
<td>1.0024</td>
</tr>
<tr>
<td>$\alpha_2$ (2.00)</td>
<td>2.0023</td>
<td>0.1460</td>
<td>1.7452</td>
<td>2.2598</td>
<td>0.002279</td>
<td>0.5009</td>
<td>1.0188</td>
</tr>
<tr>
<td>$\lambda_1$ (0.15)</td>
<td>0.1617</td>
<td>0.0294</td>
<td>0.1114</td>
<td>0.2234</td>
<td>-0.01169</td>
<td>0.4953</td>
<td>1.0311</td>
</tr>
<tr>
<td>$\lambda_2$ (0.15)</td>
<td>0.1668</td>
<td>0.0209</td>
<td>0.1193</td>
<td>0.1982</td>
<td>0.013536</td>
<td>0.5054</td>
<td>1.0100</td>
</tr>
<tr>
<td>$\gamma_1$ (0.90)</td>
<td>0.9262</td>
<td>0.0463</td>
<td>0.8313</td>
<td>0.9958</td>
<td>0.013469</td>
<td>0.5053</td>
<td>1.0120</td>
</tr>
<tr>
<td>$\gamma_2$ (0.90)</td>
<td>0.9357</td>
<td>0.0401</td>
<td>0.8488</td>
<td>0.9987</td>
<td>-0.00941</td>
<td>0.4962</td>
<td>1.0023</td>
</tr>
<tr>
<td>$\theta$ (0.50)</td>
<td>0.4518</td>
<td>0.0991</td>
<td>0.3072</td>
<td>0.6710</td>
<td>-0.00171</td>
<td>0.4993</td>
<td>1.0053</td>
</tr>
<tr>
<td>$\beta_0$ (0.05)</td>
<td>0.0508</td>
<td>0.0383</td>
<td>-0.0189</td>
<td>0.1260</td>
<td>0.009731</td>
<td>0.5038</td>
<td>1.0351</td>
</tr>
</tbody>
</table>

results were somewhat similar. Table (3.1) presents the estimates, the credible intervals, the Gelman-Rubin convergence statistic and the Geweke test for all the parameters of the Model V based on the simulation study. Table (3.2) gives the estimates, the credible intervals, the Gelman-Rubin convergence statistic and the Geweke test for all the parameters of the Model VI based on the simulation study.

From Table (3.1) and Table (3.2), it is observed that estimated values are close to real value, standard errors are quite small for the Model V and Model VI. We present simulation study only for the Model V and Model VI which are frailty models. Gelman-Rubin convergence statistic values are nearly equal to one and also Geweke test values are quite small and the corresponding p-values are large enough to say the chain attains stationary distribution.

### 3.6 Analysis of Kidney Infection data

To illustrate the Bayesian estimation procedure for inverse Gaussian frailty models we use kidney infection data of McGilchrist and Aisbett (1991), which
is used in previous chapter. First we check goodness of fit of the data for the inverse Gaussian frailty distributions with two baseline distributions and then we apply the Bayesian estimation procedure. To check goodness of fit of kidney data set, we consider Kolmogrove-Smirnov (K-S) test for two baseline distributions.

Table 3.3: p-values of K-S Statistics for goodness of fit test for Kidney Infection data set.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>recurrence time</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>first</td>
<td>second</td>
</tr>
<tr>
<td>Model V</td>
<td>0.20519</td>
<td>0.56964</td>
</tr>
<tr>
<td>Model VI</td>
<td>0.36541</td>
<td>0.73492</td>
</tr>
</tbody>
</table>

Table (3.3) gives the p-values of goodness of fit test for Models V and VI. Thus from p-values of K-S test we can say that there is no statistical evidence to reject the hypothesis that data are from Model V and Model VI in the univariate case and we assume that they also fit for bivariate case. Figure 3.1 show the parametric plot vs non parametric plot.

As in case of simulation, here also we assume same set of prior distributions. We run two parallel chains for both models using two sets of prior distributions with the different starting points using the Metropolis-Hastings algorithm and the Gibbs sampler based on normal transition kernels. We iterate both the chains for 100000 times. As seen in the simulation study here also we got nearly the same estimates of parameters for both the set of prior, so estimates are not dependent on the different prior distributions. The convergence rate of the Gibbs sampler for both the prior sets is almost the same. Also both the chains shows somewhat similar results, so we present here the analysis for only one chain with $G(a_1, a_2)$ as prior for the baseline parameters, for the models.

The Gelman-Rubin convergence statistic values are nearly equal to one and the Geweke test statistic values are quite small and the corresponding
Figure 3.1: Survival function plots for (K-M survival and parametric survival).

p-values are large enough to say that the chains attain stationary distribution. The posterior mean and the standard error with 95% credible intervals, the Gelman-Rubin statistics values and the Geweke test values with p-values for Model V and VI are presented in Table (3.4) and (3.5). The AIC, BIC and DIC values for both models are given in Table (3.6). The Bayes factor for all frailty models are given in Table (3.7).

The comparison between four proposed models is done using AIC, BIC and DIC values given in Table (3.6). The smallest AIC value is model VI (generalized Weibull distribution with frailty). Same result hold for BIC and DIC value. To take the decision about Models V to VI, we use the Bayes
Table 3.4: Posterior summary for Kidney Infection data set Model V.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Lower Credible Limit</th>
<th>Upper Credible Limit</th>
<th>Geweke p</th>
<th>Gelman &amp; Rubin Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>burn in period = 2500; autocorrelation lag = 300</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>2.0179</td>
<td>0.1613</td>
<td>1.7370</td>
<td>2.3644</td>
<td>0.00891</td>
<td>0.5035</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0.0059</td>
<td>0.0026</td>
<td>0.0147</td>
<td>0.0019</td>
<td>0.00687</td>
<td>0.4972</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>1.3312</td>
<td>0.1673</td>
<td>1.0261</td>
<td>1.6449</td>
<td>0.00485</td>
<td>0.5019</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>3.1338</td>
<td>0.4839</td>
<td>2.2835</td>
<td>4.0667</td>
<td>0.00296</td>
<td>0.5012</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0.0019</td>
<td>0.0008</td>
<td>0.0005</td>
<td>0.0036</td>
<td>0.00661</td>
<td>0.5026</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>1.3083</td>
<td>0.1148</td>
<td>1.0993</td>
<td>1.5341</td>
<td>-0.00284</td>
<td>0.4988</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.0014</td>
<td>0.0004</td>
<td>0.0007</td>
<td>0.0024</td>
<td>-0.00563</td>
<td>0.4977</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.0126</td>
<td>0.0077</td>
<td>-0.0023</td>
<td>0.0265</td>
<td>0.00377</td>
<td>0.5015</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-1.4645</td>
<td>0.3377</td>
<td>-2.1195</td>
<td>-0.8457</td>
<td>-0.01279</td>
<td>0.4948</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>-0.1826</td>
<td>0.2401</td>
<td>-0.6365</td>
<td>0.2756</td>
<td>0.00097</td>
<td>0.5003</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>-0.0198</td>
<td>0.0361</td>
<td>-0.0911</td>
<td>0.0449</td>
<td>0.00042</td>
<td>0.5001</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>-1.4177</td>
<td>0.5540</td>
<td>-2.4133</td>
<td>-0.4529</td>
<td>-0.00486</td>
<td>0.4981</td>
</tr>
</tbody>
</table>

Table 3.5: Posterior summary for Kidney Infection data set Model VI.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Lower Credible Limit</th>
<th>Upper Credible Limit</th>
<th>Geweke p</th>
<th>Gelman &amp; Rubin Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>burn in period = 2000; autocorrelation lag = 350</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>2.3871</td>
<td>0.3045</td>
<td>1.7227</td>
<td>2.99634</td>
<td>-0.00627</td>
<td>0.49749</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0.1516</td>
<td>0.0248</td>
<td>0.10304</td>
<td>0.20115</td>
<td>-0.00532</td>
<td>0.49787</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>0.7598</td>
<td>0.0487</td>
<td>0.66827</td>
<td>0.84953</td>
<td>0.00391</td>
<td>0.50156</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>5.1831</td>
<td>0.5290</td>
<td>4.28258</td>
<td>6.09560</td>
<td>-0.00261</td>
<td>0.49895</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0.2398</td>
<td>0.0777</td>
<td>0.11749</td>
<td>0.45018</td>
<td>-0.01154</td>
<td>0.49539</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>0.4723</td>
<td>0.0847</td>
<td>0.49494</td>
<td>0.82401</td>
<td>0.01020</td>
<td>0.50407</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.4154</td>
<td>0.1891</td>
<td>0.07942</td>
<td>0.78744</td>
<td>0.01322</td>
<td>0.50527</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-0.0004</td>
<td>0.0043</td>
<td>-0.00829</td>
<td>0.00076</td>
<td>0.00434</td>
<td>0.50173</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-1.5368</td>
<td>0.3448</td>
<td>-2.19859</td>
<td>-0.80260</td>
<td>5.396e-05</td>
<td>0.50173</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.0556</td>
<td>0.0351</td>
<td>-0.00240</td>
<td>0.12378</td>
<td>0.004227</td>
<td>0.50168</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>-0.0127</td>
<td>0.0138</td>
<td>-0.03931</td>
<td>0.01285</td>
<td>8.22557e-05</td>
<td>0.50003</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.0123</td>
<td>0.0177</td>
<td>-0.02136</td>
<td>0.04364</td>
<td>-0.005018</td>
<td>0.49799</td>
</tr>
</tbody>
</table>

factor. The Bayesian test based on the Bayes factors for Model V against Model II is 4.90 and Model VI against Model IV is 3.80 which support Model...
Table 3.6: Comparison of AIC, BIC, DIC.

<table>
<thead>
<tr>
<th></th>
<th>AIC</th>
<th>BIC</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLLD</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>With Frailty</td>
<td>M V</td>
<td>689.144</td>
<td>708.795</td>
</tr>
<tr>
<td>Without Frailty</td>
<td>M II</td>
<td>696.9474</td>
<td>714.9608</td>
</tr>
<tr>
<td>GWD</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>With Frailty</td>
<td>M VI</td>
<td>686.771</td>
<td>706.422</td>
</tr>
<tr>
<td>Without Frailty</td>
<td>M IV</td>
<td>690.2814</td>
<td>708.2949</td>
</tr>
</tbody>
</table>

V and Model VI for kidney infection data set compared to their corresponding models without frailty \((\theta = 0)\) and frailty is significant in Model V and Model VI. Some patients are expected to be vary prone to infection compared to others with same covariate value. This is not surprising, as seen in the data set there is a male patient with infection time 8 and 16, and there is also male patient with infection time 152 and 562. Table (3.7) shows that frailty models are better than without frailty models and Model VI is better then Model V. From Table (3.6) and (3.7), we can observe that, Model VI is the best. To check the adequacy of the Model V, Model II, Model VI and Model IV firstly we have constructed 99%, 95%, 90%, 75% and 50% equal tailed predictive intervals of the generated random sample from the predictive distribution and counted the total number of intervals in which the \(r^{th}\) observation falls in their respective intervals. Details are given in Table (3.6).

Table (3.6) shows that all four models (with frailty and without frailty) are adequate for the kidney infection Data. We can observe that regression coefficients for all the four models are different. The only credible interval of the regression coefficient \(\beta_2\) does not contain zero which indicates that the
Table 3.7: Bayes factor values and decision for models fitted to kidney infection data set.

<table>
<thead>
<tr>
<th>Numerator model against denominator model</th>
<th>$2\log(B_{jk})$</th>
<th>Range</th>
<th>Evidence against model in denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M\ V$ against $M\ II$</td>
<td>4.90</td>
<td>2 to 6</td>
<td>positive</td>
</tr>
<tr>
<td>$M\ VI$ against $M\ I$</td>
<td>3.47</td>
<td>2 to 6</td>
<td>positive</td>
</tr>
<tr>
<td>$M\ V$ against $M\ IV$</td>
<td>0.34</td>
<td>&lt; 2</td>
<td>no difference</td>
</tr>
<tr>
<td>$M\ VI$ against $M\ II$</td>
<td>8.37</td>
<td>6 to 10</td>
<td>strong</td>
</tr>
<tr>
<td>$M\ VI$ against $M\ IV$</td>
<td>3.80</td>
<td>2 to 6</td>
<td>positive</td>
</tr>
</tbody>
</table>

$B_{jk} = 2 \times \log_e \left( \frac{M_j}{M_k} \right)$

Table 3.8: Predictive interval for four models fitted to kidney infection data.

<table>
<thead>
<tr>
<th>Model</th>
<th>99%</th>
<th>95%</th>
<th>90%</th>
<th>75%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model V</td>
<td>76</td>
<td>76</td>
<td>73</td>
<td>64</td>
<td>59</td>
</tr>
<tr>
<td>Model II</td>
<td>76</td>
<td>74</td>
<td>69</td>
<td>60</td>
<td>44</td>
</tr>
<tr>
<td>Model VI</td>
<td>75</td>
<td>74</td>
<td>72</td>
<td>65</td>
<td>56</td>
</tr>
<tr>
<td>Model IV</td>
<td>75</td>
<td>71</td>
<td>68</td>
<td>60</td>
<td>42</td>
</tr>
</tbody>
</table>

covariate sex is significant for the Model II, VI and VI. But in Model V, $\beta_2$ and $\beta_5$ are significant. Negative value of $\beta_2$ indicates that the female patients have a slightly lower risk for infection. All other covariates other than sex are insignificant except in Model V. For Model V disease type PKD is also significant.

3.7 Conclusion

In this chapter we discuss results for inverse Gaussian shared frailty models with two different base line distributions. We use the generalized log logistic type II and the generalized Weibull as a baseline distributions. Main
aim of our study is to check which distribution (with inverse Gaussian frailty or without frailty) fits better. Models V and VI are with inverse Gaussian frailty and Models II and IV are without frailty. We perform simulation study and also to analyze kidney infection data by using R. We use Bayesian approach to estimate the parameters. The entire estimation procedure using Bayesian approach took large amount of computational time but the time was more or less the same for all the four models. In terms of convergence rate, we had faster convergence of Model V as compared to Model II, Model VI and Model IV.

Different prior gives the same estimates of the parameters. The convergence rate of the Gibbs sampling algorithm does not depend on these choices of the prior distributions in our proposed model for kidney infection data. The estimate of $\theta$ (Model V, $\theta = 0.0014$; Model VI, $\theta = 0.4154$) from Model VI show that there is a strong evidence of high degree of heterogeneity in the population of patients. The Bayes factor is used to test the frailty parameter $\theta = 0$ and it is observed that frailty is present and models with frailty fit better than without frailty models. The covariate sex is the only covariate which is significant for all models. Negative value of regression coefficient ($\beta_2$) of covariate sex indicates that the female patients have a slightly lower the risk of infection.

The comparison between four proposed models is done using AIC, BIC and DIC values. The smallest AIC value is Model VI (generalized Weibull distribution with frailty). The same result holds for BIC and DIC value. But these difference are not much significant. To take the decision about Model V, Model II, Model VI and Model IV, we use the Bayes factor. We observe that, the Model VI is best. In this case we can conclude that the inverse Gaussian frailty models are better than without frailty models. Also we can conclude that the shared inverse Gaussian frailty with the generalized Weibull distribu-
tion as the baseline distribution is a better fit than the shared frailty model with the generalized log-logistic type II distribution. By referring all the above analysis now we are in a position to say that, we have suggested a new shared inverse Gaussian frailty model with the generalized Weibull distribution as the baseline distribution which is the best in the proposed models for modeling of kidney infection data.