CHAPTER 6

WEAKLY π-REGULAR NI NEAR-RINGS

6.1 Introduction

In this Chapter, we characterize 2-primal near-rings by using its minimal 0-prime ideals and NI near-rings by using its minimal strongly 0-prime ideals. Kim and Kwak [32] asked one question that, "Is a ring $R$ 2-primal if $O_P \subseteq P$ for each $P \in m\text{Spec}(R)$?". In Section 6.2, we prove that if $O_P$ has the IFP for each $P \in m\text{Spec}(N)$, then $O_P \subseteq P$ for each $P \in m\text{Spec}(N)$ if and only if $N$ is a 2-primal near-ring.

In [29], Hong et al. observed that the condition "$R/N \times (R)$ is right weakly π-regular" in Proposition 18 cannot be replaced by the condition "$R$ is right weakly π-regular" in the case of rings. In Section 6.3, we show that in the case of NI near-ring which satisfies $(CZ2)$ the condition "$N/N \times (N)$ is left weakly π-regular" can be replaced by the condition "$N$ is left weakly π-regular".
6.2 Characterization of 2-primal near-rings

In this section, we give some characterization of 2-primal near-ring by using its 0-prime ideals.

**Proposition 6.2.1.** For each $P \in \text{Spec}(N)$, $O(P)$ and $N(P)$ are ideals of $N$.

**Proof.** Let $P$ be a 0-prime ideal of $N$ and let $a_1, a_2 \in O(P)$. Then $a_1N <b_1> = 0$ for some $b_1 \in N \mid P$ and $a_2N <b_2> = 0$ for some $b_2 \in N \mid P$.

Since $b_1, b_2 \in N \mid P$ and $N \mid P$ is an $m$–system, there exist $b'_1 \in <b_1>$ and $b'_2 \in <b_2>$ such that $b'_1b'_2 \in N \mid P$. Let $b_3 = b'_1b'_2$. For any $n \in N$ and $x \in <b_3>$, $(a_1 - a_2)nx = 0$ implies $a_1 - a_2 \in O(P)$. Let $x \in O(P)$. Then $xN <b> = 0$. Thus for $n, n', n_1 \in N$ and $b' \in <b>$, we have $(n(n' + x) - nn')n_1b' = 0$ implies $n(n' + x) - nn' \in O(P)$ and $(xn)n_1b' = 0$ implies $xn \in O(P)$. Thus $O(P)$ is an ideal of $N$. Similarly, $N(P)$ is an ideal of $N$. \hfill $\square$

**Example 6.2.2.**

$$
\begin{bmatrix}
D & D \\
0 & D
\end{bmatrix}
$$

Let $N = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ where $D$ is a division ring.

Then $N$ is a near-ring under addition and multiplication of matrices. We also observe the following:

(i) $P = \begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}$

are 0-prime ideals of $N$;
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(ii) \[ I = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} \]

is a two sided ideal of \( N \), but not a 0-prime ideal of \( N \) since \( PQ \subseteq I \) but \( P \not\in I \) and \( Q \not\in I \);

(iii) \[ D \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

is a left ideal but not a right ideal of \( N \);

(iv) \[ \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \]

is a right ideal but not a left ideal of \( N \);

(v) \[ D \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

is neither a left nor a right ideal of \( N \);

(vi) \[ P_0(N) = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} = N(N). \]

Therefore \( N \) is a 2-primal near-ring.

Also \( O(P) = (0); \overline{O}(P) = P_0(N); \)

\[ O_P = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \]

\[ \overline{O}_P = O_P; \ N(P) = \overline{N}(P) = N_P = \overline{N}_P = P. \]

\[ O(Q) = \overline{O}(Q) = O_Q = \overline{O}_Q = Q \text{ and } N(Q) = \overline{N}(Q) = N_Q = \overline{N}_Q = P_0(N). \]
The following results might be helpful for the criterion for a certain class of rings to be 2-primal.

**Theorem 6.2.3.** For a near-ring $N$, the following statements are equivalent:

(i) $N$ is 2-primal;

(ii) $P_0(N)$ is a completely semiprime ideal of $N$;

(iii) $N(P)$ is a completely semiprime ideal of $N$ for each $P \in m\text{Spec}(N)$;

(iv) $\overline{N_P} = \overline{N(P)} = N(P)$ for each $P \in m\text{Spec}(N)$;

(v) $N(P) = N_P$ for each $P \in m\text{Spec}(N)$;

(vi) $N_P \subseteq P$ for some $P \in m\text{Spec}(N)$;

(vii) $N_P/P_0(N) \subseteq P/P_0(N)$ for each $P \in m\text{Spec}(N)$.

*Proof.* (i) $\Rightarrow$ (ii): Since $P_0(N) = N(N)$, for any $x$ in $N$, $x^2 \in P_0(N)$ implies $x^2$ is nilpotent and hence $x \in N(N) = P_0(N)$. Therefore, $P_0(N)$ is a completely semiprime ideal of $N$.

(ii) $\Rightarrow$ (iii): Let $P$ be a minimal 0-prime ideal of $N$. Let $x \in N$ be such that $x^2 \in N(P)$. Then $x^2 N <b> \subseteq P_0(N)$ for some $b \in N \setminus P$. Since $P_0(N)$ is a completely semiprime ideal of $N$, it has the IFP. So $xN <b> \subseteq P_0(N)$ which implies $xN <b> \subseteq P_0(N)$. Thus $x \in N(P)$ and hence $N(P)$ is completely semiprime.

(iii) $\Rightarrow$ (i): Let $a \in N(N)$. Then $a^n = 0$ for some positive integer $n$. If $a \notin P_0(N)$, then there exists a minimal 0-prime ideal $P$ of $N$ such that $a \notin P$. Since $N(P)$ is a completely semiprime ideal, $a^n = 0 \in N(P)$ implies $a \in N(P) \subseteq P$, a contradiction. Hence $a \in P_0(N)$.
(ii) ⇒ (iv) : Let $P$ be a minimal 0-prime ideal of $N$ and let $a \in \overline{N}_P$. Then $a^n \in N_P$ for some positive integer $n$. Thus $a^n b \in P_0(N)$ for some $b \in N \mid P$. Since $P_0(N)$ is completely semiprime ideal of $N$, it has the IFP. By Theorem 1.2.18, $ab \in P_0(N)$. Therefore $aN < b > \subseteq P_0(N)$ for some $b \in N \mid P$ and so $a \in N(P)$. Thus $\overline{N}_P \subseteq N(P)$. But $N(P) \subseteq N_P \subseteq \overline{N}_P$ and $\overline{N}(P) \subseteq \overline{N}_P$.

Therefore, $\overline{N}_P = \overline{N}(P) = N(P)$ for each $P \in \text{mSpec}(N)$.

(iv) ⇒ (v) ⇒ (vi) : These are obvious.

(vi) ⇒ (vii) : Let $P$ be a minimal 0-prime ideal of $N$. Let $\overline{N} = N/P_0(N)$ and $\overline{P} = P/P_0(N)$. Let $\overline{a} = a + P_0(N) \in N_P$ for some $a \in N$. Then there exists $\overline{b} \in \overline{N} \mid \overline{P}$ such that $\overline{a} \overline{b} \in P_0(\overline{N}) = \overline{0}$. Thus $ab \in P_0(N)$ and so $a \in N_P \subseteq P$. Therefore $\overline{a} \in \overline{P}$ and hence $N_P \subseteq \overline{P}$.

(vii) ⇒ (i) : Suppose that $\overline{N} = N/P_0(N)$ is not reduced. Then there exists $\overline{a} \in \overline{N}$ such that $\overline{a}^2 = \overline{0}$ and $\overline{a} = \overline{0}$. Thus $a \not\in P_0(N)$ and hence $a \not\in P$ for some $P \in \text{mSpec}(N)$. Then $\overline{a} \not\in \overline{P}$ and so $\overline{a} \in \overline{N} \mid \overline{P}$. But since $\overline{a}^2 = \overline{0}$, we obtain $\overline{a} \in N_P \subseteq \overline{P}$, which is a contradiction. Therefore $P_0(N) = N(N)$ and hence $N$ is 2-primal.

**Corollary 6.2.4.** For a near-ring $N$, assume that $N$ is 2-primal. If $P = N(P)$ for each $P \in \text{Spec}(N)$, then $P$ is completely prime ideal of $N$.

**Proof.** Suppose that $N$ is a 2-primal near-ring. Let $xy \in P = N(P)$. Then there exists $b \in N \mid P$ such that $(xy)N < b > \subseteq P_0(N)$. Since $P_0(N)$ has the IFP, we have $(xNy)N < b > \subseteq P_0(N) \subseteq P$ and so $xNy \subseteq P$ since $b \not\in P$. Hence $x \in P$ or $y \in P$ since $P$ is a 0-prime ideal of $N$. Therefore, $P$ is a completely prime ideal of $N$.  

**Proposition 6.2.5.** For a near-ring $N$, we have the following:
Theorem 6.2.6. For a near-ring \( N \), assume that \( N \) is 2-primal. Then for each \( P \in \text{Spec}(N) \), the following statements are equivalent:

(i) \( P \in \text{mSpec}(N) \);

(ii) \( N(P) = P \).

Proof. (i) \( \Rightarrow \) (ii) : Let \( P \) be a minimal 0-prime ideal of \( N \) and let \( a \in P \). Suppose \( a \notin N(P) \). Let \( S = \{a, a^2, a^3, \cdots \} \). If \( 0 \in S \), then \( a^k = 0 \) for some positive integer \( k \) and hence \( a \in N(N) = P_0(N) \), which implies that \( a \in N(P) \) by...
Proposition 6.2.5, a contradiction. So \( 0 \not\in S \). Thus \( S \) is a multiplicative system that does not contain 0. Let \( L = N \mid P \), i.e., \( L \) is an \( m \)-system. Let \( T \) be the set of all non-zero elements of \( N \) of the form \( a^{t_0} x_1 a^{t_1} x_2 \cdots a^{t_{n-1}} x_n a^{t_n} \), where \( x_i \in L \) and the \( t_i \)'s are positive integers with \( t_0 \) and \( t_n \) allowed to be zero. Clearly, \( L \subseteq T \). Let \( M = T \cup S \). We show that \( M \) is an \( m \)-system. Let \( x, y \in M \). If \( x, y \in S \), then \( xy \in S \subseteq M \) and we are done. Let \( x \in S \) and \( y \in T \), say \( x = a^s \) and \( y = a^{t_0} y_1 a^{t_1} y_2 a^{t_2} \cdots y_n a^{t_n} \). If \( xy = 0 \), then \( xy \in T \). Suppose \( xy = 0 \). Since \( y_1, y_2 \in L \), there exist \( y'_1 \in <y_1> \) and \( y'_2 \in <y_2> \) such that \( y'_1 y'_2 \in L \). Since \( y'_1 y'_2, y_3 \in L \), there exist \( y'_{12} \in <y'_1 y'_2> \subseteq (<y_1> <y_2>) \) and \( y'_3 \in <y_3> \) such that \( y'_{12} y'_3 \in L \). Continuing this process, we get \( y'_{123\cdots n-2} y'_{n-1}, y_n \in L \). Then there exist \( y'_{123\cdots n-1} \in (y'_{123\cdots n-2} y'_{n-1}) \subseteq (\cdots (((y_1) (y_2)) (y_3)) \cdots (y_{n-1})) \) and \( y'_n \in <y_n> \) such that \( w = y'_{123\cdots n-1} y'_n \in L \). Since \( xy = 0 \), \( xy \in P_0(N) \). Thus \( a^s a^{t_0} y_1 a^{t_1} y_2 \cdots y_n a^{t_n} \in P_0(N) \). Since \( P_0(N) = N(N) \), \( P_0(N) \) is completely semiprime ideal of \( N \) and hence \( y_1 y_2 \cdots y_n a^{s+t_0+t_1+\cdots+t_n} \in P_0(N) \). Choose \( m = s + t_0 + t_1 + \cdots + t_n \). Then \( y_1 y_2 \cdots y_n a^m \in P_0(N) \). Since \( P_0(N) \) has the IFP, \( <y_1> <y_2> \cdots <y_n> <a^m> \subseteq P_0(N) \). Continuing this process, we obtain \( <\cdots <<y_1> <y_2> > > <y_3> > \cdots <y_{n-1} > > <y_n> > <a^m> \subseteq P_0(N) \) and so \( y'_{123\cdots n-1} y'_n a^m \in P_0(N) \). Hence \( wa^m \in P_0(N) \), where \( w = y'_{123\cdots n-1} y'_n \). Since \( P_0(N) \) is a completely semiprime ideal, \( (aw)^m \in P_0(N) \) and hence \( aw \in P_0(N) \). Thus \( a \in N_P = N(P) \), which is a contradiction. Therefore, if \( x \in S \), \( y \in T \), then \( xy = 0 \) and so \( xy \in T \).

Similarly, one can show that if \( x, y \in T \) then \( xy = 0 \) and \( xy \in T \). This shows that \( M \) is an \( m \)-system that is disjoint from (0). Hence, by Proposition 1.2.15 there is a 0-prime ideal \( Q \) that is disjoint from \( M \) such that \( a \not\in Q \) and \( Q \subseteq P \). Since \( P \) is a minimal 0-prime ideal, \( P = Q \). Therefore \( a \not\in P \), which
is a contradiction. Consequently, $a \in N(P)$.

(ii) $\Rightarrow$ (i) : If $Q \subseteq P$ for $Q \in m\text{Spec}(N)$, then

$$N(P) \subseteq N(Q) \subseteq Q \subseteq P = N(P)$$

Therefore, $P \in m\text{Spec}(N)$. $\Box$

**Theorem 6.2.7.** For a near-ring $N$, the following statements are equivalent:

(i) $N$ is 2-primal;

(ii) $\overline{O}_P \subseteq P$ for each $P \in m\text{Spec}(N)$;

(iii) $N(N) = \bigcap_{P \in m\text{Spec}(N)} \overline{O}_P = P_0(N)$.

**Proof.** (i) $\Rightarrow$ (ii) : Note that $\overline{N}_P = N(P) \subseteq P$ and therefore, $\overline{O}_P \subseteq P$ for each $P \in m\text{Spec}(N)$.

(ii) $\Rightarrow$ (iii) : Since $\overline{O}_P \subseteq P$ for each $P \in m\text{Spec}(N)$, 

$$\bigcap_{P \in m\text{Spec}(N)} \overline{O}_P \subseteq P_0(N).$$

Let $a \in N(N)$. Then $a^m = 0 \in O(P)$ for some integer $m$ and any $P \in m\text{Spec}(N)$. Hence $a \in \bigcap_{P \in m\text{Spec}(N)} \overline{O}_P$. Thus

$$N(N) \subseteq \bigcap_{P \in m\text{Spec}(N)} \overline{O}_P \subseteq P_0(N) \subseteq N(N)$$

(iii) $\Rightarrow$ (i) : It is obvious. $\Box$

**Proposition 6.2.8.** Assume that $O(P)$ is a 0-prime ideal of near-ring $N$ for each $P \in m\text{Spec}(N)$. Then $O(P)$ has the IFP for each $P \in m\text{Spec}(N)$ if and only if $N$ is a 2-primal near-ring.
Proof. Assume that $N$ is a 2-primal near-ring. Let $P$ be a minimal 0-prime ideal of $N$ such that $O(P)$ is a 0-prime ideal of $N$. Let $xy \in O(P)$ for $x, y \in N$. This implies that $xyN <z> = 0$ for $z \in N \setminus P$. Then $xyN <z> \subseteq P$. Since $z \not\in P$ and $P$ is 0-prime, $xy \in P$. Therefore, $O(P) \subseteq P$. Since $P$ is a minimal 0-prime ideal, $O(P) = P$. Since $N$ is 2-primal and $P \in mSpec(N)$, $N(P) = P$ by Theorem 6.2.6. Therefore $P$ is completely prime by Corollary 6.2.4. Since $P = O(P)$, $O(P)$ is completely prime. In particular, $O(P)$ has the IFP.

Conversely, suppose that $O(P)$ has the IFP for each $P \in mSpec(N)$. Let $x \in N(N)$. This implies that $x^n = 0$ for some positive integer $n$. So that $x^n \in O(P)$. If $x \not\in P_0(N)$, then there exists a minimal 0-prime ideal $P$ of $N$ such that $x \not\in P$. Since $P$ is a 0-prime ideal, there exist $r_1, r_2, \cdots, r_{n-1} \in N$ such that $xr_1x \cdots xr_{n-1}x \not\in P$. But since $O(P)$ has the IFP, $xr_1x \cdots xr_{n-1}x \in O(P)$. Since $O(P) \subseteq P$, $xr_1x \cdots xr_{n-1}x \in P$, a contradiction. Thus $x \in P_0(N)$. Therefore $N(N) \subseteq P_0(N)$. Always $P_0(N) \subseteq N(N)$. Hence $N(N) = P_0(N)$.

\[ \Box \]

Proposition 6.2.9. If $O(P)$ has the IFP for each $P \in mSpec(N)$, then for every $P \in mSpec(N)$, $O(P)$ is a 0-prime ideal if and only if $O(P)$ is a completely prime ideal of $N$.

Proof. Suppose that $O(P)$ is a 0-prime ideal for every $P \in mSpec(N)$. Let $xy \in O(P)$ for $x, y \in N$. If $x \in O(P)$, we have done. Suppose $x \not\in O(P)$. Since $xy \in O(P)$ and $O(P)$ has the IFP, $xNy \subseteq O(P)$. This implies that $xNyN <z> = 0$ for $z \in N \setminus P$. This implies that $xNyN <z> \subseteq P$. Since $P$ is 0-prime, $xNy \subseteq P$ and therefore $x \in P$ or $y \in P$. By Proposition 6.2.8, $P = O(P)$. Since $x \not\in O(P)$, $x \not\in P$. Therefore $y \in P = O(P)$. Hence $O(P)$
is completely prime. The Converse is obvious. □

**Proposition 6.2.10.** Let \( N \) be a near-ring with unity. Let \( O(P) \) be a 0-prime ideal of \( N \) for each \( P \in \text{mSpec}(N) \). Then the following are equivalent:

(i) \( N \) is a 2-primal near-ring;

(ii) \( O(P) \) has the IFP for each \( P \in \text{mSpec}(N) \);

(iii) \( O(P) \) is a completely semiprime ideal for each \( P \in \text{mSpec}(N) \);

(iv) \( O(P) \) is a symmetric ideal for each \( P \in \text{mSpec}(N) \);

(v) \( xy \in O(P) \) implies \( yN x \subseteq O(P) \) for \( x, y \in N \) and for each \( P \in \text{mSpec}(N) \).

Proof. (i) \( \Rightarrow \) (ii): It follows from Proposition 6.2.8.

(ii) \( \Rightarrow \) (iii): By Proposition 6.2.9, \( O(P) \) is a completely prime ideal and hence \( O(P) \) is completely semiprime.

(iii) \( \Rightarrow \) (iv): Suppose that \( O(P) \) is a completely semiprime ideal for each \( P \in \text{mSpec}(N) \). Therefore it has the IFP. Let \( a, b, c \in N \) be such that \( abc \in O(P) \). We shall prove that \( acb \in O(P) \). Since \( abc \in O(P) \), there exists \( s \in N \mid P \) such that \( abcN <s> = 0 \). So that \( abcN <s> \subseteq O(P) \). Since \( O(P) \) has the IFP, \( acbcN <s> \subseteq O(P) \). Suppose that \( cN <s> \notin O(P) \). If \( acb \notin O(P) \), since \( O(P) \) is 0-prime there exists some \( n \in N \) such that \( acbncN <s> \notin O(P) \). It contradicts the IFP of \( O(P) \). Therefore \( acb \in O(P) \).

Suppose that \( cN <s> \subseteq O(P) \). Since \( O(P) \) has the IFP, \( cbN <s> \subseteq O(P) \). Since \( O(P) \) is 0-prime and \( s \notin P = O(P) \), \( cb \in O(P) \). Therefore \( acb \in O(P) \). Hence \( O(P) \) is a symmetric ideal in \( N \).
(iv) \(\Rightarrow\) (v) : Suppose that \(xy \in O(P)\) for \(P \in m\text{Spec}(N)\). Since \(O(P)\) is symmetric and \(N\) has unity, \(yx \in O(P)\). Since \(O(P)\) has the IFP, \(yNx \subseteq O(P)\).

(v) \(\Rightarrow\) (i) : Let \(x \in N(N)\). Then \(x^r = 0\) for some \(r\). So that \(x^r \in O(P)\) for \(P \in m\text{Spec}(N)\). Suppose that \(x \notin P_0(N)\). Since \(P_0(N) = \bigcap_{P \in \text{Spec}(N)} P\), \(x \notin P\). Since \(P\) is a 0-prime ideal, there exist \(n_1, n_2, \ldots, n_{r-1} \in N\) such that \(xn_1x \cdots xn_{r-1}x \notin P\). Since \(xy \in O(P)\), by hypothesis \(yNx \subseteq O(P)\). Therefore \(xn_1x \cdots xn_{r-1}x \in O(P) \subseteq P\), a contradiction. Thus \(x \in P_0(N)\). Hence \(N(N) \subseteq P_0(N)\). Always \(P_0(N) \subseteq N(N)\) and consequently \(N\) is a 2-primal near-ring.

\[\square\]

**Theorem 6.2.11.** Let \(O(P)\) be a 0-prime ideal for each \(P \in m\text{Spec}(N)\). Then the following are equivalent:

(i) \(N\) is a 2-primal near-ring;

(ii) \(O(P)\) has the IFP;

(iii) Every minimal 0-prime ideal of \(N\) is a completely prime ideal of \(N\).

**Proof.** (i) \(\Rightarrow\) (ii) : It follows from Proposition 6.2.8.

(ii) \(\Rightarrow\) (iii) : Let \(P\) be a minimal 0-prime ideal of \(N\). Let \(a, b \in N\) be such that \(ab \in P\). If \(b \in P\), we have done. Suppose that \(b \notin P\). Since \(O(P) = P\), \(ab \in O(P)\). Since \(O(P)\) has the IFP, \(aNb \subseteq O(P) = P\). Since \(P\) is 0-prime and \(b \notin P\), \(a \in P\). Hence, \(P\) is a completely prime ideal.

(iii) \(\Rightarrow\) (i) : Let \(x \in N(N)\). Then \(x^r = 0\) for some \(r\). So that \(x^r \in P\), where \(P \in m\text{Spec}(N)\). Since every minimal 0-prime ideal is completely prime, \(x \in P\) for every \(P \in m\text{Spec}(N)\).

Since \(P_0(N) = \bigcap_{P \in \text{Spec}(N)} P\), \(x \in P_0(N)\). Thus \(N(N) \subseteq P_0(N)\). \[\square\]
Theorem 6.2.12. Let \( O(P) \) be a 0-prime ideal of \( N \) for every \( P \in \text{mSpec}(N) \). Then \( N \) is a 2-primal near-ring if and only if \( P = \overline{O}(P) \) for every minimal 0-prime ideal \( P \) of \( N \).

Proof. Suppose that \( N \) is a 2-primal near-ring. Then \( O(P) \) is a completely prime ideal of \( N \) by Proposition 6.2.8. Let \( a \in \overline{O}(P) \). Then \( a^m \in O(P) \). Since \( O(P) \) is completely prime, \( a \in O(P) \). Therefore \( \overline{O}(P) \subseteq O(P) \). Clearly, \( O(P) \subseteq \overline{O}(P) \). Thus \( O(P) = \overline{O}(P) \). Since \( O(P) \) is a 0-prime ideal of \( N \), \( P = O(P) \). Hence \( P = \overline{O}(P) \).

Conversely, assume that \( P = \overline{O}(P) \) for every minimal 0-prime ideal \( P \) of \( N \). Let \( x \in N(N) \). This implies that \( x^n = 0 \) for some \( n \). So \( x^n \in P \) for every \( P \in \text{mSpec}(N) \). Since \( P = \overline{O}(P) = O(P) \), \( x^n \in O(P) \). Since \( O(P) \) is completely prime, \( x \in O(P) = \overline{O}(P) = P \). This implies that \( x \in P_0(N) \). Thus \( N(N) \subseteq P_0(N) \) and consequently \( N \) is a 2-primal near-ring.

In [32], Kim and Kwak asked one question that "Is a ring \( R \) 2-primal if \( O_P \subseteq P \) for each \( P \in \text{mSpec}(R) \)?". Here we prove the following theorem for near-rings.

Theorem 6.2.13. If \( O_P \) has the IFP for each \( P \in \text{mSpec}(N) \), then \( O_P \subseteq P \) for each \( P \in \text{mSpec}(N) \) if and only if \( N \) is a 2-primal near-ring.

Proof. Let \( x \in N(N) \). Then \( x^n = 0 \) for some \( n \). So that \( x^n \in O(P) \subseteq O_P \). Suppose \( x \notin P_0(N) \). Since \( P_0(N) = \bigcap_{P \in \text{mSpec}(N)} P \), there exists \( P \in \text{mSpec}(N) \) such that \( x \notin P \). Since \( P \) is a 0-prime ideal, there exist \( r_1, r_2, \ldots, r_{n-1} \in N \) such that \( x r_1 x \cdots x r_{n-1} x \notin P \). But since \( O_P \) has the IFP, \( x r_1 x \cdots x r_{n-1} x \in O_P \). Again since \( O_P \subseteq P \), \( x r_1 x \cdots x r_{n-1} x \in P \), a contradiction. Thus \( x \notin P_0(N) \). Hence \( N(N) \subseteq P_0(N) \).
Conversely, assume that \( N \) is a 2-primal near-ring. By Theorem 6.2.7, \( \mathcal{O}_P \subseteq P \) for each \( P \in m\text{Spec}(N) \). Since \( O_P \subseteq \mathcal{O}_P \), \( O_P \subseteq P \) for each \( P \in m\text{Spec}(N) \).

### 6.3 NI near-rings which are weakly \( \pi \)-regular

In this section, we show that if \( N \) is an NI near-ring which satisfies the condition (CZ2), then (i) every strongly 0-prime ideal is maximal if and only if \( N \) is left weakly \( \pi \)-regular. (ii) \( P \in m\text{SSpec}(N) \) if and only if \( P = \mathcal{O}(P) \).

For a near-ring \( N \), let \((m)\text{SSpec}(N)\) be the set of all (minimal) strongly 0-prime ideals of \( N \). For \( P \in \text{SSpec}(N) \), we have

\[
\mathcal{O}(P) = \{ a \in N \mid aN < b \geq 0 \text{ for some } b \in N \mid P \},
\]

\[
\mathcal{O}(P) = \{ a \in N \mid a^m \in \mathcal{O}(P) \text{ for some positive integer } m \},
\]

\[
O_P = \{ a \in N \mid ab = 0 \text{ for some } b \in N \mid P \},
\]

\[
O_P = \{ a \in N \mid a^m \in O_P \text{ for some positive integer } m \},
\]

\[
N(P) = \{ a \in N \mid aN < b \subseteq N^*(N) \text{ for some } b \in N \mid P \}.
\]

**Example 6.3.1.** Consider the near-ring \((N, +, \cdot)\) defined on the Klein’s four group \((N, +)\) with \( N = \{0, a, b, c\} \) where \( \cdot \) is defined as follows (as per scheme 2,p.408 [35]).

\[
\begin{array}{ccc}
    \cdot & 0 & a & b & c \\
    0 & 0 & 0 & 0 & 0 \\
    a & 0 & 0 & a & a \\
    b & 0 & a & b & b \\
    c & 0 & a & c & c \\
\end{array}
\]
Clearly \( \{0, a\} \) is a strongly 0-prime ideal, since the ideals are \( \{0\} \), \( \{0, a\} \), and \( \{0, a, b, c\} \). Let \( P = \{0, a\} \). Then \( O\{P\} = O_P = \{0\} \) and \( \overline{O(P)} = \overline{O_P} = N\{P\} = \overline{N(P)} = N \) \( P = \overline{N_P} = P \).

Also we observe that \( N(N) = \{0, a\} = N^*(N) \). Therefore \( N \) is a NI near-ring.

**Theorem 6.3.2.** For a near-ring \( N \) the following are equivalent:

(i) \( N \) is an NI near-ring;

(ii) Every minimal strongly 0-prime ideal of \( N \) is completely prime.

**Proof.** (i) \( \Rightarrow \) (ii) : Let \( P \) be a minimal strongly 0-prime ideal of \( N \). Let \( a, b \in N \) be such that \( ab \in P \) and \( b \notin P \). We will show that \( a \in P \).

Case (i): Suppose that \( (ab)^k = 0 \) for some \( k \). Then \( (ab)^k \in N^*(N) \). Since \( N^*(N) \) is completely semiprime by Theorem 1.2.17, \( a^k b^k \in N^*(N) \). Since \( b \notin P \), there exist \( z_1, z_2, z_3, \ldots, z_{k-1} \in N \) such that \( bz_1 b z_2 \cdot \cdot \cdot z_{k-1} b \notin P \) and since \( N^*(N) \) has the IFP, \( a^k N(bz_1 b z_2 \cdot \cdot \cdot z_{k-1} b) \in N^*(N) \). Again by completely semiprimeness of \( N^*(N) \), we have \( aN b z_1 b z_2 \cdot \cdot \cdot z_{k-1} b \in N^*(N) \). Hence \( a \in N(P) \subseteq P \).

Case (ii): Suppose that \( (ab)^k = 0 \) for all \( k > 0 \). Let \( S = \{(ab)^r / r \geq 1\} \), \( L = N \mid P \) and \( T = \{n \in N \mid n = 0, n = (ab)^{t_0} x_1 (ab)^{t_1} x_2 (ab)^{t_2} x_3 \cdot \cdot \cdot (ab)^{t_r}, \) where \( t_i \geq 1, i = 1, 2, \ldots, r - 1, t_i \geq 0, i = 0, n \) and \( x_i \in L \) for all \( i \} \). Clearly, \( S = \{0\} \) and \( L \subseteq T \). Let \( M = S \cup T \). We shall prove that \( M \) is an \( m \)-system in \( N \mid \{0\} \). Let \( x, y \in M \). If \( x, y \in S \), then \( xy \in S \subseteq M \). If \( x \in S \) and \( y \in T \), then let \( x = (ab)^q \) for some \( q > 0 \) and \( y = (ab)^{t_0} x_1 (ab)^{t_1} x_2 \cdot \cdot \cdot x_r (ab)^{t_r} \). Suppose that \( xy = 0 \). Since \( x_1, x_2 \in L \), there exist \( x_1' \in \langle x_1 \rangle \) and \( x_2' \in \langle x_2 \rangle \) such that \( x_1' x_2 \in L \). Since \( x_1' x_2, x_3 \in L \), there exist \( x_{12}' \in \langle x_1' x_2 \rangle \leq \langle x_1 \rangle < \langle x_2 \rangle \) and \( x_3' \in \langle x_3 \rangle \) such that \( x_1' x_2 x_3' \in L \).
\[ x^r_{12}x^r_3 \in L. \] Continuing this process, we get \( x^r_{123 \ldots r-2}x^r_{r-1} \), \( x_r \in L \). Then there exist \( x^r_{123 \ldots r-1} \subseteq x^r_{123 \ldots r-2}x^r_{r-1} \rangle \langle \cdots \rangle \langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle \cdots \langle x_{r-1} \rangle \rangle \) and \( x_r \in \langle x_r \rangle \) such that \( w = x^r_{123 \ldots r-1}x^r_r \in L \). Since \( xy = 0 \), \( xy \in N^*(N) \). Thus \( (ab)^i(ab)^jx^i_1x^j_2 \cdots \langle x_r \rangle_0(ab)^{t_r} \in N^*(N) \). Since \( N^*(N) \) is completely semiprime ideal of \( N \), \( x_1x_2 \cdots \langle x_r \rangle_0(ab)^{t_0} \cdots + t_r \in N^*(N) \). Choose \( m = q + t_0 + \cdots + t_r \). Then \( x_1x_2 \cdots \langle x_r \rangle_0(ab)^m \in N^*(N) \). Since \( N^*(N) \) has the IFP, \( \langle x_1 \rangle \langle x_2 \rangle \cdots \langle x_r \rangle \langle (ab)^m \rangle \subseteq N^*(N) \). This implies that \( \langle \cdots \rangle \langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle \cdots \langle x_{r-1} \rangle \rangle \langle x_r \rangle \langle (ab)^m \rangle \subseteq N^*(N) \) and so \( x^r_{123 \ldots r-1}x^r_r(ab)^m \in N^*(N) \). Hence \( w(ab)^m \in N^*(N) \), where \( w = x^r_{123 \ldots r-1}x^r_r \). Since \( N^*(N) \) is completely semiprime, \( (ab)w^m \in N^*(N) \) and hence by case(i) \( a \in P \). Suppose that \( xy = 0 \). Then, clearly, from the definition of \( T \), \( xy \in T \subseteq M \). Similarly, we can show that if \( x, y \in T \) then \( xy \in T \subseteq M \). Thus we have \( M \) is an \( m \)-system in \( N \mid \{0\} \). By Proposition 1.2.15, there is a 0-prime ideal \( Q \) that is disjoint from \( M \) such that \( \langle ab \rangle \notin Q \) and \( Q \subseteq P \). Now we claim that \( Q \) is strongly 0-prime. Suppose \( I/Q \) is a nonzero nil ideal of \( N/Q \). Since \( Q \subseteq I, I \cap M = \{0\} \). If \( (ab)^m \in I \) for some positive integer \( m \), then \( (ab)^m + Q \) is a nilpotent element in \( N/Q \). Thus \( (ab)^{mk} \in Q \) for some positive integer \( k \), which is a contradiction. So we choose \( x \in I \cap T \). Then \( x \in T \) implies \( 0 = x^t \in T \) for any positive integer \( t \). Since \( x + Q \) is nilpotent in \( N/Q \), \( x^s \in Q \) for some positive integer \( s \) which is again a contradiction. Therefore, \( Q \) is a strongly 0-prime ideal of \( N \) such that \( \langle ab \rangle \notin Q \) and \( Q \subseteq P \). Since \( P \in mSSpec(N) \), \( Q = P \). Therefore \( \langle ab \rangle \notin P \) which is a contradiction and consequently \( a \in P \).

(ii) \( \Rightarrow \) (i): Let \( x^n = 0 \) for some positive integer \( n \). Then \( x^n \in P \) for all minimal strongly 0-prime ideal \( P \) of \( N \). Since every minimal strongly 0-prime
ideal of $N$ is completely prime, $x \in P$. Thus $x \in \bigcap_{P \in \text{mSpec}(N)} P = N^\ast(N)$ by Lemma 1.2.16. Therefore, $N$ is an NI near-ring.

The following definitions are critical to our characterization of minimal strongly 0-prime ideals.

**Definition 6.3.3.** Let $x, y \in N$ and $n$ a positive integer. We say $N$ satisfies

1) $(CZ1)$ condition if whenever $(xy)^m = 0$ then $x^m y^m = 0$, for some positive integer $m$.

2) $(CZ2)$ condition if whenever $(xy)^m = 0$ then $<x>^m N <y>^m = 0$, for some positive integer $m$.

Hong et al. [29] observed that the condition "$R/N \ast(R)$ is right weakly $\pi$-regular" in Proposition 18 cannot be replaced by the condition "$R$ is right weakly $\pi$-regular " in the case of rings. But the following theorem shows that in the case of NI near-ring which satisfies $(CZ2)$ the condition "$N/N \ast(N)$ is left weakly $\pi$-regular" can be replaced by the condition "$N$ is left weakly $\pi$-regular".

**Theorem 6.3.4.** Let $N$ be an NI near-ring with unity satisfying $(CZ2)$. Then the following are equivalent:

(i) $N$ is left weakly $\pi$-regular;

(ii) $N/N \ast(N)$ is left weakly $\pi$-regular;

(iii) Every strongly 0-prime ideal of $N$ is maximal.

**Proof.** (i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (iii) : Let $P \in S\text{Spec}(N)$. Then there exists a minimal strongly 0-prime ideal $I \subseteq P$ which is completely prime by Theorem 1.2.18. Let $\overline{N} = N/I$. Then $\overline{N}$ is an integral left weakly $\pi$-regular near-ring. Let $x$ be a non zero element in $\overline{N}$. There exists a positive integer $k$ such that $x^k \subseteq <x^k > x^k$. Then $x^k = yx^k$, where $y \in <x^k >$. Therefore $x^k - yx^k = 0$ which implies that $(1 - y)x^k = 0$. Hence $N/I$ is a simple near-ring. Thus $I$ is a maximal ideal and so is $P$.

(iii) $\Rightarrow$ (i) : Suppose that $N$ is not left weakly $\pi$-regular. Then there exists an element $a \in N$ such that $a$ is not left weakly $\pi$-regular. So we have $a^k \not\subseteq <a^k > a^k$ for every positive integer $k$. Hence $a^k = 0$ for all $k > 0$ and $a \not\subseteq <a > a$. Then $<a >$ is contained in a maximal ideal which is also a strongly 0-prime ideal. Let $T$ be the union of all strongly 0-prime ideals which contain $a$. Let $S = N \setminus T$. Since every strongly 0-prime ideal is maximal, every strongly 0-prime ideal is minimal. Since every minimal strongly 0-prime ideal is completely prime, by Theorem 6.3.2, $S$ is multiplicatively closed. Let $F = \{a^{i_0}b_1a^{i_1}b_2 \cdots b_na^{i_n} = 0/b_i \in S$ and $t_i \in \{0\} \cup N, \text{ where } N \text{ is the set of all positive integers}\}$. Let $L = \{a, a^2, \ldots \}$ and let $M = L \cup F$. Clearly, $S \subseteq F \subseteq M$. We shall claim that $M$ is an $m$-system in $N \setminus \{0\}$. Let $x, y \in M$. Assume $x \in L$ and $y \in F$. Suppose that $xy = 0$. Take $x = a^i$ and $y = a^{i_0}b_1a^{i_1}b_2 \cdots b_na^{i_n}$. Since $xy = 0$, $a^i a^{i_0}b_1a^{i_1}b_2 \cdots b_na^{i_n} = 0$. Since $b_1, b_2 \in S$, there exist $b'_1 \subseteq <b_1 >$ and $b'_2 \subseteq <b_2 >$ such that $b'_1b'_2 \in S$. Since $b'_2, b_3 \in S$, there exist $b'_{12} \subseteq <b'_1b'_2 > \subseteq <b'_1 > <b'_2 >$ and $b'_3 \subseteq <b_3 >$ such that $b'_{12}b'_3 \subseteq S$. Continuing this process, we get $b'_{12} \cdots b'_{n-1}b'_n \in S$. Then there exist $b'_{12} \cdots b'_{n-1} \subseteq <b'_{12} \cdots b'_{n-2}b'_{n-1} > \subseteq \cdots <b'_1 > <b'_2 > <b'_3 > <b'_4 > <b'_5 > \cdots <b'_{n-1} >$ and $b'_n \subseteq <b_n >$ such that $b'_{12} \cdots b'_{n-1}b'_n \subseteq S$. 

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Since $xy = 0$ and $N^*(N)$ has the IFP, $a^{r+t_0+t_1+\cdots+t_n}b_1b_2\cdots b_n \in N^*(N)$. Choose $m = r + t_0 + t_1 + \cdots + t_n$. Then $a^mb_1b_2\cdots b_n \in N^*(N)$. Since $N^*(N)$ has the IFP, $< a^m > < b_1 > < b_2 > \cdots < b_n > \subseteq N^*(N)$ and so $a^mb'_1b'_2\cdots b'_{n-1}b'_n \in N^*(N)$. Hence $a^mw \in N^*(N)$, where $w = b'_1b'_2\cdots b'_{n-1}b'_n$. Since $N^*(N)$ is completely semiprime, $(wa)^m \in N^*(N)$ and hence $(wa)^k = 0$ for some $k$. Since $N$ satisfies (CZ2), $< w^k > N < a^k > = 0$ for some $k$. By the definition of $S$ and $T$, a strongly 0-prime ideal cannot contain both $a^k$ and $w^k$. Hence

$$< a^k > + < w^k > = N.$$

So

$$< a^k > a^k + < w^k > a^k = Na^k.$$

Since $< w^k > N < a^k > = 0$, $< w^k > a^k = 0$. Therefore $a^k \in < a^k > a^k$. This shows that $a$ is left weakly $\pi$-regular, a contradiction. Hence $0 = xy \in M$. Similarly, we can prove that if $x, y \in F$, then $0 = xy \in M$. Thus $M$ is an $m-$system in $N | \{0\}$. By Proposition 1.2.15, there is a 0-prime ideal $Q$ that is disjoint from $M$ such that $a \notin Q$. As in the proof of Theorem 6.3.2, we obtain $Q$ is a strongly 0-prime ideal of $N$. Since $a \notin Q$, $Q + < a > = N$. Hence $1 = b + c$ for some $b \in Q$ and $c \in < a >$. This gives $b \notin T$. So that $b \in S \subseteq F \subseteq M$ which implies that $Q \cap M = \emptyset$, a contradiction and consequently $N$ is left weakly $\pi$-regular.

\[\square\]

**Corollary 6.3.5.** Let $N$ be a 2-primal near-ring with unity satisfying condition (CZ2). Then the following are equivalent:

(i) $N$ is left weakly $\pi$-regular;
(ii) $N/N^+(N)$ is left weakly $\pi$-regular;

(iii) $N/P_0(N)$ is left weakly $\pi$-regular;

(iv) Every 0-prime ideal of $N$ is maximal;

(v) Every strongly 0-prime ideal of $N$ is maximal.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (v) follows from Theorem 6.3.4.

(i) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv) : Let $P$ be a 0-prime ideal of $N$. Then, by Corollary 6.2.4 and Theorem 6.2.6, there exists a minimal 0-prime ideal $X \subseteq P$ which is completely prime. Let $\overline{N} = N/X$. Then $\overline{N}$ is an integral left weakly $\pi$-regular near-ring. Let $a$ be a non-zero element in $\overline{N}$. There exists a positive integer $k$ such that $a^k \in a^k < a^k$. Then $a^k = ya^k$ where $y \in a^k >$. Therefore $a^k - ya^k = 0$ which implies that $(1 - y^k)a^k = 0$. Hence $N/X$ is a simple near-ring. Thus $X$ is a maximal ideal and so is $P$.

(iv) $\Rightarrow$ (v) : Since every strongly 0-prime ideal is 0-prime and every 0-prime ideal is maximal, every strongly 0-prime ideal is maximal. \(\square\)

We now characterize the minimal strongly 0-prime ideal $P$ of $N$ in terms of $\overline{O}_P$ and $\overline{O}(P)$.

**Theorem 6.3.6.** Let $N$ be an NI near-ring and $P$ a strongly 0-prime ideal of $N$.

1. If $N$ satisfies $(CZ1)$, then $P$ is a minimal strongly 0-prime ideal of $N$ if and only if $P = \overline{O}_P$.

2. If $N$ satisfies $(CZ2)$, then $P$ is a minimal strongly 0-prime ideal of $N$ if and only if $P = \overline{O}(P)$.
Proof. 1) Let \( P \) be a minimal strongly 0-prime ideal of \( N \). Then by Theorem 6.3.2, \( P \) is completely prime and so \( S = N \mid P \) is an \( m \)-system. For \( a \in P \), if we suppose that \( a^k = 0 \) for some \( k > 0 \), then there is nothing to prove. Assume that \( a^k = 0 \) for all \( k > 0 \). Construct \( M \) as in the proof of Theorem 6.3.4. Let \( x, y \in M \). Then either \( xy = 0 \) or \( xy = 0 \). By the similar method to that of Theorem 6.3.4, \( (aw)^k = 0 \) for some \( k, w \in N \mid P \) or \( Q \cap M = \emptyset \) for some strongly 0-prime ideal \( Q \). Suppose the latter is true. Then \( Q \subseteq P \). Since \( P \) is minimal strongly 0-prime, \( Q = P \). So that \( a \in Q \) and hence \( Q \cap M = \emptyset \), a contradiction. Thus \( (aw)^k = 0 \) for some \( k > 0 \). Since \( N \) satisfies (CZ1), \( a^q w^q = 0 \) for some \( q > 0 \). Hence \( a^q \in O_P \), because \( w^q \in S \) and consequently \( a \in \overline{O}_P \). Hence \( P \subseteq \overline{O}_P \).

On the other hand, let \( x \in \overline{O}_P \). Then there exist a positive integer \( n \) and \( s \in N \mid P \) such that \( x^n s = 0 \) and \( x^n s \in N^*(N) \). Since \( N \) is NI, \( N^*(N) \) is completely semiprime and therefore we obtain \( xs \in N^*(N) \). Since \( N^*(N) \) has the IFP, \( xNs \subseteq N^*(N) \subseteq P \). Since \( P \) is strongly 0-prime, \( x \in P \). Therefore \( \overline{O}_P \subseteq P \). Thus \( \overline{O}_P = P \).

Conversely, assume that \( \overline{O}_P = P \). We have to show that \( P \) is a minimal strongly 0-prime ideal of \( N \). Suppose that there is a strongly 0-prime ideal \( Q \) of \( N \) such that \( Q \subseteq P \). Then \( P = \overline{O}_P \subseteq \overline{O}_Q \subseteq Q \). So that \( P = Q \). Therefore, \( P \) is a minimal strongly 0-prime ideal of \( N \).

2) Let \( P \) be a minimal strongly 0-prime ideal of \( N \) and let \( a \in P \). By a similar method used in part (1), we obtain \( (aw)^k = 0 \) for some \( k > 0 \) and \( w \in N \mid P \). Since \( N \) satisfies (CZ2), \( < a^q > N < w^q > = 0 \) for some \( q > 0 \). Since \( P \) is minimal strongly 0-prime, \( P \) is completely prime by Theorem 6.3.2. Hence \( w^q \in N \mid P \) and so that \( a^q \in O(P) \) and consequently \( a \in \overline{O}(P) \).
the reverse inclusion, let \(x \in \mathcal{O}(P)\). Then \(x^n N <s> = 0\) for some \(n > 0\) and \(s \in N \mid P\). Hence \(x^n N <s> \subseteq N^*(N)\). Since \(N^*(N)\) is completely semiprime, \(xN <s> \subseteq N^*(N) \subseteq P\). Since \(P\) is strongly 0-prime, \(x \in P\). Therefore \(\overline{\mathcal{O}(P)} \subseteq P\). Thus \(P = \overline{\mathcal{O}(P)}\). The converse is similar to the converse of part (1).

\[\square\]

**Corollary 6.3.7.** Let \(N\) be a near-ring with unity which satisfies the condition (CZ1). Then \(N\) is NI if and only if \(P = \overline{\mathcal{O}_P}\) for every minimal strongly 0-prime ideal \(P\) of \(N\).

**Proof.** Suppose that \(P = \overline{\mathcal{O}_P}\) for every minimal strongly 0-prime ideal \(P\) of \(N\). Then

\[
N^*(N) = P \quad P = \overline{\mathcal{O}_P}
\]

Let \(x^n = 0\) for some \(n > 0\). Then \(x^n \in \mathcal{O}_P\) for all \(P \in \mathfrak{mSSpec}(N)\). This implies that \(x \in N^*(N)\). Thus \(N\) is NI. The converse follows from Theorem 6.3.6.

\[\square\]