CHAPTER III

ON OPERATORS OF ASCENT AND DESCENT 0 OR 1

In this chapter, we study operators in the class $\mathcal{A}$ whose descent is also 0 or 1. Since the condition that an operator should be of ascent 0 or 1 is rather very general, it is natural to impose certain further restrictions on operators in the class $\mathcal{A}$, so as to obtain stronger results. An operator may belong to $\mathcal{A}$ without having finite descent. An example is provided by the unilateral shift $U$ defined on the space $l^2$ of square-summable complex sequences by $U(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots)$. Since $U$ is isometric, $N(U) = N(U^2) = (0)$, so that $U \in \mathcal{A}$. But since $R(U^2)$ is a proper subspace of $R(U)$, the descent of $U$ is not 0 or 1, and hence by Theorem 5.41-E of Taylor [47], the descent is not finite.

Although the condition for an operator in $\mathcal{A}$ to have
descent 0 or 1 may appear heavy, our first theorem in this chapter justifies the study of such operators in the general case when X is not finite-dimensional.

**Theorem 12.** Let \( \dim X < \infty \), then \( T \in A \) if and only if \( T \) is of descent 0 or 1.

**Proof.** It is clear that every operator \( T \) on a finite-dimensional space \( X \) is a Fredholm operator of index zero. Indeed, \( R(T) \), being finite-dimensional, is closed and \( n(T) \) and \( d(T) \) are both finite. Also the relations \( \dim X = \dim N(T) + \dim R(T) \) and \( \dim X = \dim R(T) + \dim R(T)^- \) yield \( n(T) = d(T) \). It now follows from Theorem 4.5(c) and (d) of Taylor [48] that \( T \in A \) if and only if \( s(T) = 0 \) or 1.

The proof of Theorem 12 depends on the fact that in a finite-dimensional space, every operator is a Fredholm operator of index zero. Since not every operator is Fredholm if \( X \) is not finite-dimensional, it is natural to ask for conditions under which an operator is a Fredholm operator of index zero. One such condition is given in the following

**Theorem 13.** Let \( T \in A \) and \( s(T) \) be finite. Then \( T \) is a Fredholm operator of index zero if and only if there exists an operator \( A \) such that \( AT - I \) is compact.
We shall show this theorem follows as a consequence of the following lemma which is contained in [18, Theorem 1.1].

**Lemma 13.1.** The following conditions on an operator $T$ are equivalent:

(a) Either $R(T)$ is not closed or $N(T)$ is not finite-dimensional.

(b) There cannot exist any operator $A$ such that $AT - I$ is compact.

**Proof of Theorem 13.** It will be shown in Theorem 17 that if $T \in \mathcal{A}$ and $\delta(T)$ is finite, then $R(T)$ is closed. Thus if $AT - I$ is compact for some operator $A$, then by Lemma 13.1, it follows that $n(T)$ is finite. But then by Corollary 4.4 of Taylor [48], $d(T)$ is also finite and in fact, $n(T) = d(T)$. Conversely, if $T$ is a Fredholm operator of index zero, then again in virtue of the Lemma, the finiteness of $n(T)$ and the closedness of $R(T)$ yield an operator $A$ such that $AT - I$ is compact.

As was seen at the beginning of this chapter, an operator may belong to $\mathcal{A}$ without having finite descent. In our next three theorems, we discuss conditions under which an operator in $\mathcal{A}$ should be of descent 0 or 1.
THEOREM 14. Let $X$ be reflexive. If $T$ is an operator of ascent 0 or 1, then $R(T^*) = R(T^{*2})$.

Consequently, if $X = H$, a Hilbert space, then $T \in \mathcal{A}$ if and only if $R(T^*) = R(T^{*2})$.

PROOF. Since $X$ is reflexive, the relation $(N^*)^\perp = N^*$ holds for any subspace $N^*$ of $X^*$.

[18, Theorem II.3.5]. Also $R(T^*) = (N^*)^\perp$ for any operator $T$. [18, Theorem II.3.8]. Thus

$$R(T^*) = (N^*)^\perp = (N^{T^2})^\perp = (R(T^{*2}))^\perp = R(T^{*2}).$$

Since a Hilbert space is reflexive, it follows in virtue of Theorem 1 (Chapter I) that $T \in \mathcal{A}$ if and only if $R(T^*) = R(T^{*2})$.

For any operator $T$, $R(T)$ is closed if and only if $R(T^*)$ is closed [18, Theorem IV.1.2]. Also $T^2$ is Fredholm whenever $T$ is [18, Theorem IV.2.7]. This leads to the following

COROLLARY 14.1. Let $T$ be a Fredholm operator. Then $T \in \mathcal{A}$ if and only if $T^*$ is of descent 0 or 1.

THEOREM 15. (i) If $T$ is an operator such that $T^*$ is of ascent 0 or 1, then $R(T) = R(T^2)$. If, in particular, $T$ is Fredholm, then $\delta(T) = 0$ or 1.
(ii) Let $T$ be a Fredholm operator of index zero. Then $T \in \mathcal{A}$ if and only if $\delta(T) = 0$ or 1.

**Proof.** (i) Since $R(T)^\perp = R(T_\perp) = N(T^*)$ by [18, Theorem II.3.7], we obtain $R(T) = \frac{1}{2}(R(T)^\perp) = \frac{1}{2}N(T^*) = \frac{1}{2}N(T^2) = \frac{1}{2}(R(T^2)^\perp) = R(T^2)$. If, now, $T$ is Fredholm, then $R(T)$ and $R(T^2)$ are both closed and we are done.

(ii) Since $n(T)$ and $d(T)$ are finite and equal, invoking Theorem 4.5(c) and (d) of [48], it follows that $T \in \mathcal{A}$ if and only if $\delta(T)$ is 0 or 1.

As an immediate consequence we obtain the following

**Corollary 15.1.** If $T$ and $T^*$ have both ascent (or descent) equal to 0 or 1, then $n(T) = n(T^2)$ and $d(T) = d(T^2)$. If, in addition, $T$ is a Fredholm operator of index zero, then all the four quantities $n(T)$, $n(T^2)$, $d(T)$, and $d(T^2)$ are equal.

Our next theorem gives a necessary and sufficient condition that a normal operator should be of descent 0 or 1.

**Theorem 16.** Let $T$ be a normal operator on $H$. $T$ is of descent 0 or 1 if and only if $T^*T$ is of descent 0 or 1.
PROOF. It has been shown in [12, Theorem 2.2] that for any operator $A$, $R(A) = R((AA^*)^{1/2})$. This fact, together with our hypothesis yields $R(T^2) = R(T^2T^2)^{1/2} = R(T^2T) = R((T^*T)^2) = R((T^4)^{1/2}) = R(T^4)$. Since $R(T^4) \subset R(T^3) \subset R(T^2)$, it follows that $R(T^2) = R(T^3)$. Thus $\delta(T) \leq 2$. From Theorem 5.41-E of Taylor [47] it follows that $\delta(T) = \alpha(T) = 0$ or 1. Conversely if $\delta(T) = 0$ or 1, then clearly $R(T^2) = R(T^4)$. By another application of [12, Theorem 2.2] we obtain $R(T^*T) = R((T^*T)^2)$. This completes the proof.

Remark. The above theorem does not remain true for a non-normal operator even if it is hyponormal. In fact, if $U$ is the unilateral shift on the space $l^2$, then $U$ is isometric so that $\alpha(U) = 0$. Also $U^*U = I$ so that $\delta(U^*U) = \delta(I) = 0$; but the descent of $U$ is not finite.

THEOREM 17. Let $T \in \mathcal{A}$ and $\delta(T)$ be finite. Then $R(T)$ is closed.

PROOF. By our hypothesis and Theorem 5.41-G of Taylor [47], it follows that $\alpha(T) = \delta(T) = 0$ or 1 and that $I = N(T) \supset R(T)$. It has been proved in
that if \( N \) is a closed subspace of \( X \) such that \( R(T) \oplus N \) is closed, then \( R(T) \) is closed. Since \( N(T) \) is always closed, the closedness of \( R(T) \) follows.

**COROLLARY 17.1.** Let \( T \) be an operator such that \( T - \lambda I \in \mathcal{A} \) for all scalars \( \lambda \) and \( \delta(T - \lambda I) \) be finite for all \( \lambda \neq 0 \). If one of the quantities \( n(T - \lambda I) \) and \( d(T - \lambda I) \) is finite, then \( T \) is a Riesz operator.

**PROOF.** Caradus [8] has characterized Riesz operators \( T \) by the condition that for each non-zero scalar \( \lambda \), \( T - \lambda I \) should have finite ascent, descent, nullity, defect and closed defect. By Theorem 17, \( R(T - \lambda I) \) is closed, so that \( i(T - \lambda I) = \delta(T - \lambda I) \) for each \( \lambda \neq 0 \). The corollary follows now by the above mentioned characterization of Caradus.

Ch. Constantin [10] has shown that any restriction-normaloid Riesz operator is normal. This leads to the following

**COROLLARY 17.2.** If \( T \) is a restriction-normaloid operator satisfying the conditions of Corollary 17.1, then \( T \) is normal.

In the following theorem we obtain a relation between
operators in the class $\mathcal{A}$ and commutators of operators.

**THEOREM 18.** Let $T$ be an operator of ascent 0 or 1 on $H$. If $\delta(T)$ is not finite, then $T$ is a commutator.

**PROOF:** Assume, to the contrary, that $T$ is not a commutator. Then by the well known Brown-Pearcy characterization of commutators [6], $T$ can be expressed in the form $\lambda + C$ where $\lambda$ is a non-zero scalar and $C$ is compact. But then by Theorem 5.5-E of [47], we obtain $\sigma(T) = \delta(T)$. This contradiction proves the theorem.

It was proved in [37, Theorem 3] that if $T \in \mathcal{A}$ and $R(T)$ is not closed, then $T$ is a commutator. This result is immediate in virtue of the following stronger result.

**THEOREM 19.** If $T$ is an operator on $H$ whose range is not closed then $T$ is a commutator.

**PROOF.** Berberian [3] has shown that if $0 \in w(T)$, $w(T)$ being the Weyl's spectrum of $T$, then $T$ is a commutator. Thus it suffices to show that if $R(T)$ is not closed then $0 \in w(T)$. In fact, if $R(T)$ is not closed then clearly $T$ is not a Fredholm operator of index zero. The result now follows from Schechter's characterization of $w(T)$.
as the set \( w(T) = \{ \lambda | T - \lambda I \text{ is not a Fredholm operator of index zero} \} \).

We obtain a corresponding result for self-commutators in the following

**Theorem 20.** If \( T \) is a self-adjoint operator on a separable Hilbert space \( H \) such that \( R(T) \) is not closed, then \( T \) is a self-commutator.

**Proof.** Radjavi [33] has characterized self-commutators on a separable Hilbert space \( H \) as follows: An operator \( T \) on \( H \) is a self-commutator if and only if \( T \) cannot be expressed as \( \lambda + C \) where \( \lambda \) is a non-zero positive scalar and \( C \), a compact operator. If, now, \( T \) is not a self-commutator, then by Radjavi's characterization, there exists a non-zero positive number \( \lambda \) and a compact operator \( C \) such that \( T = \lambda + C \). This implies in virtue of [47, Theorem 55.E] and Theorem 17 that \( R(T) \) is closed, a contradiction.

In the following theorems we discuss for an operator \( T \) of descent (ascent) 0 or 1, on \( H \), conditions under which the compactness of some power of \( T \) implies that of \( T \).

**Theorem 21.** Let \( T \in \mathcal{A} \) and \( \delta(T) \) be finite. If
$T^k$ is compact for some $k$, then $T$ is finite-dimensional.

**Proof.** It is clear from the hypothesis that $T^k$ has ascent and descent equal to 0 or 1 and that

$$R(T^k) = R(T).$$

Further, if the range of a compact operator is closed, then it is finite-dimensional [18, Theorem III.1.12]. Thus it only remains to observe that, in virtue of Theorem 17 and the compactness of $T^k$, $R(T^k)$ is finite-dimensional.

We obtain a more general result in the following

**Theorem 22.** Let $T \in c^f$ and $R(T^k)$ be closed for some $k$. If $T^k$ is compact then $T$ is finite-dimensional.

**Proof.** We first observe that an operator $A$ is finite-dimensional (compact) if and only if $A^*$ is finite-dimensional (compact). [47, Theorem 5.5-B]. Further, if $A$ is a compact operator with closed range, then $A$ is finite-dimensional. Since $N(T^j)$ are equal for all $j \geq 1$ and $R(T^{*k})$ is closed it follows that

$$R(T^*) = R(T^{*2}) = \ldots = R(T^{*k}) = R(T^{*k}).$$

Also $T^{*k}$ is finite-dimensional, being a compact operator with closed range; so that
\[ \dim R(T^*) \leq \dim R(T^*) = \dim R(T^{*k}) < \infty. \]

This proves that \( T^* \), and hence \( T \), is finite-dimensional.

**Remarks.** 1. As was already observed, for the unilateral shift \( U \) the range of every power of \( U \) is closed, but the descent of \( U \) is not finite. Thus Theorem 22 is a proper generalization of Theorem 21.

2. It has been proved in [12, Theorem 2.5] that if \( V \) is a linear subspace of \( H \), then any operator \( A \) on \( H \) with \( R(A) \subseteq V \) is compact if and only if \( V \) contains no closed infinite-dimensional subspace of \( H \). From this result we deduce that if \( T \) is an operator of descent 0 or 1 and \( T^k \) is compact, then \( T \) is finite-dimensional. In fact since

\[ R(T) = R(T^2) = \ldots = R(T^k) \subseteq R(T^k) = V, \]

and \( T^k \) is compact we obtain \( \dim R(T) = \dim R(T^k) \leq \dim R(T^k) < \infty \).