Chapter 7

Similarity Equations of Steady Three-Dimensional Boundary-Layer Flow of an Incompressible Second-Order Fluid Past a Flat Plate

In last chapter we studied two-dimensional boundary-layer flows of second order fluids using the similarity techniques. To demonstrate the extension and applicability of the proposed method, this chapter is devoted to investigate generalized three-dimensional second order fluid flows over a flat plate.

This chapter is organized as follows: Sections 7.1 and 7.2 contains the introduction and mathematical formulation of steady three-dimensional boundary-layer flow of an incompressible second-order fluid past a flat plate. The similarity analysis of the governing equations using the proposed method is performed in Section 7.3. Finally, conclusions are provided in Section 7.4.
7.1 Introduction

The boundary layer theory emerged at first from the very inspired ideas of Prandtl (1904) when he formulated the boundary-layer equations. In many engineering and industrial applications of fluid flows, one often deals with fluids whose behavior cannot be described. It is well-known that a number of fluids which occur in practical applications, such as molten plastic, polymers, pulps, foods, etc. exhibit a non-Newtonian fluid behavior.

Two-dimensional boundary layer flows for non-Newtonian fluids (especially the second-order fluids) have received considerable attention in the last few decades. Moreover, three-dimensional boundary layer flows for such fluids appear to be rare and received relatively little attention in the literature in comparison with their counterparts two-dimensional boundary-layer flows so far; that is because that system of equations for a such fluids are very complex and highly nonlinear. This makes the analysis of three-dimensional problems difficult if not impossible to solve. This is may be a reason why the theoretical development of unsteady three dimensional boundary layer flows of second-order fluids could not progress much.

However, is the literature survey of the work done in this direction, Sacheti and Chandran [127] have studied the steady three-dimensional boundary layer flow of a certain kind of second order fluid over a flat plate, while Seshadri et al [131] investigated the unsteady three-dimensional stagnation point flow of a viscoelastic fluid. Both nodal and saddle point regions of flow have been considered.

In recent years, three-dimensional flow of a second grade fluid near the stagnation point of an infinite flat plate moving parallel to itself with constant velocity was considered by Baris [21]. Further, Nazar and Latip [103] investigated the numerical solution of three-dimensional flow over a stretching surface in a viscoelastic fluid. Besides that, a study related to this topic was also carried out by Nabil et al [102] with mass and heat transfer over an infinite horizontal stretching sheet under heat generation (absorption) and chemical reaction. Recently, the steady flow of a second grade fluid past an obstacle in three space dimensions was discussed by Pawel
Konieczny and Ondrej Kreml [75]. Al-Saheli [1] have obtained similarity representation of two-dimensional thermal boundary layer equations on a vertical plate and steady three-dimensional boundary layer flow over a flat plate for a second order with the help of the free parameter method. We consider the same problem as considered by Al-Saheli [1] to show the effectiveness of the our method in obtaining similarity analysis to steady three-dimensional boundary layer flow over a flat plate.

### 7.2 Mathematical Formulation

We choose cartesian coordinates as our reference axes. Assume that the fluid is bounded by an infinite flat plate. Since the flow is three-dimensional, the velocity field must be depending upon the three space variables, i.e. $V = (u(x, y, z), v(x, y, z), w(x, y, z))$. Thus the continuity and momentum equations for three-dimensional flow of an incompressible viscoelastic fluid under the usual boundary layer approximations take the form [1]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7.1}
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - U \frac{\partial U}{\partial x} - W \frac{\partial U}{\partial z} - \frac{\partial^2 u}{\partial y^2} &= -\mu_1 \left( u \frac{\partial^3 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial y^3} + w \frac{\partial^3 u}{\partial z \partial y^2} + \frac{\partial u}{\partial y} \left( \frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 w}{\partial z \partial y} \right) \right) \\
&\quad + \frac{\partial^2 u}{\partial y^2} \left( \frac{\partial u}{\partial x} - 2 \frac{\partial w}{\partial z} \right) + \frac{\partial w}{\partial y} \left( \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 w}{\partial z \partial y} \right) + \frac{\partial^2 w}{\partial y^2} \left( \frac{\partial u}{\partial x} + 2 \frac{\partial w}{\partial z} \right)
\end{align*} \tag{7.2}
\]

\[
\begin{align*}
\frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} - U \frac{\partial W}{\partial x} - W \frac{\partial W}{\partial z} - \frac{\partial^2 w}{\partial y^2} &= -\mu_2 \left( \frac{\partial u}{\partial y} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial z} \right) - 2 \frac{\partial u \partial w}{\partial y \partial z} \right) + \frac{\partial \left( \frac{\partial u}{\partial y} \right)}{\partial z} + \frac{\partial \left( \frac{\partial w}{\partial y} \right)}{\partial z} \left( \frac{\partial u}{\partial y} \right)^2 = 0 \\
\end{align*} \tag{7.3}
\]
Subject to the boundary conditions:

\[ u = 0, \ v = 0, \ w = 0 \text{ at } y = 0; \]

\[ u = U, \ w = W \text{ as } y \to \infty \]  

(7.4)

where \( U \) and \( W \) are the mainstream velocity components in the \( x \) and \( z \) directions, respectively; \( \mu_1 \) and \( \mu_2 \) are the coefficients of viscoelasticity and the cross viscosity respectively.

### 7.3 The Application of the Proposed Method

#### 7.3.1 Group of Transformation

The system (7.1)-(7.4) contain three independent variables \( x, y \) and \( z \) and five dependent variables \( u, v, w, U \) and \( W \). Therefore to convert this system into system of ordinary differential equations one needs to apply two-parameter group of transformation. As mentioned previously, a class of group transformation of the form (5.17), with two real parameters \( a_1, a_2 > 0 \) where the \( C's \) and \( K's \) are real values at least differentiable in these arguments \( (a_1, a_2) \) is good enough to this task. Thus, we consider,

\[
G_2 = \begin{cases} 
\tilde{x} = C^x(a_1, a_2)x + K^x(a_1, a_2) \\
\tilde{y} = C^y(a_1, a_2)y + K^y(a_1, a_2) \\
\tilde{z} = C^z(a_1, a_2)z + K^z(a_1, a_2) \\
\tilde{u} = C^u(a_1, a_2)u + K^u(a_1, a_2) \\
\tilde{v} = C^v(a_1, a_2)v + K^v(a_1, a_2) \\
\tilde{w} = C^w(a_1, a_2)w + K^w(a_1, a_2) \\
\tilde{U} = C^U(a_1, a_2)U + K^U(a_1, a_2) \\
\tilde{W} = C^W(a_1, a_2)W + K^W(a_1, a_2).
\end{cases}
\]  

(7.5)

#### 7.3.2 Invoking the Similarity Transformations

These groups provide us transformations of the form (5.30) which is useful to reduce a given system of nonlinear partial differential equations to three-dimensional...
second order fluid flows over a flat plate. More convenient, in terms of the notations of a given problem these transformations can be written in the following general form:

\[
\begin{align*}
\eta &= \frac{y}{\pi(x,z)} \\
u(x,y,z) &= \Gamma^1(x,z)F_1(\eta) \\
v(x,y,z) &= \Gamma^2(x,z)F_2(\eta) \\
w(x,y,z) &= \Gamma^3(x,z)F_3(\eta) \\
U(x,z) &= \Gamma^4(x,z)F_4(\eta) \\
W(x,z) &= \Gamma^5(x,z)F_5(\eta)
\end{align*}
\] (7.6)

Since \(\Gamma_4(x,z), \Gamma_5(x,z), U(x,z)\) and \(W(x,z)\) are independent of \(y\), whereas \(\eta\) depends on \(y\), it follows that \(F_4\) and \(F_5\) in (7.6) must be equal to a constant, \(F_4 = U_0\) and \(F_5 = W_0\) (say).

Next examine whether the system (7.1)-(7.4) is expressible in terms of the new variables which rise in these transformation.

### 7.3.3 The Invariance Analysis and Reductions to Ordinary Differential Equations

As the general analysis proceeds, the established forms of the transformations generated from the absolute invariants of group are used to obtain ordinary differential equations. According to this method the invariance in the form of (7.1)-(7.4) takes place if they are express in terms of new variables which appear in these transformations, namely, \(\eta\) as independent variable and \(F_1(\eta), F_2(\eta) & F_3(\eta)\) as dependent variables otherwise no similarity representation is possible. First we will start with the auxiliary conditions (7.4), i.e., we will examine whether the auxiliary conditions are appropriate to (7.6) because that these auxiliary conditions are simpler in nature. Substituting (7.6) into (7.4) to get

\[
\begin{align*}
F_1(0) &= 0, \quad F_2(0) = 0, \quad F_3(0) = 0 \quad \text{at} \quad \eta = 0, \\
F_1(\eta) &= U_0 \frac{\Gamma_4(x,z)}{\Gamma_1(x,z)} \quad F_3(\eta) = W_0 \frac{\Gamma_5(x,z)}{\Gamma_3(x,z)} \quad \text{as} \quad \eta \to \infty
\end{align*}
\] (7.7)
which is appropriate to the transformations (7.6) if

\[
\Gamma_4(x, z) = \Gamma_1(x, z) \\
\Gamma_5(x, z) = \Gamma_3(x, z)
\]  

(7.8)

Next, inserting (7.6) in (7.1)-(7.3) after using the above results and dividing the three equations by \( n(x, z)T_2(x, z) \), \( n(x, z)T_2(x, z)T_3(x, z) \) respectively, further simplifications, yields,

\[
F_2' + C_4F_1 + C_2F_3 + C_5\eta F_1' + C_1\eta F_3' = 0, 
\]

(7.9)

\[-\mu_1 F_2 F''_1 + C_4C_5(F_1^2 - 1) + C_5C_6(F_1F_3 - 1) + C_3C_5\eta F_1 F'_1 - C_9F''_1
\]

\[+ C_5F_2 F'_1 + C_1C_5\eta F_3 F'_1 - C_6[\mu_1 F_3 F''_1 + (\mu_2 + \mu_1)(F_1 F'_3 + 2F_1 F''_3)]
\]

\[- C_4[(F'_1)^2(2\mu_2 + 3\mu_1) + 2\mu_1 F_1 F'_1] + C_2[(\mu_1 + \mu_2)(F_1 F'_3 + 2F_3 F''_3)]
\]

\[- C_7(\mu_2 + 2\mu_1)[(F_3^2 + F_3 F'_3)] - C_8(\mu_2 + 2\mu_1)(F_3^2 + 2\eta F_2 F'_3)
\]

\[- C_3[(2\mu_2 + 3\mu_1)(F'_1)^2 + 2\mu_1 F_1 F'_1 + (2\eta \mu_2 + 4\eta \mu_1)F_1 F''_1 + \eta \mu_1 F_1 F'']
\]

\[- C_1[(\mu_2 + \mu_1)F_1 F'_3 + 2\mu_1 F_3 F''_3 + \eta \mu_1 F_3 F''] = 0 \text{ and }
\]

\[-\mu_1 F_2 F''_3 + C_1 C_5\eta [F_3 F'_3] + C_2 C_5[F_3^2 - 1] + C_3 C_5[\eta F_1 F'_3] + C_5 F_2 F'_3
\]

\[+ C_6 C_5(F_1 F_3 - 1) - C_10[\mu_2 + \mu_1](2F'_1 F'_3 + F_3 F''_3) + \mu_1 F_1 F'_3\]

(7.11)

\[- C_11(\mu_2 + 2\mu_1)[(F'_1)^2 + 2\eta F_1 F'_1] - C_12(\mu_2 + 2\mu_1)[(F'_1)^2 + F_1 F'_3]
\]

\[+ C_4[(\mu_1 + \mu_2)(2F_1 F'_3 + F'_1 F'_3)] - C_2[(2\mu_2 + 3\mu_1)(F'_3)^2 + 2\mu_1 F_3 F''_3]
\]

\[- C_3[(\mu_2 + \mu_1)F_1 F'_3 + 2\mu_1 F_3 F''_3 + \eta \mu_1 F_1 F''] - C_9 F_3
\]

\[- C_1[(2\mu_2 + 3\mu_1)(F'_3)^2 + 2\eta \mu_2 + 2\mu_1)F_3 F''_3 + 2\eta \mu_1(F_3 F''_3) + \eta \mu_1 F_3 F'''_3)] = 0
\]

where the primes refer to differentiation with respect to \( \eta \), and \( C_1, ..., C_{12} \) are as following:

\[
C_1 = \frac{\Gamma_3}{\pi^2 \Gamma_2} \frac{\partial \pi}{\partial z}, \quad C_2 = \frac{1}{\pi^2 \Gamma_2} \frac{\partial \Gamma_3}{\partial z}, \quad C_3 = \frac{\Gamma_1}{\pi^2 \Gamma_2} \frac{\partial \pi}{\partial x}, \quad C_4 = \frac{1}{\pi^2 \Gamma_2} \frac{\partial \Gamma_1}{\partial x}, \quad C_5 = \frac{1}{\pi^2}
\]

(7.12)

\[
C_6 = \frac{\Gamma_3}{\pi^2 \Gamma_2 \Gamma_1} \frac{\partial \Gamma_1}{\partial z}, \quad C_7 = \frac{\Gamma_3}{\pi^2 \Gamma_2 \Gamma_1} \frac{\partial \Gamma_3}{\partial x}, \quad C_8 = \frac{\Gamma_2}{\pi^2 \Gamma_2 \Gamma_1} \frac{\partial \pi}{\partial x}, \quad C_9 = \frac{1}{\pi^2 \Gamma_2}
\]

\[
C_{10} = \frac{\Gamma_1}{\pi^2 \Gamma_3} \frac{\partial \Gamma_3}{\partial x}, \quad C_{11} = \frac{\Gamma_2^2}{\pi^2 \Gamma_2 \Gamma_3} \frac{\partial \pi}{\partial z} \text{ and } C_{12} = \frac{\Gamma_1}{\pi^2 \Gamma_2 \Gamma_3} \frac{\partial \Gamma_1}{\partial z}
\]
We have to mention that most of the above expressions were calculated automatically using Mathematica.

Inasmuch as the first terms of (7.9)-(7.11) have the constant coefficients. So to reduce (7.9)-(7.11) to an expressions in single independent variable \( \eta \), it is necessary that the remaining coefficients be constants or functions of \( \eta \) alone. Thus, since \( \pi \) and \( \Gamma' \)s are independent of \( y \), the \( C' \)s are constants. and to be determined for each individual case corresponding to \( \pi \) and \( \Gamma' \)s. The equations \( C_5 = \frac{1}{\pi_3} \) and \( C_9 = \frac{1}{\pi_5} \), state that the \( \pi \) and \( \pi_2 \) are constants so the absolute invariant \( \eta \) take only one case when \( \pi = \text{Constant} \). By considering \( C_5 \) and \( C_9 \) may be taken to be unity, it follow that \( \pi = 1, \Gamma_2 = 1 \) and \( C_1 = C_3 = C_8 = C_{11} = 0 \). Moreover, let

\[
F_1(\eta) = F'(\eta) \tag{7.13}
\]

\[
F_5(\eta) = G'(\eta)
\]

Equations (7.9) gives

\[
F_2(\eta) = -C_4F(\eta) - C_2G(\eta) \tag{7.14}
\]

Inserting all these results in equations (7.10)-(7.11), with simplifying further we get

\[
-\mu_1[C_4(3(F'')^2 + 2F' F'' - F''')F) - C_2(F''')^2 + F''G' + 2G'F'')
\]

\[
+C_6(2F''G'' + F'G'' + G'F''') + 2C_7((G''')^2 + G'G'') - C_2G'F' - F'''
\]

\[
-\mu_2[2C_4(F'')^2 - C_4(F''G'' + 2G'F'') + C_6(2F''G'' + F'G''')
\]

\[
+C_7((G''')^2 + G'G'')] + C_4((F')^2 - FF'' - 1) + C_6(F'G' - 1) = 0
\]

\[
-\mu_1[-C_4(2F'G'' + F''G' + FG''') + C_2(3(G'')^2 + 2G'G'' - GG''')
\]

\[
+C_6(2F''G'' + G'F'' + F'G''') + 2C_12((F'')^2 + F'F''') - C_4G'F - G'''
\]

\[
-\mu_2[2C_3(G'')^2 - C_4(2F'G'' + F''G') + C_10(2F''G'' + F'G'')
\]

\[
+C_12((F'')^2 + F'F'') + C_2[(G'')^2 - G''G - 1] + C_10(F'G' - 1) = 0
\]

(7.15)

with the boundary conditions

\[
F'(\eta) = 0, \quad -C_4F(\eta) - C_2G(\eta) = 0, \quad G'(\eta) = 0 \quad \text{at} \quad \eta = 0,
\]

\[
F'(\eta) = U_0, \quad G'(\eta) = W_0, \quad \text{as} \quad \eta \to \infty \tag{7.17}
\]
7.3. The Application of the Proposed Method

Our target now is to find the expressions of \( \Gamma_1 \) and \( \Gamma_3 \) which satisfy (7.12). Using the classical techniques we solve (7.12) to obtain the following expression for \( \Gamma_1 \) and \( \Gamma_3 \):

\[
\Gamma_1 = \Gamma_3 = C_4 x + C_2 z + K \tag{7.18}
\]

with \( C_2 = C_6 = C_{12} \), \( C_4 = C_7 = C_{10} \) and \( K \) is arbitrary constant.

It is obvious from (7.18) that \( \Gamma_1 \) and \( \Gamma_3 \) are functions for both \( x, z \). Subcases appear when they are functions for only one variable.

### 7.3.4 Similarity Representation when \( \Gamma_1 = \Gamma(x) \) and \( \Gamma_3 = \Gamma_3(x) \)

For this sub-case with (7.12) we find that \( C_2 = C_6 = C_{12} = 0 \) and

\[
\Gamma_1 = C_4 x + K_1 \\
\Gamma_3 = K_2 (C_4 x + K_1) \frac{C_4}{C_6}
\]

where \( K_1 \) and \( K_2 \) are integration constants and \( C_4 = C_{10}, \frac{C_7}{C_{10}} = K_2 \). By substituting the above-obtained values of \( C \)'s and \( \Gamma_1 \) & \( \Gamma_3 \) into Equations (7.15) and (7.16), we get:

\[
\begin{align*}
\mu_1 [C_4 (F''' F - 3(F'')^2 - 2F' F''') - 2C_7 (G''^2 + G' G''')] - F''' \\
+ \mu_2 [2C_4 (F''^2 + C_7 (G''^2 + G' G''')) + C_4 ((F')^2 - FF'' - 1)] = 0
\end{align*}
\tag{7.19}
\]

\[
\begin{align*}
\mu_1 [C_4 (FG'''' + 2F' G'''' + F'' G''') - C_{10} (2F'' G'' + C' F'''' - F' G''')] - C_4 G'' F \\
+ \mu_2 [-C_4 (2F' G'''' + F'' G''') + C_{10} (2F'' G'' + G' F'')] + C_{10} (F' G' - 1) - G'''' = 0
\end{align*}
\tag{7.20}
\]

with the boundary conditions,

\[
F'(\eta) = F(\eta) = 0, \ G'(\eta) = 0 \ \text{at} \ \eta = 0, \\
F'(\eta) = U_0, \ G'(\eta) = W_0, \ \text{as} \ \eta \to \infty
\]
7.3. The Application of the Proposed Method

7.3.5 **Similarity Representation when** $\Gamma_1 = \Gamma_1(z)$ and $\Gamma_3 = \Gamma_3(z)$

For this sub-case with (7.12) we find that $C_4 = C_7 = C_{10} = 0$ and

$$\Gamma_1 = K_2(C_2 z + C_6) K_1$$
$$\Gamma_3 = C_2 z + K_1,$$

where $K_1$ and $K_2$ are integration constants and $C_2 = C_6$, $C_6^2 = K_2$. Inserting the above-obtained values of $C$'s and $\Gamma_1$ & $\Gamma_3$ into Equations (7.15) and (7.16), we get:

$$\mu_1 [C_2 (F''') G + F''' G'' + 2G' F'') - C_6 (2F'' G'' + F' G'' + G' F''')] - C_2 GF''$$
$$+ \mu_2 [-C_2 (F'' G'' + 2G' F'') + C_6 (2F'' G'' + F' G''')] + C_6 (F' G' - 1) - F''' = 0$$

(7.21)

$$\mu_1 [C_2 (G G''' - 3(G'')^2 - 2G' G'') - 2C_12 (F''')^2 + F'' F''') - C_2 GF''$$
$$- \mu_2 [2C_2 (G'' G - C_12 (F'')^2 + F' F'')] + C_2 (C' G'' - 1) = 0$$

(7.22)

with the boundary conditions,

$$F'(\eta) = G(\eta) = 0, \ G'(\eta) = 0 \ at \ \eta = 0,$$
$$F'(\eta) = U_0, \ G'(\eta) = W_0, \ as \ \eta \to \infty$$

7.3.6 **Similarity Representation when** $\Gamma_1 = \Gamma_1(x)$ and $\Gamma_3 = \Gamma_3(z)$

For this sub-case with (7.17) we got $C_6 = C_7 = C_{10} = C_{12} = 0$ and

$$\Gamma_1 = (C_4 x + K_1)$$
$$\Gamma_3 = C_2 z + K_2$$

where $K_1$ and $K_2$ are integration constants. By substituting the above-obtained values of $C$'s and $\Gamma_1$ & $\Gamma_3$ into Equations (7.15) and (7.16), we get:

$$\mu_1 [C_2 (F''') F - 3(F'')^2 - 2F' F'') + C_2 (F'' G + F' G'' + 2G' F'')] - F'''$$
$$+ \mu_2 [2C_4 (F'')^2 - C_2 (F'' G'' + 2G' F'')] + C_4 ((F'')^2 - FF'' - 1) - C_2 GF'' = 0$$

(7.23)
\[ \mu_1[C_4(FG''' + 2F'G'' + F''G') + C_2(GG''' - 3(G'G')^2 - 2G'G'')] - G'' \\
- \mu_2[2C_2(G')^3 - C_4(2F'G'' + F''G'')] + C_2[(G')^2 - G''G - 1] - C_4G'F = 0 \] (7.24)

with the boundary conditions (7.17).

### 7.4 Conclusion

In this chapter, we have considered steady three-dimensional boundary-layer flow of an incompressible second-order fluid past a flat plate. The governing system of non-linear partial differential equations is transformed into the system of non-linear ordinary differential equations using our proposed method. The carried out analysis, in this problem shows the effectiveness of the method in obtaining similarity representation for the system of partial differential equations easily with less steps. It is interesting to note that the proposed method here is well suited over to huge problems and enables us to use the Mathematica to carry out the calculations.

A paper based on the work described in this chapter has been published in Al-Salihi [6].