CHAPTER 5

BIASED-MODEL REDUCTION BY SIMPLIFIED ROUTH APPROXIMATION METHOD (SRAM)
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5.1 INTRODUCTION

Most of the available model reduction methods including the Routh Approximation Methods [18-21,24,25] for linear time-invariant systems are based on producing models which retain the first \( k \) time-moments of the original high-order system. By retaining the first \( k \) time-moments of the original system in the \( k \)th-order reduced model, they ensure good approximation of steady-state part of time response of the original system. But it has been shown by Shamash [33] and Pal [34] among others, that it is sometimes desirable to retain initial Markov-parameters of the original system along with the time-moments in the reduced order model. This produces more accurate initial transient response approximation than that obtained by retaining only time-moments. These models which are capable of retaining some of the first time-moments as well as Markov-parameters of the original high-order system are known as biased models.

Therefore it is obvious that it is always useful to have an approximation of high-order systems with the biased model which retains \( t \) first 't' time-moments and 'm' Markov-parameters of the original system where \( t + m = k \) (order of the model). These biased models give better approximation of steady state as well as transient parts of the time response of the original system.
5.2 LIMITATIONS OF OTHER METHODS

5.2.1 Limitations of the Existing Routh Approximation Methods for obtaining biased models

The existing Routh Approximation methods like RAM, DRAM [19,20,24,25] are all based on both $\alpha$ and $\beta$ parameters and involve recursive formulae to develop reduced order models. These models are capable of retaining only the first $k$ time-moments of the original system. It becomes very laborious to use these methods to develop procedures for biased model reduction because of the presence of $\beta$-parameters. There is no procedure available in the literature for biased model reduction using these Routh Approximation Methods.

5.2.2 Limitations of Factor Division Method of biased model reduction

One of the available methods for biased model reduction given by Lucas [35] using Factor division method is also a laborious procedure. The Factor division method gives a procedure to generate biased models which retain some of the initial time-moments as well as Markov-parameters of the original system. But it gives a procedure to develop a numerator of the biased model from an already available stable denominator obtained by using any other method. Factor division method cannot generate the denominator of the biased model on its own.

Even then, to produce the numerator of the biased model, after using the stable denominator obtained by some other method, the Factor division method needs formulation of two Routh type arrays. This involves more computations and the method becomes very tedious.
5.3 AIMS OF BIASED MODEL REDUCTION METHOD USING SRAM [36]

In this chapter a new simple procedure to reduce biased models for high order systems is proposed and discussed. The model reduction by the simplified Routh Approximation method (SRAM) is extended to produce biased models with the following aims such that it
(a) Should use only one Routh type array to generate both numerator and denominator polynomials of the kth-order model.
(b) Should avoid the computation of the time-moments and Markov-parameters of the retained system beforehand and solving the Padé equation for the reduced numerator.
(c) Should generate reduced order biased models retaining the initial time-moments as well as Markov-parameters of the original high-order system.
(d) Should be applicable to obtain the biased order model numerator directly if the stable reduced order denominator is available using any other method.

5.4 DEVELOPMENT OF THE NEW METHOD [36]

The Simplified Routh Approximation Method (SRAM) is extended to develop a new method with less computational effort to produce biased models. The procedural steps involved in the new method are shown by thick lines in Fig.5.1.
Let the original nth-order system be:

\[
G(s) = \frac{B_0 + B_1s + \ldots + B_{n-1}s^{n-1}}{A_0 + A_1s + \ldots + A_ns^n}
\]  

...(5.1)

Then the reduced model of order k, formed to retain the first 't' time-moments and 'm' Markov-parameters of \( G(s) \) with \( k = t + m \) is defined by:

\[
N_k(t,m)(s) = \frac{R_k(s)}{D_k(s)}
\]

...(5.2)

**Development of Denominator \( D_k(s) \)**

By using the Simplified Routh Approximation Method (SRAM), the denominator of the kth-order model is defined as:

\[
D_k(s) = s^k + \sum_{j=0}^{k-1} A_js^j
\]  

...(5.3)

The parameters \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are obtained from the new table of SRAM as proposed and discussed in section 4.2 of the thesis.

**Development of Numerator \( N_k(s) \)**

For any given \( D_k(s) \), the numerator of the biased model which will retain the first 't' time-moments and 'm' Markov-parameters of
G(s) is defined as

\[ N_k(t, m)(s) = N_{kt}(s) + N_{km}(s) \quad \text{with } k = t + m \]

\[ = T_1 + T_2 s + \ldots + T_t s^{k-m+1} \]

\[ + M_m s^{k-m} + \ldots + M_2 s^{k-2} + M_1 s^{k-1} \quad \ldots(5.4) \]

Where \( T_1, T_2, \ldots, T_t \) and \( M_1, M_2, \ldots, M_m \) are obtained by using the proposed relations as given below:

\[ T_1 = \frac{a_0}{A_0} - B_0 \]

\[ T_2 = \frac{a_0}{A_0} - B_1 \]

\[ \ldots \]

\[ \ldots \]

and so on.

In general,

\[ T_t = \frac{a_0}{A_0} - B_{t-1} \quad \ldots(5.5) \]

Similarly,

\[ M_1 = \frac{1}{A_n} (B_{n-1} a_k) \]

\[ M_2 = \frac{1}{A_n} (B_{n-1} a_k - B_{n-2} a_k - M_1 A_{n-1}) \]
In general

\[ M_m = \frac{1}{A_n} \left\{ \sum_{i=1}^{m} B_{n-i} a_{k-(m-i)} - \sum_{j=0}^{m-1} M_j A_{n-(m-j)} \right\} \]

with \( M_0 = 0 \) \(...(5.6)\)

The proposed method is illustrated with numerical examples in the following section.

5.5 ILLUSTRATIVE EXAMPLES

Example 5.1 Consider the 5th-order system given by [35]

\[ G(s) = \frac{10s^4 + 82s^3 + 264s^2 + 369s + 156}{2s^5 + 21s^4 + 84s^3 + 173s^2 + 148s + 40} \]

To obtain the denominator of the biased model of order 3, using SRAM we need the \( a \)-parameters. The \( a \)-parameters are obtained by using \( a \)-table of SRAM as given below:

\( a \)-table (SRAM)

<table>
<thead>
<tr>
<th>40</th>
<th>148</th>
<th>173</th>
<th>84</th>
<th>21</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>148</td>
<td>173</td>
<td>84</td>
<td>21</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>148</td>
<td>150.297</td>
<td>84</td>
<td>20.459</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

\( a_1 = 0.27027 < \)

\( \alpha_1 = \frac{0.27027}{148} = 0.27027 < \)

\( \alpha_2 = 0.98471 < \)

\( \alpha_2 = \frac{0.98471}{150.297} = 0.98471 < \)
The denominator of the third order biased models \((k = 3)\) from eqn. (5.3) will be

\[
D_3(s) = s^3 + \frac{a_1 a_2 a_3}{a_0} (A_0 + A_1 s + A_2 s^2)
\]

\[
= s^3 + 2.7091s^2 + 2.3176s + 0.6264
\]

\[
= a_3 s^3 + a_2 s^2 + a_1 s + a_0
\]

Therefore

\[
a_0 = 0.6264
\]

\[
a_1 = 2.3176
\]

\[
a_2 = 2.7091
\]

\[
a_3 = 1.0
\]

Then to obtain the numerator of the third order biased models using eqn.(5.4), the values of \(T_1, T_2, T_3, M_1\) and \(M_2\) are to be calculated.
Using eqn. (5.5),

\[ T_1 = \frac{a_0}{A_0} B_0 = 2.4429 \]

\[ T_2 = \frac{a_0}{A_0} B_1 = 5.7785 \]

\[ T_3 = \frac{a_0}{A_0} B_2 = 4.1342 \]

Similarly using eqn. (5.6),

\[ M_1 = \frac{1}{A_5} (B_{5-1} a_3 - (1-1)) = \frac{1}{A_5} B_4 a_3 = 5 \]

\[ M_2 = \frac{1}{A_5} (B_{5-1} a_3 - (2-1) + B_{5-2} a_3 - (2-2) - M_1 A_5 - (2-1)) \]

\[ = \frac{1}{A_5} (B_4 a_2 + B_3 a_3 - M_1 A_4) = 2.045 \]

Then using eqn. (5.4), the numerator of the third order biased model retaining the first 3 time-moments only (and no Markov-parameters) is,

For \( t = 3, m = 0 \):

\[ N_{3}(3,0)(s) = T_1 + T_2 s + T_3 s^2 \]

\[ = 2.4429 + 5.7785 s + 4.1342 s^2 \]

The numerator of the third order biased model retaining the first 2 time-moments and one Markov-parameter is,
For $t = 2, m = 1$;
\[ N_{3(2,1)}(s) = T_1 + T_2s + M_1s^2 \]
\[ = 2.4429 + 5.7785s + 5s^2 \]
The numerator of the third order biased model retaining the first one
time-moment only and two Markov-parameters is,

For $t = 1, m = 2$;
\[ N_{3(1,2)}(s) = T_1 + M_2s + M_1s^2 \]
\[ = 2.4429 + 2.0450s + 5s^2 \]
The three stable third order reduced biased models which retain first
't' time-moments and 'm' Markov-parameters from G(s), where $t+m = k = 3$
are given by

t = 3; m = 0
\[ R_{3(3,0)}(s) = \frac{4.1342s^2 + 5.7785s + 2.4429}{s^3 + 2.7091s^2 + 2.3176s + 0.6264} \]
t = 2; m = 1
\[ R_{3(2,1)}(s) = \frac{5s^2 + 5.7785s + 2.4429}{s^3 + 2.7091s^2 + 2.3176s + 0.6264} \]
t = 1; m = 2
\[ R_{3(1,2)}(s) = \frac{5s^2 + 2.0450s + 2.4429}{s^3 + 2.7091s^2 + 2.3176s + 0.6264} \]
The time-moments and Markov-parameters of the original system and the
third order biased models obtained by using SRAM are compared below:
Expanding around $s = 0$:

\[ G(s) = 3.90 - 5.205s + 8.991s^2 + \ldots \]

\[ R_3(3,0)(s) = 3.90 - 5.205s + 8.991s^2 + \ldots \]

\[ R_3(2,1)(s) = 3.90 - 5.205s + 10.374s^2 + \ldots \]

\[ R_3(1,2)(s) = 3.90 - 4.3805s + 2.8733s^2 + \ldots \]

Expanding around $s = \infty$:

\[ G(s) = 5s^{-1} - 11.5s^{-2} + \ldots \]

\[ R_3(3,0)(s) = 4.1342s^{-1} - 5.421s^{-2} + \ldots \]

\[ R_3(1,2)(s) = 5s^{-1} - 7.767s^{-2} + \ldots \]

\[ R_3(1,2)(s) = 5s^{-1} - 11.5s^{-2} + \ldots \]

Fig. 5.2 illustrates the different biases by comparing the step responses of all the reduced order models with that of the original system.

Example 5.2 Consider the 5th-order system given by [37]

\[
G(s) = \frac{20s^4 + 190s^3 + 852.5s^2 + 1200s + 300}{s^5 + 25.2s^4 + 132.5s^3 + 626.5s^2 + 300s + 60}
\]

To obtain the denominator of the biased model of order 3, using SRAM we need the $\alpha$-parameters. The $\alpha$-parameters are obtained by using $\alpha$-table of SRAM as given below:

$\alpha$ - table (SRAM)

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20000&lt;</td>
<td>60</td>
<td>300</td>
<td>626.5</td>
<td>132.5</td>
<td>25.2</td>
</tr>
</tbody>
</table>

\[ a_1 = 0.20000< \]

\[ 300 \quad 626.5 \quad 132.5 \quad 25.2 \quad 1 \]
The denominator of the third order biased models \((k = 3)\) obtained from eqn. (5.3) is

\[
D_3(s) = s^3 + \frac{a_1 a_2 a_3}{A_0} (A_0 + A_1 s + A_2 s^2)
\]

\[
= s^3 + 5.22083^2 + 2.5000s + 0.5
\]

\[
= a_3 s^3 + a_2 s^2 + a_1 s + a_0
\]

Therefore

\[
a_0 = 0.5
\]

\[
a_1 = 2.5
\]
\[ a_2 = 5.22083 \]
\[ a_3 = 1.0 \]

Then to obtain the numerators of the third order biased models using eqn. (5.4), we have to find the values of \( T_1, T_2, T_3, M_1 \) and \( M_2 \).

Using eqn. (5.5),

\[ T_1 = \frac{a_0}{A_0} B_0 = 2.5 \]

\[ T_2 = \frac{a_0}{A_0} B_1 = 10.0 \]

\[ T_3 = \frac{a_0}{A_0} B_2 = 7.104166 \]

Similarly using eqn. (5.6),

\[ M_1 = \frac{1}{A_5} (B_5 - 1 a_3 - (1 - 1)) = \frac{1}{A_5} B_4 a_3 = 20 \]

\[ M_2 = \frac{1}{A_5} (B_5 - 1 a_3 - (2 - 1) + B_5 - 2 a_3 - (2 - 2) - M_1 A_5 - (2 - 1)) \]

\[ = \frac{1}{A_5} (B_4 a_3 + B_3 a_3 - M_1 A_4) = -209.5 \]

Then using eqn. (5.4), the numerator of the third order biased model to retain the first 3 time-moments only (and no Markov-parameters) is given by
For $t = 3$, $m = 0$;

$$N_{3(3,0)}(s) = T_1 + T_2s + T_3s^2$$

$$= 2.5 + 10s + 7.104166s^2$$

The numerator of the third order biased model to retain the first two time-moments and one Markov-parameter is given by

For $t = 2$, $m = 1$;

$$N_{3(2,1)}(s) = T_1 + T_2s + M_1s^2$$

$$= 2.5 + 10s + 20s^2$$

The numerator of the third order biased model to retain the first one time-moment only and two Markov-parameters is given by

For $t = 1$, $m = 2$;

$$N_{3(1,2)}(s) = T_1 + M_2s + M_1s^2$$

$$= 2.5 - 209.5s + 20s^2$$

The three stable third order reduced biased models which retain first $t$ time-moments and $m$ Markov-parameters from $G(s)$, where $t+m = k = 3$ are given by

For $t = 3$, $m = 0$:

$$R_{3(3,0)}(s) = \frac{7.104166s^2 + 10s + 2.5}{s^3 + 5.22083s^2 + 2.5s + 0.5}$$

For $t = 2$, $m = 1$:

$$R_{3(2,1)}(s) = \frac{20s^2 + 10s + 2.5}{s^3 + 5.22083s^2 + 2.5s + 0.5}$$
The time-moments and Markov-parameters of the original system and the third order biased models obtained by using SRAM are compared below:

Expanding around $s = 0$:

$$G(s) = 5.00 - 5.000s - 13.00s^2 + \ldots$$

$$R_3(3,0)(s) = 5.00 - 5.000s - 13.00s^2 + \ldots$$

$$R_3(2,1)(s) = 5.00 - 5.000s - 12.7917s^2 + \ldots$$

$$R_3(1,2)(s) = 5.00 - 394.00s + 1957.79s^2 + \ldots$$

Expanding around $s = \infty$:

$$G(s) = 20 + 314s^{-1} + \ldots$$

$$R_3(3,0)(s) = 7.10416 + 27.0896s^{-1} + \ldots$$

$$R_3(2,1)(s) = 20 + 94.4166s^{-1} + \ldots$$

$$R_3(1,2)(s) = 20 + 314.0s^{-1} + \ldots$$

5.6 COMPARISON OF PROPOSED METHOD WITH FACTOR DIVISION METHOD

In a recent paper [35], Lucas suggested a new method for biased model reduction using Factor division. But the Factor division method involves more computations and requires formation of two Routh-type arrays even to develop the numerator of the biased model. It suggests no procedure to find the denominator of the biased model. It is capable of only developing numerator of biased models only when a stable denominator is available using any other method. The proposed method can also be generalized and be applied to the biased models for
any available stable reduced denominator using any other technique like in the case of Factor division method. However to obtain the numerator of the biased model, the proposed method needs no formation of Routh-type arrays unlike the Factor division method. In this section we compare the two methods to show that the proposed biased model reduction method based on SRAM is simpler in computation, more effective and straightforward.

Steps involved in Factor Division Method [35]

Let the original nth-order stable transfer function be given by

\[ G(s) = \frac{b_0 + b_1 s + ... + b_{n-1} s^{n-1}}{a_0 + a_1 s + ... + a_n s^n} \]  

...(5.7)

A reduced kth-order model is given by

\[ R_k(s) = \frac{\alpha_0 + \alpha_1 s + ... + \alpha_{t-1} s^{t-1} + \beta_1 s + ... + \beta_{m} s^{m-1}}{D(s)} \]  

...(5.8)

\[ \frac{N(s)}{D(s)} = \frac{N_t(s)}{D(s)} + \frac{N_m(s)}{D(s)} \]  

...(5.9)

where \( D(s) \) is the reduced stable denominator which is to be found by one of the other available techniques and \( N(s) \), the reduced numerator is formed so that \( R_k(s) \) retains the first \( t \) time-moments and \( m \) Markov-parameters of \( G(s) \), where \( t + m = k \).
the $\alpha'$-parameters are obtained by forming a Routh-type array from the coefficients of $[N(s) \; D(s)]$ and $\beta'$-parameters are obtained from another Routh-type array formed by the coefficients of $[N_m(s) \; D(s)]$.

Example 5.3 To compare the Factor division method and the proposed method (SRAM), consider the 5th-order system given by [35],

$$G(s) = \frac{10s^4 + 82s^3 + 264s^2 + 369s + 156}{2s^4 + 21s^3 + 84s^2 + 173s + 148s + 40}$$

Using Factor Division Method

The denominator of the 3rd order biased model is obtained by using Routh method of Shamash [33],

$$D_3(s) = s^3 + 10.5s^2 + 10.073s + 2.723$$

Using eqn. (5.9),

$$N_t(s) \; D(s) = \left( 156 + 369s + 264s^2 + \ldots \right) \left( 2.723 + 10.073s + 10.5s^2 + \ldots \right) = 424.788 + 2576.175s + 6073.809s^2 + \ldots$$

and the $\alpha'$-parameters needed for a 3rd order model i.e. $\alpha_0', \alpha_1'$ and $\alpha_2'$ are given by the following array [35].
It can be observed that the biased models obtained by the proposed method using SRAM are same as those obtained by the Factor division method by Lucas [35]. Thus the method is simple and straightforward.

5.7. MAIN FEATURES OF BIASED MODEL REDUCTION METHOD BY SRAM

a. It uses SRAM to generate biased models using only one Routh-array.

b. It can generate both numerator and denominator polynomials of the biased models using only \( \infty \)-parameters by applying SRAM.

c. It is a simple and direct method as it needs no time-moments of the system beforehand.

d. It can be applied to generate biased models using the stable denominator of the model reduced by any other method, involving less computational effort.

5.8. CONCLUSIONS

Model reduction by the Simplified Routh-Approximation Method (SRAM) has been extended to generate biased models. The new method is simple and uses only one Routh-array to develop both numerator and
denominator of the kth-order model. Computation of the time-moments and Markov-parameters of the original system beforehand and solving Padé equation for the reduced numerator are avoided. It is a straightforward method to reduce biased models which retain first k time-moments and Markov-parameters of the system.
PROPOSED METHOD (SRAM) 
FACTOR DIVISION METHOD OF LUCAS [35] 

\[ f_t \] 
PARAMETERS 
\[ f_{tl} \ldots k \]

\[ r^* \]

\[ N(s) \] 
USING BOTH 
\[ D(s) \] 
\[ N_k(s) \] 
AND \[ D_k(s) \] 
\[ R_k(s) = \frac{N_k(s)}{D_k(s)} \]

\[ N_k(s) \] 
AND \[ D_k(s) \] 
\[ R_k(s) = \frac{N_k(s)}{D_k(s)} \]

\[ N_k(s) \] 
USING BOTH 
\[ \alpha_i \] 
\[ \beta_i \] 
\[ \alpha_i \] 
\[ \beta_i \]

\[ D_k(s) \] 
USING ANY OTHER 
METHOD

\[ \alpha' \] 
PARAMETERS 
\[ \alpha'_i \] 
\[ j = 1, 2, \ldots, k \]

\[ \beta' \] 
PARAMETERS 
\[ \beta'_j \] 
\[ j = 1, 2, \ldots, k \]

FIGURE 5.1 COMPARISON OF STEPS INVOLVED IN BIASED MODEL REDUCTION BY SRAM AND FACTOR DIVISION METHOD
FIGURE 5.2 COMPARISON OF STEP RESPONSES EXAMPLE 5.1