Chapter 6

Applications of the New Method for Two-Dimensional Boundary Layer Flow of Second-Order Fluid

In this chapter, the method discussed in two previous chapters is applied to selected equations in second order fluids. The equations considered are, steady two-dimensional boundary layer equations for this fluid past a stretching sheet and the two-dimensional thermal boundary layer equations on a vertical plate, while the steady three-dimensional boundary layer flow of a second order fluid over a flat plate will be taken in next chapter.

According to the proposed method, similarity transformations (new variable) under somewhat general group of transformation are evoked and attempt to express the given representations in terms of these new variables, if it is possible. The result be reduction of a given representations to a representations with less independent variables and hence an invariance of the given representations is achieved.

This chapter is organized as following: The first section contains an introduction to second order fluids. In Sections 6.2 and 6.3 two selected problems of the second order fluids show the applicability and accuracy of the proposed method. Numerical solutions for some reduced equations are presented in section 6.4 while Sections 6.5 and 6.6 contain results, discussion and the concluding remarks.
6.1 Introduction

There are many different characteristics that affect the behavior of a fluid and that may also be used to provide a categorization of all fluids. Of primary importance in determining the flow behavior of a fluid is the viscosity. On the basis of viscosity, fluids can be roughly subdivided into Newtonian and Non-Newtonian fluids. For Newtonian fluids the viscosity is taken to be constant over some range of shear rate. For Non-Newtonian fluids the viscosity varies with the shear rate; and may depend on the shear rate itself.

The theory of viscoelastic fluids has become a field of very active research for the last few decades as this class of fluids represents many important fluids used in industries such as plastic films, waste fluids, synthetic fibers, foodstuffs, as well as in some flows of polymer solutions. As special class of viscoelastic fluids is the second-order fluid which are one of the simplest non-Newtonian continuum model fluid which has a constant shear viscosity but nonzero first and second normal stress differences. These normal stress differences are unique viscoelastic features to all non-Newtonian fluids.

The so-called second-order fluid is a particular case of the Rivlin-Ericksen fluid of complexity two. The stress tensor for such fluids in given by

\[ T_{ij} = -p\varepsilon_{ij} + \mu A_1 + \mu_1 A_2 + \mu_2 A_1^2 \]  \hspace{1cm} (6.1)

where \( \mu \) is the coefficient of viscosity, \( \mu_1 \), the coefficient of viscoelasticity, and \( \mu_2 \) is the coefficient of cross viscosity; and the kinematic first two Rivlin-Ericksen tensors \( A_1 \) and \( A_2 \) are given by

\[ A_1 = (\text{grad} V) + (\text{grad} V)^T \]
\[ A_2 = \frac{d}{dt} A_1 + A_1 (\text{grad} V) + (\text{grad} V)^T A_1 \]

where \( \frac{d}{dt} \) denotes the material time derivative, \( V \) is the velocity field and grad is the gradient operator. The employment of those tensor in the momentum equations results in highly nonlinear equations of motion.
Over the past twenty years, there have been very extensive studies of such fluids. An extensive literature is available for boundary layer flows of second-order fluids with different physical features such as heat conduction and MHD. A brief account of studies is given below.

Some important studies dealing with the boundary layer flow of second-order fluids over a stretching sheet are made by Bujurke et al. [34], Ariel [18], Singh and Agarwal [141], Mamaloukas et al. [84], Lawrence and Rao [79], Shit and Haldar [138], Hsiao [59], Dhanalaxmi and Shanker [42], Cortell [41], Alharbi et al. [12], Rollins and Vajravelu [124], Mushtaq et al. [97], Siddappa and Abel [139], Ahmad [11].

Further, the boundary layer flow of second-order fluids over a vertical plate and vertical channel has also been considered by some authors, such as; Kasim et al. [71] who studied the free convection boundary layer flow of a viscoelastic fluid in the presence of heat generation, Uwanta et al. [149] consider a viscoelastic fluid past an infinite vertical plate with heat dissipation, Kumar and Sivaraj [77] studied the effect of dufour and chemical reaction of MHD mixed convective viscoelastic fluid flow in a permeable vertical channel. Choudhury and Deb [36] investigated heat and mass transfer for visco-elastic MHD boundary layer flow past a vertical flat plate, Sivaraj and Kumar [142] and Ghosh and Shit [67] studied the mixed convection MHD flow of viscoelastic fluid in a porous medium past the channel and hot vertical plate, respectively. Recently, Kasim et al. [72] considered the natural convection boundary layer flow of a viscoelastic fluid on solid sphere with newtonian heating. There are situations in which the unsteady flow considered such as the unsteady free convection heat and mass transfer in a Walters-B viscoelastic flow past a semi-infinite vertical plate: a numerical study by Prasad et al. [117] in 2012.

6.2.1 Governing Equations

The new method has applications to a wide range of physical problems. However, to explain it, we consider the problem of the flow of an incompressible second-order fluid past a stretching sheet (see [120]). The steady two-dimensional boundary layer equations for this fluid were derived by Beard [22]. In usual notation these equations are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - k \left( \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial y} \right) + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} + \nu \frac{\partial^3 u}{\partial y^3} \right)
\]

where \( u, v \) are the velocity components in the \( x, y \) directions, respectively, \( \nu = \frac{v}{\nu} \) the kinematic viscosity, and \( k = -\frac{\eta}{\nu} \) the elasticity parameter. \( \eta, \alpha_1(< 0) \) and \( \rho \) are coefficients of viscosity, coefficient of viscoelasticity and the density of the fluid respectively.

For deriving these equations it was assumed that in addition to the usual boundary layer approximations the contribution due to normal stresses to be of the same order of magnitude as that due to shear stresses. The subjected boundary conditions are

\[ u = cx, \quad v = 0 \quad \text{at} \quad y = 0; \quad u \to 0 \quad \text{as} \quad y \to \infty, \quad \delta \]

where \( c > 0 \) is a constant and \( \delta \) is the velocity boundary layer thickness.

6.2.2 The Application of the Proposed Method

6.2.2.1 Group of Transformations

Equations (6.2) and (6.3) contain two dependent variables \( u \) and \( v \) - the velocity components depend on the variables \( x \) & \( y \). Hence to reduce this equations we need to
apply one-parameter group. We introduce a class of one-parameter group of the form:

\[
G_1 : \begin{cases}
  \bar{x} = C'(a)x + K'(a), \\
  \bar{y} = C'(a)y + K'(a), \\
  \bar{u} = C''(a)u + K''(a), \\
  \bar{v} = C'(a)v + K'(a).
\end{cases}
\] (6.5)

Where \(C's\) and \(K's\) are real-valued and at least differentiable in the real argument \(a\).

6.2.2.2 Invoking the Similarity Transformations

As pointed out earlier that the groups can provide us with the transformations which convert above system of equations (6.2) and (6.3) into ordinary differential equations with meaningful boundary conditions. One-parameter group with the form (6.5) provides us sets of transformations of the forms (5.36), (5.37), (5.39) and (5.40) for case \(n = 2\) (number of the independent variables) and \(m = 2\) (number of the dependent variables), namely,

Set 1: If \(\alpha_1 \neq 0\) the transformations derived from \(G_1\), are

\[
\eta = A(x + \frac{\beta_1}{\alpha_1})^\frac{a_2}{a_1} (y + \frac{\beta_2}{\alpha_2}), \quad u = R_1 \text{ and } v = R_2, \quad \text{or}
\]

\[
\eta = A \ln(x + \frac{\beta_1}{\alpha_1})^\frac{-a_2}{a_1} (y + \frac{\beta_2}{\alpha_2}), \quad u = R_1 \text{ and } v = R_2
\] (6.6) (6.7)

where \(R_1\) and \(R_2\) take one of the following forms

\[
R_1 = \begin{cases}
  \frac{1}{B}(x + \frac{\beta_1}{\alpha_1})^{a_3} F_1(\eta) - \frac{\beta_3}{\alpha_3}, & \text{if } \alpha_3 \neq 0 \\
  \frac{1}{B} \ln(x + \frac{\beta_1}{\alpha_1})^{a_3} F_1(\eta), & \text{if } \alpha_3 = 0 \\
  F_1(\eta), & \text{if } \alpha_3 = \beta_3 = 0
\end{cases}
\]

\[
R_2 = \begin{cases}
  \frac{1}{B}(x + \frac{\beta_1}{\alpha_1})^{a_4} F_2(\eta) - \frac{\beta_4}{\alpha_4}, & \text{if } \alpha_4 \neq 0 \\
  \frac{1}{B} \ln(x + \frac{\beta_1}{\alpha_1})^{a_4} F_2(\eta), & \text{if } \alpha_4 = 0 \\
  F_2(\eta), & \text{if } \alpha_4 = \beta_4 = 0
\end{cases}
\]

Set 2: If \(\alpha_1 = 0\) the transformations derived from \(G_1\), are

\[
\eta = A e^{\frac{a_2}{\beta_1}} \left( y + \frac{\beta_2}{\alpha_2} \right), \quad u = R_1 \text{ and } v = R_2 \quad \text{or}
\]

\[
\eta = A (\beta_1 y - \beta_2 x), \quad u = R_1 \text{ and } v = R_2
\] (6.8) (6.9)
where $R_1$ and $R_2$ take one of the following forms

$$
R_1 = \begin{cases} 
\frac{1}{6} \epsilon^{\frac{a_3}{a_1}} F_1(\eta) - \frac{\beta_3}{a_3}, & \text{if } \alpha_3 \neq 0 \\
\frac{1}{p_1} \left( \beta_3 x - \frac{1}{6} F_1(\eta) \right) & \text{if } \alpha_3 = 0 \\
F_1(\eta) & \text{if } \alpha_3 = \beta_3 = 0
\end{cases}
$$

$$
R_2 = \begin{cases} 
\frac{1}{6} \epsilon^{\frac{a_4}{a_1}} F_2(\eta) - \frac{\beta_4}{a_4}, & \text{if } \alpha_4 \neq 0 \\
\frac{1}{p_1} \left( \beta_4 x - \frac{1}{6} F_2(\eta) \right) & \text{if } \alpha_4 = 0 \\
F_2(\eta) & \text{if } \alpha_4 = \beta_4 = 0
\end{cases}
$$

where $A$ and $B$ are arbitrary constants.

### 6.2.2.3 Invariance Analysis and the Reduction to ODEs

In order to investigate the invariant in form of the equations (6.2)-(6.4); it is enough to express them in term of the new variables which appear in the similarity transformations (6.6)-(6.9). Using one of these transformations, for example

$$
\eta = A(x + \frac{B_1}{\alpha_1})^{\frac{a_3}{a_1}} (y + \frac{B_2}{\alpha_2})
$$

$$
u = \begin{cases} 
B(x + \frac{B_1}{\alpha_1})^{\frac{a_3}{a_1}} F_1(\eta) - \frac{\beta_3}{a_3}, & \text{if } \alpha_3 \neq 0 \\
F_1(\eta) & \text{if } \alpha_3 = 0
\end{cases}
$$

$$
\nu = \begin{cases} 
C(x + \frac{B_1}{\alpha_1})^{\frac{a_4}{a_1}} F_2(\eta) - \frac{\beta_4}{a_4}, & \text{if } \alpha_4 \neq 0 \\
F_2(\eta) & \text{if } \alpha_4 = 0
\end{cases}
$$

where $\alpha_1 \neq 0$ and $A$, $B$ and $C$ are arbitrary constants. According to our method, the system (6.2) and (6.3) with auxiliary conditions should be expressible in terms of the new variables $\eta$ as independent variable and $F_1(\eta)$ & $F_2(\eta)$ as dependent variables otherwise no similarity solution is possible. We will start by examining the appropriateness of the auxiliary conditions (6.4) to the transformations (6.10) because they are not complicated in nature. This is, by using the transformations (6.10), the boundary conditions (6.4) transform to

$$
F_1(\eta) = 1, \quad F_2(\eta) = 0 \quad \text{as } \eta \to \infty
$$

$$
F_1(\eta) = 0 \quad \text{as } \eta \to \infty
$$

when \( \alpha_3 = \alpha_1, B = c, \beta_2 = 0 \) and \( \beta_1 = 0 \) (the restrictions under which the boundary conditions proportion with the transformations). Making use of values \( \alpha_3, B, \beta_1 \) and \( \beta_2 \) obtained here and introducing the transformation (6.10) into Equations (6.2) and (6.3). Farther dividing the resulting equations by \( B(x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_3}{\alpha_1} - 1} \) and \( B^2(x + \frac{\beta_1}{\alpha_1})^{\frac{3\beta_2}{\alpha_1} - 1} \) respectively, and rearranging the terms, one gets,

\[
\alpha_3 F_1 - \alpha_2 \eta F' + \left( \frac{AC}{B} \right) \alpha_1 (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} F'_2 = 0 \tag{6.12}
\]

\[
\frac{\alpha_3}{\alpha_1} F'_2 - \frac{\alpha_2}{\alpha_1} \eta F'_1 F' + \frac{AC}{B} (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} F_2 F'_1 - \frac{A^2}{B} \nu (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} F''_1
\]

\[
+ k \frac{C A^3}{B} (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} \alpha_2 \eta (F'_1 F'' + F_1 F''') + 2(\alpha_3 - \alpha_2) F_1 F''
\]

\[
+ k \frac{CA^3}{B} (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} (F'_1 F'' + F_1 F''')
\]

\[
- \left( x + \frac{\beta_1}{\alpha_1} \right) \frac{\beta_3}{B \alpha_1} \left( F_1 - \frac{\alpha_2}{\alpha_3} \eta F'_1 \right) = \frac{A \beta_4}{B \alpha_4} (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} F'_1
\]

\[
+ k \frac{A^2 \beta_3}{B \alpha_1} (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} \left( \frac{2 \alpha_2}{\alpha_3} - 1 \right) F''_1 + \frac{\alpha_2}{\alpha_3} \eta F'''
\]

\[
- \frac{A^3}{k} \frac{\beta_4}{B \alpha_4} (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} F'''_1 = 0 \tag{6.13}
\]

For (6.12) and (6.13) to be reduced as expression in a single independent invariant, it is necessary that the coefficients should be constants or functions of \( \eta \) alone. Thus,

\[
(x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} = c_1
\]

\[
(x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} = c_2
\]

\[
(x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} = c_3
\]

\[
(x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} = c_4
\]

\[
(x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} = c_5
\]

\[
\frac{\beta_3}{\alpha_3} (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} = c_6
\]

\[
\frac{\beta_4}{\alpha_4} (x + \frac{\beta_1}{\alpha_1})^{\frac{\alpha_4}{\alpha_1} - \frac{\alpha_3}{\alpha_1} - 1} = c_7
\]
Hence, the only possible values of \( \alpha' \)'s and \( \beta' \)'s are

\[
\alpha_2 = \alpha_4 = 0
\]
\[\beta_3 = \beta_4 = 0.\]  

(6.15)

In the light of the above results, the transformations (6.10) take the following form

\[
\eta = Ay
\]

\[
u = B(x + \frac{\beta_1}{\alpha_1})^{\frac{a_3}{a_1}} F_1(\eta), \quad \alpha_3 \neq 0
\]

(6.16)

which are called the similarity transformations. And the equations (6.12) and (6.13), readily give the similarity representation,

\[
F' + F_2 = 0 \quad \text{and}
\]

\[
F_1^2 + F_1 F'_1 - F'_2 + k_1[2F_1 F''_1 + F'_1 F''_2 + F_2 F'''_2] = 0.
\]

(6.17)

(6.18)

with the boundary conditions follow from (6.11). Using (6.17); (6.18) can be expressed in the terms of \( F_2 \) alone, i.e. we get

\[
F'_2 + F_2 F''_2 - F'''_2 + k_1[2F_2 F''_2 + F''_2 + F_2 F'''_2] = 0.
\]

(6.19)

and the following boundary conditions

\[
F'_2(\eta) = 1 \quad F_2(\eta) = 0 \quad \text{at} \quad \eta = 0
\]

\[
F'_2(\infty) = 0 \quad \text{as} \quad \eta = \infty
\]

(6.20)

where \( A = (\xi)^{\frac{1}{2}} \), \( B = c \), \( AC = c \) and \( k_1 = k(\xi) \). Thus the transformations (6.16) are useful in getting similarity solution.

Examination the other transformations in (6.6)-(6.9) reveals that they are disproportionately with the auxiliary conditions. So they do not represent similarity transformations. For example, under the transformation,

\[
\eta = Ae^{-\frac{\alpha_1}{2}y}
\]

\[
u = \begin{cases} 
\frac{1}{B^{\frac{\alpha_1}{2}}} F_1(\eta) - \frac{\theta_0}{\alpha_3}, & \text{if } \alpha_3 \neq 0 \\
F_1(\eta) & \text{if } \alpha_3 = \beta_3 = 0
\end{cases}
\]

\[
u = \begin{cases} 
Ce^{\frac{\alpha_1}{2}y} F_2(\eta) - \frac{\theta_0}{\alpha_4}, & \text{if } \alpha_4 \neq 0 \\
F_2 & \text{if } \alpha_4 = \beta_4 = 0
\end{cases}
\]

(6.21)
where $A$, $B$ and $C$ are arbitrary constants, the first part of boundary conditions, i.e. $u = cx$ at $y = 0$ takes the form:

$$\
\left( \frac{1}{B} e^{\frac{y}{2} F_1(\eta)} - \frac{\beta_3}{\alpha_3} \right) = cx \quad \text{at } \eta = 0
\right.
$$

which is impossible to express it in terms of a single independent variable $\eta$ under any assumptions.

### 6.3 Two-Dimensional Thermal Boundary Layers for Second Order Fluids

#### 6.3.1 Governing Equation

The governing equations for the laminar boundary layer equations with heat convection on a vertical plate can be written as: (see Al-Saheli [74])

$$u_x + v_y = 0 \quad \text{(6.22)}$$

$$\rho(u_t + uu_x + vv_y) = -p_x + g\beta(T - T_\infty) + \mu u_{yy} + (3\mu_1 + 2\mu_2)u_yu_{xy} + \mu_1[u_{yy} + uu_{xy} + vu_{yy} + u_xu_{yy}]$$

$$\quad + \frac{1}{\rho}(3\mu_1 + 2\mu_2)u_yu_{xy}$$

$$\quad + \frac{\mu_1}{\rho}[u_{yy} + uu_{xy} + vu_{yy} + u_xu_{yy}]$$

$$\quad + \frac{\mu_1}{\rho}[u_{yy} + uu_{xy} + vu_{yy} + u_xu_{yy}]$$

$$\quad + \rho \beta(T_t + uT_x + vT_y) - \varepsilon T_{yy} \quad \text{(6.24)}$$

with the boundary conditions

$$u = v = 0 \quad T = T_\infty \quad \text{at } y = 0;$$

$$u \rightarrow U, \quad T \rightarrow T_\infty \quad \text{as } y \rightarrow \infty. \quad \text{(6.25)}$$

or

$$u_x + v_y = 0 \quad \text{(6.26)}$$

$$u_t + uu_x + vv_y = U_t + UU_x + \frac{g\beta}{\rho}(T - T_\infty) + \frac{\mu}{\rho}u_{yy} + \frac{1}{\rho}(3\mu_1 + 2\mu_2)u_yu_{xy}$$

$$+ \frac{\mu_1}{\rho}[u_{yy} + uu_{xy} + vu_{yy} + u_xu_{yy}]$$

$$+ \frac{\mu_1}{\rho}[u_{yy} + uu_{xy} + vu_{yy} + u_xu_{yy}]$$

$$+ \rho \beta(T_t + uT_x + vT_y) - \varepsilon T_{yy} \quad \text{(6.27)}$$
\[ \rho c_p(T_t + uT_x + vT_y) = kT_{yy} \]  

(6.28)

with the boundary conditions

\[ u = v = 0 \quad T = T_w \quad \text{at} \quad y = 0; \]

(6.29)

\[ u \to U, \quad T \to T_\infty \quad \text{as} \quad y \to \infty . \]

Dimensionalize the variables according to:

\[ \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L Re^\frac{1}{2}}, \quad \bar{u} = \frac{u}{U_0}, \quad \bar{v} = \frac{v}{U_0 Re^\frac{1}{2}}, \quad \bar{t} = \frac{t}{U_0 L}, \quad \bar{U} = \frac{U}{U_0} \]

(6.30)

\[ \bar{\mu}_2 = \frac{Re}{L^2 Pr \mu_2} \]

In dimensionalized form

The basic equations are:

\[ \bar{u}_x + \bar{v}_y = 0 \]  

(6.31)

\[ \bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y = \bar{U}_t + \bar{U}\bar{U}_x + (\bar{T} - \bar{T}_\infty) + \bar{u}\bar{u}_{yy} + (3\bar{\mu}_1 + 2\bar{\mu}_2)\bar{u}_y\bar{u}_{xy} \]

(6.32)

\[ + \bar{\mu}_1 [\bar{u}\bar{u}_{yy} + \bar{u}\bar{u}_{xy} + \bar{v}\bar{u}_{yy} + \bar{u}_x\bar{u}_{xy}] \]

\[ \bar{T}_t + \bar{u}\bar{T}_\bar{x} + \bar{v}\bar{T}_\bar{y} = \frac{1}{Pr} \bar{T}_{\bar{y}y} \]  

(6.33)

with the boundary conditions

\[ \bar{u} = \bar{v} = 0 \quad \bar{T} = \bar{T}_w \quad \text{at} \quad \bar{y} = 0; \]

\[ \bar{u} \to \bar{U}, \quad \bar{T} \to \bar{T}_\infty \quad \text{as} \quad \bar{y} \to \infty . \]  

(6.34)

(6.31) suggest existence of stream function \( \psi(t, x) \), such that; \( u = \psi_y, \quad v = -\psi_x \), and the non-dimensional temperature defined by \( \theta = \frac{\bar{T}_\infty - \bar{T}_w}{\bar{T}_w - \bar{T}_1} \). Therefore, (6.31-6.31) become

\[ \psi_{ty} + \psi_t\psi_{xy} - \psi_x\psi_{yy} = U_t + UU_x + \theta T_1 + \psi_{yyy} + (3\mu_1 + 2\mu_2)\psi_y\psi_{xy} \]

\[ \mu_1 [\psi_{yy} + \psi_t\psi_{xyy} - \psi_x\psi_{yyy} + \psi_{xy} \psi_{yy}] \]

(6.35)

\[ \theta_t T_1 + \theta \frac{\partial T_1}{\partial t} + \psi_t (6_x T_1 + \theta \frac{\partial T_1}{\partial x}) - \psi_x \theta_t T_1 = \frac{1}{Pr} T_1 \theta_{yy} \]  

(6.36)
with the boundary conditions

\[
\begin{align*}
\psi_y = \psi_x &= 0 \text{ and } \theta = 1 \text{ at } y = 0; \\
\psi_y - U(t, x) &= 0, \; \theta = 0 \text{ as } y \to \infty.
\end{align*}
\]

(6.37)

In the above equations, the overbars have been dropped for convenience. We will solve the reduced equations (6.35)-(6.37) using our method.

### 6.3.2 The Application of the Proposed Method

#### 6.3.2.1 Invoking the Similarity Transformations

The system of nonlinear partial differential equations (6.35)-(6.37) contains three independent variables \( t, x \) and \( y \), so to reduce this system to system of ordinary differential equations we need to apply two-parameter group. Therefore, due to the group (5.17), when \( r = 2 \) (two-parameter group) and in terms of the notations of a given problem the similarity transformations (5.29) can be written as

\[
\begin{align*}
\eta &= \frac{y}{\pi(x,t)} \\
\psi &= \Gamma_1(x,t)F_1(\eta) \\
\theta &= \Gamma_2(x,t)F_2(\eta) \\
U &= U_0\Gamma_3(x,t) \\
T_1 &= T_0\Gamma_4(x,t)
\end{align*}
\]

(6.38)

where \( U_0 \) and \( T_0 \) are arbitrary constants.

#### 6.3.2.2 Invariance Analysis and the Reduction to ODEs

In order to achieve the invariance and the reduction of nonlinear system (6.35)-(6.37) we employ the proposed method (Refer Chapter 4). Accordingly, The invariance in the form of (6.35)-(6.37) take place if they are expressible in terms of the new variables appear in the transformations (6.38).
Due the above transformations (6.38). Equations (6.35)-(6.37) readily transform to

\[-k_2 - k_3 - C_1 F_2 + C_3 (C_2 + C_7) F_1' + (C_3 C_4 + C_5) (F_1')^2 + (C_2 C_3 \eta - C_2 F_1) F_1'' + (2C_4 + C_6)(3\mu_1 + 2\mu_2) (F_1'')^2 + [2C_4 \eta(2\mu_1 + \mu_2) F_1'' - \mu_1(3C_2 + C_7 + 2C_4 F_1') - 1] F_1'' + \mu_1(C_6 F_1 - C_2 \eta)(F_1'') = 0\]

\[(C_{10} C_3 + C_{8} k) F_2 + (C_{11} + C_9 k) F_2 F_1' + C_2 C_3 k \eta F_2' - C_5 k F_1 F_2' - \frac{1}{Pr} k F_2'' = 0\]

with the boundary conditions

\[F_1' = F_1 = 0 \text{ and } F_2 = \frac{1}{k} \text{ at } \eta = 0;\]

\[F_1' = k_1 \text{ } F_2 = 0 \text{ as } \eta \rightarrow \infty .\]

where the primes refer to differentiation with respect to \(\eta\), the expressions for \(C_1, ..., C_{11}\) and \(k, k_1, k_2, k_3\) were obtained using Mathematica- a computer algebra system; to get,

\[C_1 = \frac{T_0 \Gamma_2 \Gamma_4}{\pi^3 \Gamma_1},\]  
\[C_2 = \frac{1}{\pi} \frac{\partial \pi}{\partial t},\]  
\[C_3 = \frac{1}{\pi^2},\]  
\[C_4 = \Gamma_1 \frac{\partial \Gamma_1}{\partial x},\]  
\[C_5 = \frac{1}{\pi} \frac{\partial \Gamma_1}{\partial x},\]  
\[C_6 = \pi \frac{\partial \Gamma_1}{\partial x},\]  
\[C_7 = \frac{1}{\Gamma_1} \frac{\partial \Gamma_1}{\partial t},\]  
\[C_8 = \frac{1}{\pi^2 \Gamma_4} \frac{\partial \Gamma_4}{\partial t},\]  
\[C_9 = \frac{\Gamma_1}{\pi \Gamma_4} \frac{\partial \Gamma_4}{\partial x},\]  
\[C_{10} = \frac{\partial \Gamma_2}{\partial t},\]  
\[C_{11} = \frac{\Gamma_1}{\pi} \frac{\partial \Gamma_2}{\partial x}.\]
\[ k = \Gamma_2, \quad (6.43a) \]
\[ k_1 = \frac{U_0 \Gamma_3}{\pi \Gamma_1}, \quad (6.43b) \]
\[ k_2 = \frac{U_0}{\pi^3 \Gamma_1} \frac{\partial \Gamma_3}{\partial t} = C_3 k_1 \frac{1}{\Gamma_3} \frac{\partial \Gamma_3}{\partial t} \quad \text{and} \quad (6.43c) \]
\[ k_3 = \frac{U_0^2 \Gamma_3}{\pi^2 \Gamma_1} \frac{\partial \Gamma_3}{\partial x} = C_3 k_1 \frac{\partial \Gamma_3}{\partial x}. \quad (6.43d) \]

More importantly, equations (6.39)-(6.41) can be expressed in terms of the single variable \( \eta \) as independent variable and \( F_1 \) & \( F_2 \) as dependent variables, if \( k, k_1, k_2, k_3 \) and \( C_i (i = 1, ..., 11) \) are constants and hold (6.42) and (6.43). If so,

- \( C_3 = \frac{1}{n^2} \) in (6.42c), state that \( \pi \) is constants, namely \( \pi = \sqrt{\frac{1}{C_3}} \). this leads to \( C_2 = C_4 = 0, C_3 \neq 0 \)
- also \( k = \Gamma_2 \Rightarrow C_{10} = C_{11} = 0 \), since \( k \) is constant.

And by considering the other conditions in (6.42) and (6.43), we get from (6.42e) that

\[ \Gamma_1 = \frac{C_5}{\sqrt{C_3}} x + f(t) \quad (6.44) \]

Using (6.44) in (6.42g) we have

\[ C_7 \frac{C_8}{\sqrt{C_3}} x = f'(t) - C_7 f(t) \quad (6.45) \]

Since the left-hand side of this equation is a function of \( x \) only and the right-hand is a function of \( t \) only, it follows that, (6.45) can be true without loss of generality if both sides are equal to zero. Consequently, (6.45) gives two cases, namely,

- Case 1: \( C_7 = 0 \Rightarrow f(t) = A_0 = \text{constant} \) (Steady Case)
- Case 2: \( C_5 = 0 \Rightarrow f(t) = e^{C_7 t + B} \) (Unsteady Case)
6.3. Two-Dimensional Thermal Boundary Layers for Second Order Fluids

► Steady Case:

Substitute of \( f(t) = A_0 \) into (6.42), (6.43) and (6.44) we get,

\[
\Gamma_1 = Ax + A_0 \\
\Gamma_4 = B_1(Ax + A_0), \\
\Gamma_3 = B_2(Ax + A_0),
\]

\[
C_6C_3 = C_5, \quad C_8 = 0, \quad C_9 = A \sqrt{C_3}, \quad k_2 = 0, \quad k_3 = k_1C_3B_2A
\]

where \( A = \frac{C_5}{\sqrt{C_3}}, \quad B_1 = \left(\frac{1}{C_3}\right)^{\frac{3}{2}} \frac{C_5}{\eta_0} \) and \( B_2 = \frac{U_1}{U_0 \sqrt{C_3}} \). Thus, the similarity representations become,

\[
-k_3 - C_1F_2 + C_5((F')^3 - F_1F'') + C_6(3\mu_1 + 2\mu_2)(F''')^2 - F''' + \mu_1C_6F_1F'''' = 0 \tag{6.47}
\]

\[
C_9F_2F_1 - C_3F_1F_2' - \frac{1}{Pr} F_2'' = 0 \tag{6.48}
\]

with the boundary conditions

\[
F'_1 = F_1 = 0 \quad \text{and} \quad F_2 = \frac{1}{k} \quad \text{at} \quad \eta = 0; \\
F'_1 = k_1 \quad F_2 = 0 \quad \text{as} \quad \eta \to \infty . \tag{6.49}
\]

where \( k \neq 0, k_1, k_3 \) and \( k \) are arbitrary constants.

► Unsteady Case: \( i.e., \ C_5 = 0 \)

\[
\Gamma_1 = e^{C_1t+B} \\
\Gamma_4 = B_1e^{C_1t+B}, \\
\Gamma_3 = B_2e^{C_1t+B},
\]

\[
C_6 = 0, C_9 = k_3 = 0, C_8 = kC_3C_7, \quad k_2 = k_1C_3C_7
\]

where \( B_1 \) and \( B_2 \) as above. Thus, In light of these results, the similarity representations for this case become,

\[
-k_2 - C_1F_2 + C_3C_7F'_1 - (\mu_1C_7 + 1)F''_1 = 0 \tag{6.51}
\]

\[
C_9F_2 - \frac{1}{Pr} F''_2 = 0 \tag{6.52}
\]

with the boundary conditions (6.49).
6.4 Numerical Solution

The systems (6.47)-(6.48) and (6.51)-(6.52) involve two non-linear ODEs posed on an infinite interval and the first one has singularity at the origin because $y(0) = 0$. So the analysis is somewhat lengthy because unknown parameters can arise when solving BVPs with singularities Shampine et al. [132]. In this thesis we will confine ourself to solving BVPs without singularities.

The transformed equations (6.51) and (6.52) subject to the boundary conditions (6.49) constitute a set of highly nonlinear differential equations for which obtaining closed form solution is difficult. Hence the bvp4c coding in MatLab which uses a collocation method to solve system of ODEs in two-point or multi-point boundary value form, is used to solve this system for some values of the viscoelastic parameter $\mu_1$ and Prandtl number $Pr$.

In order to implement the bvp4c coding, the system of equations (6.51)-(6.52) are converted to a system of first order differential equations as follows: Substitute $(F_u, F_v, F_i, F_x, F_2, F_3) = (y_1, y_2, y_3, y_4, y_5)$. Then equations (6.51) and (6.52) with the boundary conditions (6.49) reduce to:

\[
\begin{align*}
  y_1' &= y_2 \\
  y_2' &= y_3 \\
  y_3' &= \frac{-1}{\mu_1 C_7 + 1} (k_2 + C_1 y_4 - C_3 C_7 y_2) \\
  y_4' &= y_5 \\
  y_5' &= Pr C_8 y_4
\end{align*}
\]

\[\tag{6.53}\]

\[y_1(0) = 0, \quad y_2(0) = 0, \quad y_2(\infty) = k_1, \quad y_4(0) = \frac{1}{k}, \quad y_4(\infty) = 0.\]

We will deal with the boundary condition at infinity by imposing it at a finite point, namely, $\eta_{\infty} = 10$. i.e. the domain of the problem is restricted to $[0, 10]$ and the value of constants $C_1 = C_3 = C_7 = C_8 = k_2 = 1$, with the hypothetical step size $\Delta \eta = 0.2041$.

First, for various values of the viscoelastic parameter $\mu_1(0, 1, 5, 15)$ with fixed Prandtl numbers $Pr = 0.75$, the problem is solved and the output plotted in a straight-
6.4. Numerical Solution

forward way by building the MatLab script M-file ch6pr2.m.

function soll = ch6pr2
    solinit = bvpinit(linspace(0,10,50),[-1 1 1.5 2 2.1]);
    sol = bvp4c(@(SysWithMu0,@bc4,solinit);
    so2 = bvp4c(@(SysWithMu1,@bc4,solinit);
    so3 = bvp4c(@(SysWithMu5,@bc4,solinit);
    so4 = bvp4c(@(SysWithMu5,@bc4,solinit);
    soll = [sol
             so2
             so3
             so4];

function ad2 = SysWithMu0(t,y)
    Pr= 0.75; C1=1; C3=1; C7=1; C8=1; k2=1; mul=0;
    ad2 = [ y(2)
            y(3)
            (1/(mul*C7+1))(-C1*y(4)-k2+C3*C7*y(2))
            y(5)
            Pr*(C8*y(4))];

function ad2 = SysWithMu1(t,y)
    Pr= 0.75; C1=1; C3=1; C7=1; C8=1; k2=1; mul=1;
    ad2 = [ y(2)
            y(3)
            (1/(mul*C7+1))(-C1*y(4)-k2+C3*C7*y(2))
            y(5)
            Pr*(C8*y(4))];
function ad2 = SysWithMu5(t,y)
Pr= 0.75; Cl=1; C3=1; C7=1; C8=1; k2=1; mul=5;
ad2 = [ y(2)
         y(3)
         (1/(mul*C7+1))(-Cl*y(4)-k2+C3*C7*y(2))
         y(5)
         Pr*(C8*y(4))] ;

function ad2 = SysWithMu15(t,y)
Pr= 0.75; Cl=1; C3=1; C7=1; C8=1; k2=1; mul=15;
ad2 = [ y(2)
         y(3)
         (1/(mul*C7+1))(-Cl*y(4)-k2+C3*C7*y(2))
         y(5)
         Pr*(C8*y(4))] ;

function bc2 = bc4(ya,yb)
kl=1; k=1;
bc2 = [ ya(1)
         ya(2)
         yb(2)-kl
         ya(4)-k
         yb(4) ];

Where the following code produces Figure (6.1),
plot(sol.x,sol.y(1,:),'k-*',so2.x,so2.y(1,:),'k-','
so3.x, so3.y(1,:), 'k-.', so4.x, so4.y(1,:), 'k:');

hleg1 = legend('\mu_l=0', '\mu_l=1', '\mu_l=5', '\mu_l=15');
set(hleg1, 'Location', 'Northwest');
xlabel('\eta');
ylabel('\mathbf{F}_1(\eta)');

The field \texttt{sol.x} contains the mesh found by the solver and the field \texttt{sol.y} (\texttt{sol, so2, so3, so4}) contains the solution on this mesh for \(\mu_l = 0, 1, 5, 15\), respectively.

Note that \texttt{(sol.y(1,:), sol.y(2,:), sol.y(3,:), sol.y(4,:), sol.y(5,:))}, in the solver refers to the solution on the mesh of \((y_1, y_2, y_3, y_4, y_5)\) or equivalently \((F_1, F'_1, F_2, F'_2)\), respectively, at viscoelastic parameter \(\mu_l = 0\). By replacing some of the values of constants and parameters in the m-file \texttt{ch6pr2.m} with rearranging the functions one can build a solution at various values of the Prandtl numbers \(Pr=(0, 0.05, 0.2, 1, 1000)\) and fixed viscoelastic parameter \(\mu_l = 15\). Various numerical solutions are to be obtained from which the various profiles are generated.

6.5 Results and Discussion

Results are obtained for various values of the viscoelastic parameter \(\mu_l = (0, 1, 5, 15)\) at fixed Prandtl numbers \(Pr = 0.75\) and for various values of the Prandtl numbers \(Pr=(0, 0.05, 0.2, 1, 1000)\) at fixed viscoelastic parameter \(\mu_l = 15\). Numerical computations of the results are depicted in Fig 6.1-6.8 and numerical data provided in Tables 6.1 to 6.9.

- Figures 6.1-6.4 show the effects of the viscoelasticity or material parameter \(\mu_l\) on the fluid flow and heat transfer characteristics, namely \(F_1(\eta), F_2(\eta), F'_1(\eta)\) and \(F'_2(\eta)\), when \(Pr = 0.75\). It is noticed that in unsteady flow of velocity distributions tended to a almost linear profiles and its magnitudes are decreased when the values of viscoelastic parameter \(\mu_l\) are increased while the temperature
distributions never affected. The same trend is also observed for \( F'_1(\eta) \) and \( F'_2(\eta) \) profiles as depicted by Figures 6.2 and 6.4.

- Figures 6.5-6.8 displayed the velocity and temperature profiles namely \( F_1(\eta), F_2(\eta), F'_1(\eta), F'_2(\eta) \), for different values of the Prandtl number, \( Pr = 0.6, 0.75, 1, 10 \), while the viscoelastic parameter \( \mu_1 = 15 \). on the velocity and temperature profiles, respectively, when Prandtl number \( Pr = 0, 0.05, 0.2, 1, 1000 \), with viscoelastic parameter \( \mu_1 = 15 \). It is found that as Prandtl number \( Pr \) increases, the Dimensionless velocity and temperature profiles decrease and also the thermal boundary layer thickness. This is because the fluid is highly conductive for small values of the Prandtl number. Physically, if Prandtl number \( Pr \) increases, the thermal diffusivity decreases and this phenomenon lead to the decreasing manner of the energy transfer ability that reduces the thermal boundary layer. It is seen from these figures that the material or viscoelastic parameter \( \mu_1 \) and the Prandtl number \( Pr \) presents opposite effects on the fluid flow and heat transfer characteristics.

![Figure 6.1: Dimensionless velocity profile \( F_1(\eta) \) at Prandtl number \( Pr=0.75 \) with various values of viscoelastic parameter \( \mu_1 \)](image-url)
6.5. Results and Discussion

Figure 6.2: Dimensionless velocity profile $F'_r(\eta)$ at Prandtl number $Pr=0.75$ with various values of viscoelastic parameter $\mu_1$

Figure 6.3: Dimensionless temperature profile $F_2(\eta)$ at Prandtl number $Pr=0.75$ with various values of viscoelastic parameter $\mu_1$
6.5. Results and Discussion

Figure 6.4: Dimensionless temperature profile $F'_{2}(\eta)$ at Prandtl number $Pr=0.75$ with various values of viscoelastic parameter $\mu_1$

Figure 6.5: Dimensionless velocity profile $F_1(\eta)$ at viscoelastic parameter $\mu_1 = 15$ with various Prandtl number $Pr$
6.5. Results and Discussion

Figure 6.6: Dimensionless velocity profile $F_1'(\eta)$ at viscoelastic parameter $\mu_1 = 15$ with various Prandtl number $Pr$

Figure 6.7: Dimensionless temperature profile $F_2(\eta)$ at viscoelastic parameter $\mu_1 = 15$ with various Prandtl number $Pr$
Figure 6.8: Temperature profile $F_2'(\eta)$ at viscoelastic parameter $\mu_1 = 15$ with various Prandtl number $Pr$
6.5. Results and Discussion

Table 6.1: Numerical solution when \( \mu_1 = 0, \ Pr = 0.75 \)

<table>
<thead>
<tr>
<th>( \mu_1 )</th>
<th>( F_1(\theta) )</th>
<th>( F_1'(\theta) )</th>
<th>( F_1''(\theta) )</th>
<th>( F_2(\theta) )</th>
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\( \mu_1 = 0, \ Pr = 0.75 \)
### Table 6.2: Numerical solution when $\mu_1 = 1$, $Pr = 0.75$

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<th>$F''_1(t)$</th>
<th>$F_2(t)$</th>
<th>$F'_2(t)$</th>
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### Table 6.3: Numerical solution when $m = 5$, $Pr = 0.75$

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### Table 6.5: Numerical solution when $Pr = 0, \mu_1 = 15$

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### 6.5. Results and Discussion

Table 6.6: Numerical solution when $P_r = 0.05, \mu_1 = 15$

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Table 6.8: Numerical solution when $Pr = 1$, $\mu = 15$

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Table 6.9: Numerical solution when $Pr = 1000, \mu_1 = 15$

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6.6 Conclusions

In this chapter, we have applied the proposed method for steady two-dimensional boundary layer equations for this fluid past a stretching sheet and the two-dimensional thermal boundary layer equations on a vertical plate for second order fluids. The governing systems of non-linear partial differential equations is transformed into the system of non-linear ordinary differential equations. The carried out analysis, in this problem, shows the effectiveness of the method in obtaining similarity representation and achieving an invariance of system of non-linear partial differential equations easily with less steps. Some obtained differential equations are solved numerically via the bvp4c solver of Matlab and the results were presented and discussed. The velocity and temperature profiles are drawn for different values of the physical parameters involved. In the light of present investigations following conclusions may be drawn.

- The dimensionless velocity profiles decrease with increase the values of viscoelastic parameter $\mu_1$, while the temperature profiles never affected (case unsteady flow).

- For smaller $Pr$ number the dimensionless velocity profiles are higher and the temperature are higher due to higher thermal conductivity of the medium. Increase in $Pr$ number, decreases the thermal boundary layer.

Here we dealt with the two-dimensional boundary-layer flow of second order fluids. The situation gets more complicated if one considers three-dimensional flow instead of two-dimensional. In the coming chapter we deal with three-dimensional flow of a viscous fluid over a stretching sheet.

The work in this chapter has appeared in a number of papers by the author [2, 5].