# CHAPTER-5

**USE OF MULTIAUXILIARY INFORMATION IN ESTIMATING MEAN, VARIANCE AND COEFFICIENT OF VARIATION IN FINITE POPULATION-II.**

<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 Introduction and notations</td>
<td>89</td>
</tr>
<tr>
<td>5.2 Classes of estimators and their properties</td>
<td>92</td>
</tr>
<tr>
<td>5.3 Theoretical comparison</td>
<td>98</td>
</tr>
<tr>
<td>5.4 Use of simple weights</td>
<td>100</td>
</tr>
<tr>
<td>5.5 Multivariate extensions</td>
<td>102</td>
</tr>
<tr>
<td>5.6 Estimation of variance and coefficient of variation</td>
<td>107</td>
</tr>
<tr>
<td>Appendix B</td>
<td>112</td>
</tr>
</tbody>
</table>
5.1 INTRODUCTION AND NOTATIONS

Let $U = (1, 2, \ldots, N)$ be a finite population of $N$ units. Let $y$ and $x$ be real-valued functions defined on $U$ taking the values $y_i$ and $x_i$ for the $i$th unit of $U (1 \leq i \leq N)$. Let

$$Y_N = \frac{1}{N} \sum_{i=1}^{N} y_i, \quad \sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2 \quad \text{and} \quad C_0 = \frac{\sigma^2}{\bar{y}}$$

denote the mean, variance and coefficient of variation respectively of a study character $y$ for a finite population.
The problems of estimating mean, variance and coefficient of variation of a study character for a finite population utilizing auxiliary information have been studied by various authors. Das and Tripathi (1978a,b; 1980) have proposed the classes of estimators

\[
t_0 = \bar{y} \left( \frac{c_1^2}{s_1^2/\bar{x}_1^2} \right)^{\alpha} \quad (5.1.1)
\]

\[
t^*_0 = s_0^2 \left( \frac{c_1^2}{s_1^2/\bar{x}_1^2} \right)^{\beta} \quad (5.1.2)
\]

and

\[
t^{**}_0 = c_0 \left( \frac{c_1^2}{s_1^2/\bar{x}_1^2} \right)^{\gamma} \quad (5.1.3)
\]

respectively for estimating mean, variance and coefficient of variation of a study character \( y \) for a finite population using the knowledge of the coefficient of variation of an auxiliary character \( x \) in case of simple random sampling with replacement. Singh (1967, 1969) and Shah and Shah (1978) have proposed various ratio-cum-product estimators for estimating ratio (product) of two population
parameters, utilizing the auxiliary information in terms of ratios of their means.

In the present chapter, the problem of estimating mean, variance and coefficient of variation of a study character for a finite population using the knowledge of the coefficient of variations of auxiliary characters is considered. The classes of ratio-type, product-type and ratio-cum-product estimators for estimating mean $\bar{Y}$, variance $\sigma^2_0$ and coefficient of variation $C_o$ utilizing the knowledge of the coefficient of variations of two auxiliary characters are suggested. The biases, mean square errors (MSE's) and the minimum MSE's of the suggested estimators, upto the second order moments, are obtained. The efficiency of the suggested estimators as compared to the corresponding conventional estimators is investigated. All the results are extended to the multivariate case i.e. for $p(>2)$ auxiliary characters.

We consider the method of simple random sampling with replacement (SRSWR).

As in Chapter 4, following Isaki (1983), we
introduce the following notations.

\[ b_i = \text{Coefficient of variation of } c_i^2. \]
\[ = \text{C.V. (} c_i^2\text{); } i = 1, 2. \]

\[ f_{ij} = \text{Coefficient of correlation between } c_i^2 \text{ and } c_j^2. \]
\[ = \text{Corr. (} c_i^2, c_j^2\text{); } i \neq j = 1, 2. \]

\[ b_{ij} = \text{Coefficient of variation of } c_i^2 \text{ and } c_j^2. \]
\[ = f_{ij} b_i b_j; \ i \neq j = 1, 2. \]

Further, we note that the suffix 0 stands for the study character \( y \) and suffixes 1 and 2 stand for the auxiliary characters \( x_1 \) and \( x_2 \) respectively.

### 5.2 CLASSES OF ESTIMATORS AND THEIR PROPERTIES

The following classes of estimators for estimating mean \( \bar{y} \), variance \( \sigma_y^2 \) and coefficient of variation \( C_y \) are proposed analogously as in Chapter 4.

\[ t_1 = \bar{y} \left[ w_1 \left( \frac{c_i^2}{s_i^2 / x_i^2} \right) + w_2 \left( \frac{c_j^2}{s_j^2 / x_j^2} \right) \right]^a; \ w_1 + w_2 = 1 \quad (5.2.1) \]
\[ t_2 = \tilde{y} \left( \frac{c_1^2}{s_1^2/x_1^2} \right)^2 \left( \frac{c_2^2}{s_2^2/x_2^2} \right)^2 \]  
(5.2.2)

\[ t_3 = \tilde{y} \left[ \theta_1 \left( \frac{c_1^2}{s_1^2/x_1^2} \right) \varphi_1 + \theta_2 \left( \frac{c_2^2}{s_2^2/x_2^2} \right) \varphi_2 \right] ; \]

\[ \theta_1 + \theta_2 = 1 \]  
(5.2.3)

\[ t_1^* = s_o^2 \left[ W_1^* \left( \frac{c_1^2}{s_1^2/x_1^2} \right) + W_2^* \left( \frac{c_2^2}{s_2^2/x_2^2} \right) \right] \alpha^* ; \]

\[ W_1^* + W_2^* = 1 \]  
(5.2.4)

\[ t_2^* = s_o^2 \left( \frac{c_1^2}{s_1^2/x_1^2} \right) \beta_1^* \left( \frac{c_2^2}{s_2^2/x_2^2} \right) \beta_2^* \]  
(5.2.5)

\[ t_3^* = s_o^2 \left[ \theta_1^* \left( \frac{c_1^2}{s_1^2/x_1^2} \right) \varphi_1^* + \theta_2^* \left( \frac{c_2^2}{s_2^2/x_2^2} \right) \varphi_2^* \right] ; \]

\[ \theta_1^* + \theta_2^* = 1 \]  
(5.2.6)

\[ t_1^{**} = \alpha_0 \left[ W_1^{**} \left( \frac{c_1^2}{s_1^2/x_1^2} \right) + W_2^{**} \left( \frac{c_2^2}{s_2^2/x_2^2} \right) \right] \alpha^{**} ; \]

\[ W_1^{**} + W_2^{**} = 1 \]  
(5.2.7)
\[ t_{2}^{**} = c_{o} \left( \frac{c_{1}^{2}}{s_{1}/\bar{x}_{1}} \right) \left( \frac{c_{2}^{2}}{s_{2}/\bar{x}_{2}} \right) \tag{5.2.8} \]

\[ t_{3}^{**} = c_{o} \left[ \theta_{1}^{**} \left( \frac{c_{1}^{2}}{s_{1}/\bar{x}_{1}} \right) + \theta_{2}^{**} \left( \frac{c_{2}^{2}}{s_{2}/\bar{x}_{2}} \right) \right] ; \]

\[ \theta_{1}^{**} + \theta_{2}^{**} = 1 \tag{5.2.9} \]

where the weights \( W \)'s and \( \Theta \)'s and the constants \( \alpha, \beta \)'s and \( Y \)'s are to be determined by minimizing MSE's of the estimators.

We give below the results for the estimators \( t_{1} \), \( t_{2} \) and \( t_{3} \) of the population mean \( \bar{Y} \) and in section 5.6 indicate as to how the results for the estimators \( t_{1}^{*} \) and \( t_{1}^{**} \) \((i = 1, 2, 3)\) for \( \sigma_{o}^{-2} \) and \( C_{o} \) respectively follow by some suitable adjustment.

Letting \( \bar{y} = \bar{Y}(1+e_{o}), \bar{x}_{i} = \bar{x}_{i}(1+e_{i}), i = 1, 2 \) and \( s_{i}^{2} = \sigma_{i}^{-2}(1+e_{i})' \) \( i = 0, 1, 2 \) in (5.2.1) to (5.2.3) and expanding them by Taylor's series and taking expectations term by term, the biases and MSE's of the suggested
etimators $t_1$, $t_2$ and $t_3$, to the order of $n^{-1}$, are obtained as follows.

$$
B(t_1) = \frac{\bar{Y}}{2} \left[ (\alpha - 1)(w_1^2b_1^2 + w_2^2b_2^2 + 2w_1w_2b_{12}) - 2(w_1k_1 + w_2k_2) + 2(w_1m_1 + w_2m_2) \right] \quad (5.2.10)
$$

$$
B(t_2) = \frac{\bar{Y}}{2} \left[ \beta_1^2b_1^2 + \beta_2^2b_2^2 + 2\beta_1\beta_2b_{12} - 2(\beta_1k_1 + \beta_2k_2) + 2(\beta_1m_1 + \beta_2m_2) - (\beta_1b_1^2 + \beta_2b_2^2) \right] \quad (5.2.11)
$$

$$
B(t_3) = \frac{\bar{Y}}{2} \left[ \gamma_1(\frac{y_1}{\theta_1} + 1)b_1^2 + \gamma_2(\frac{y_2}{\theta_2} + 1)b_2^2 - 2(\gamma_1k_1 + \gamma_2k_2) + 2(\gamma_1m_1 + \gamma_2m_2) \right] \quad (5.2.12)
$$

$$
M(t_1) = V(\bar{Y}) + \bar{Y}^2 \left[ \alpha^2(w_1^2b_1^2 + w_2^2b_2^2 + 2w_1w_2b_{12}) - 2\alpha(w_1k_1 + w_2k_2) \right] \quad , (5.2.13)
$$

$$
M(t_2) = V(\bar{Y}) + \bar{Y}^2 \left[ \beta_1^2b_1^2 + \beta_2^2b_2^2 + 2\beta_1\beta_2b_{12} - 2(\beta_1k_1 + \beta_2k_2) \right] \quad (5.2.14)
$$
\[ M(t_3) = V(\bar{y}) + \bar{y}^2 \left[ y_{1b_1}^2 + y_{2b_2}^2 + 2y_1y_2b_{12} \right. \]

\[ \left. - 2(y_{1k_1} + y_{2k_2}) \right] \]

\( (5.2.15) \)

where

\[ V(\bar{y}) = \frac{\sigma^2}{n} \]

\( (5.2.16) \)

\[ k_i = \frac{\text{Cov}(\bar{y}, s_i^2)}{\bar{y} \sigma_i^2} \]

\( (5.2.17) \)

\[ = \frac{\text{Cov}(\bar{y}, s_i^2)}{\bar{y} \sigma_i^2} - 2 \frac{\text{Cov}(\bar{y}, \bar{x}_i)}{\bar{y} \bar{x}_i} ; \quad i = 1, 2. \]

\[ m_i = \frac{V(\bar{x}_i)}{\bar{x}_i^2} - 2 \frac{\text{Cov}(\bar{x}_i, s_i^2)}{\bar{x}_i \sigma_i^2} + \frac{V(s_i^2)}{\sigma_i^4} ; \]

\[ i = 1, 2. \]

\( (5.2.18) \)

On minimization of MSE's of the estimators

\( t_i \) (\( i = 1, 2, 3 \)) given in (5.2.13) to (5.2.15), we get the optimum values of \( \alpha, \beta', \gamma', \) and \( \lambda' \) as
\[ \hat{\alpha} = \frac{A_1 + A_2}{B} \quad \text{and} \quad \hat{\beta}_i = \frac{A_i}{A_1 + A_2}; \quad i = 1, 2. \] (5.2.19)

\[ \hat{\beta} = \hat{\gamma}_i = \frac{A_i}{B}; \quad i = 1, 2. \] (5.2.20)

where

\[ A_1 = k_1 b_2^2 - k_2 b_{12}. \]

\[ A_2 = k_2 b_1^2 - k_1 b_{12}. \]

and

\[ B = b_1^2 b_2^2 - b_{12}^2. \]

Substitution of the optimum values of parameters from (5.2.19) and (5.2.20) in (5.2.10) to (5.2.15) yield the resulting biases and MSE's of the estimators \( t_1, t_2 \) and \( t_3 \) as

\[ B_0(t_1) = \frac{\bar{y}}{2B} \left[ 2(A_1 m_1 + A_2 m_2) - \left( \frac{1+\hat{\gamma}}{\hat{\alpha}} \right) A \right]. \] (5.2.21)

\[ B_0(t_2) = \frac{\bar{y}}{2B} \left[ 2(A_1 m_1 + A_2 m_2) - (A_1 b_1^2 + A_2 b_2^2) - A \right]. \] (5.2.22)
\[ B_0(t_3) = \frac{\bar{Y}}{2} \left[ \hat{\gamma}_2 \left( \frac{1}{\hat{\theta}_1} + 1 \right) b_1^2 + \hat{\gamma}_2 \left( \frac{1}{\hat{\theta}_2} + 1 \right) b_2^2 \right. \]
\[ + \frac{2(A_1 m_1 + A_2 m_2)}{B} - 2 \frac{A}{B} \left. \right] \quad (5.2.23) \]

\[ M_0(t_1) = V(\bar{Y}) - \bar{Y}^2 \frac{A}{B} ; \quad i = 1, 2, 3 ; \quad (5.2.24) \]

where

\[ A = k_1 A_1 + k_2 A_2. \]

Thus, the minimum MSE's of the estimators \( t_1, t_2 \) and \( t_3 \) turn out to be identically equal.

As in Chapter 4, here also we note that in practice, the population characteristics occurring in the optimum values can be replaced by their sample counterparts.

5.3 THEORETICAL COMPARISON

From (5.2.24), it is clear that

\[ V(\bar{Y}) - M_0(t_1) = \bar{Y}^2 \frac{A}{B} \]
\[ = \bar{Y}^2 \frac{k_1^2 b_1^2 + k_2^2 b_2^2 - 2 k_1 k_2 b_{12}}{b_1^2 b_2 - b_{12}^2} ; \quad i = 1, 2, 3. \]
\[ (5.3.1) \]
Now, since $b_{12} = \rho_{12} b_1 b_2$, $|b_{12}| < b_1 b_2$ whenever $|\rho_{12}| < 1$. This shows that both numerator and denominator of R.H.S. of (5.3.1) are positive if $|\rho_{12}| < 1$, this condition being generally satisfied for sample values, the suggested estimators $t_1$, $t_2$ and $t_3$ for $\bar{Y}$ are always superior to $\bar{y}$.

Das and Tripathi (1980) have proposed the estimator $t_0$ as defined in (5.1.1) for $\bar{Y}$ using the knowledge on the coefficient of variation of an auxiliary character with minimum MSE as

$$M_0(t_0) = \nu(\bar{y}) - \bar{y}^2 \frac{k_1^2}{b_1^2}$$

(5.3.2)

Next, from (5.2.24) and (5.3.2), we find that

$$M_0(t_0) - M_0(t_i) = \bar{y}^2 \frac{(k_2 b_2^2 - k_1 b_1 b_2)^2}{b_1^2 (b_1^2 b_2^2 - b_1^2)} ; i = 1,2,3.$$  

(5.3.3)

$$\geq 0$$ provided $|\rho_{12}| < 1$.

which imply that when optimum weights are used, the
use of auxiliary information on more and more characters increases the efficiency of the estimators.

5.4 USE OF SIMPLE WEIGHTS

In various situations, due to complicated expressions of the optimum weights it is advisable to use simple weights if the loss in efficiency is not large. Now, we consider $W_1 = W_2 = \frac{1}{2}$ in (5.2.1), then the resulting estimator is given by

$$t_4 = \bar{y} \left[ \frac{1}{2} \left( \frac{C_1^2}{s_1^2/x_1^2} \right) + \frac{1}{2} \left( \frac{C_2^2}{s_2^2/x_2^2} \right) \right] \delta$$

(5.4.1)

Following the same procedure as adopted earlier, the bias and MSE of $t_4$, to the order of $n^{-1}$, are respectively given by

$$B(t_4) = \frac{\bar{y}}{2} \delta \left[ (\delta - 1)(b_1^2 + b_2^2 + 2b_{12}) + (m_1 + m_2) - (k_1 + k_2) \right]$$

(5.4.2)
and

\[ M(t_4) = V(\bar{y}) + \bar{Y}^2 \left[ \frac{1}{4} \delta^2 \left( b_1^2 + b_2^2 + 2b_{12} \right) - \delta (k_1 + k_2) \right] \]  

(5.4.3)

When \( M(t_4) \) is minimized w.r.t. \( \delta \), it gives the optimum value of \( \delta \) as

\[
\hat{\delta} = \frac{2(k_1 + k_2)}{b_1^2 + b_2^2 + 2b_{12}}
\]  

(5.4.4)

By substituting this optimum value of \( \delta \) in (5.4.2) and (5.4.3), we obtain the resulting bias and MSE of \( t_4 \) as

\[ B_0(t_4) = \frac{\bar{Y}}{2} \delta \left[ (m_1 + m_2) - \left( \frac{1 + \delta}{\delta} \right) \left( \frac{k_1 + k_2}{k_2} \right) \right] \]  

(5.4.5)

and

\[ M_0(t_4) = V(\bar{y}) - \bar{Y}^2 \frac{(k_1 + k_2)^2}{b_1^2 + b_2^2 + 2b_{12}} \]  

(5.4.6)
Next, from (5.2.24) and (5.4.6), we have

$$M_o(t_4) - M_o(t_1) = \bar{y}^2 \frac{[k_1(b_2^2 + b_{12}) - k_2(b_1^2 + b_{12})]^2}{(b_1^2 + b_2^2 + 2b_{12})(b_1^2 b_2^2 - b_{12}^2)}$$

$$i = 1,2,3.$$ (5.4.7)

which exhibits the fact that the use of simple weights is advisable only if the loss in efficiency is not of high degree.

Further, it can be verified that the minimum MSE of $t_3$ turns out to be same if either optimum or simple weights are used and hence, in that case it is better to use simple weights rather than optimum weights.

5.5 MULTIVARIATE EXTENSIONS

Motivated by the result in (5.2.24), we provide the multivariate analogues of the estimators and the expressions for their biases and MSE's when the optimum
values of parameters are substituted. As in Chapter 4, here also we take $p(>2)$ auxiliary characters.

The following notations will be used.

\[ u_i = \frac{c_i^2}{s_i^2/x_i^2}; \quad i = 1, 2, \ldots, p. \]

\[ e' = (1, 1, \ldots, 1): 1 \times p. \]

\[ b' = (b_1^2, b_2^2, \ldots, b_p^2): 1 \times p. \]

\[ k' = (k_1, k_2, \ldots, k_p): 1 \times p. \]

\[ m' = (m_1, m_2, \ldots, m_p): 1 \times p. \]

where

\[
k_i = \frac{\text{Cov}(\bar{y}, c_i^2)}{\bar{y} c_i^2} = \frac{\text{Cov}(\bar{y}, s_i^2)}{\bar{y} \sigma_i^2} - 2 \frac{\text{Cov}(\bar{y}, x_i)}{\bar{y} \bar{x}_i}; \quad i = 1, 2, \ldots, p.
\]

and

\[
m_i = \frac{V(x_1)}{x_i^2} - 2 \frac{\text{Cov}(x_i, s_i^2)}{x_i \sigma_i^2} + \frac{V(s_i^2)}{\sigma_i^4}; \quad i = 1, 2, \ldots, p.
\]
The multivariate analogues of the estimators $t_i$ ($i = 1, 2, 3, 4$) are given by

\begin{align*}
t_{1m} &= \bar{y} \left[ \sum_{i=1}^{p} w_i u_i \right]^\alpha ; \quad \sum_{i=1}^{p} w_i = 1 \quad (5.5.1) \\
t_{2m} &= \bar{y} \prod_{i=1}^{p} u_i \quad (5.5.2) \\
t_{3m} &= \bar{y} \sum_{i=1}^{p} \theta_i u_i \theta_i ; \quad \sum_{i=1}^{p} \theta_i = 1 \quad (5.5.3) \\
t_{4m} &= \bar{y} \left[ \sum_{i=1}^{p} u_i/p \right]^\delta \quad (5.5.4)
\end{align*}
The MSE's of these estimators when minimized give the optimum values of parameters as

\[ \hat{\alpha} = e'B^{-1}k \]  \hspace{1cm} (5.5.5)

\[ \hat{w} = \frac{B^{-1}k}{e'B^{-1}k} \]  \hspace{1cm} (5.5.6)

\[ \hat{\beta} = \hat{\gamma} = B^{-1}k \]  \hspace{1cm} (5.5.7)

\[ \hat{\delta} = \frac{e'k}{e'Be} \]  \hspace{1cm} (5.5.8)

On substitution of the optimum values of parameters from (5.5.5) to (5.5.8) in the biases and MSE's of the estimators \( t_{im} \) \((i = 1, 2, 3, 4)\), the resulting biases and MSE's of the estimators \( t_{1m}, t_{2m}, t_{3m} \) and \( t_{4m} \) are obtained as under.

\[ B_o(t_{1m}) = \frac{\hat{\gamma}}{2} \left[ (k'B^{-1}k)^{-1} \{ 2m - (\frac{1+\hat{\gamma}}{\alpha} \)A} \} \right] \]  \hspace{1cm} (5.5.9)

\[ B_o(t_{2m}) = \frac{\hat{\gamma}}{2} \left[ k'B^{-1}(2m - b - k) \right] \]  \hspace{1cm} (5.5.10)
Now, we consider the problem of comparison of the estimators $t_{im}$ $(i = 1, 2, 3, 4)$ as defined in (5.5.1) to (5.5.4). We note that the matrix $B$ is a variance-covariance matrix and is positive definite if $| \rho_{ij} | < 1$ for all $i \neq j = 1, 2, \ldots, p$. This condition being generally satisfied for sample counterparts, it can be easily verified from (5.5.13) and (5.5.14) that the estimators given in (5.5.1) to (5.5.4) are more efficient than mean per unit estimator $\bar{y}$.

Next, from (5.5.13) and (5.5.14), we obtain that

\[
B_o(t_{3m}) = \frac{\bar{y}}{2} \left[ k'B^{-1}(GB^{-1}k + 2m + b - 2k) \right] \tag{5.5.11}
\]

\[
B_o(t_{4m}) = \frac{\bar{y}}{2} \left[ \frac{\delta}{\delta} \{ p_m - ( \frac{1}{\delta^k} )k \} \right] \tag{5.5.12}
\]

\[
M_o(t_{1m}) = V(\bar{y}) - \bar{y}^2 k'B^{-1}k \quad ; \quad i = 1, 2, 3. \tag{5.5.13}
\]

and

\[
M_o(t_{4m}) = V(\bar{y}) - \bar{y}^2 \frac{e'k'k'e}{e'B^{-1}e} \tag{5.5.14}
\]
and using Cauchy-Schwarz inequality (see Rao (1973), p.54), it can be verified that

\[ M_0(t_{4m}) - M_0(t_{1m}) = \sqrt{\sum_{i=1}^{3} (k_i B^{-1} k_i - e'B e) }; \]

\[ i = 1, 2, 3. \quad (5.5.15) \]

which, as in two variate case, again exhibits the fact that in multivariate case also the estimator using optimum weights is better than that using simple weights.

5.6 ESTIMATION OF VARIANCE AND COEFFICIENT OF VARIATION

Now, we consider the problem of estimation of variance and coefficient of variation. In earlier sections, we have obtained the biases, MSE's and the resulting (optimum) biases and MSE's by using the optimum values of parameters for the estimators \( t_1, t_2 \) and \( t_3 \) of the population mean \( \bar{Y} \). The expressions for biases, MSE's,
the optimum values of parameters and the resulting (optimum) biases and MSE's of the estimators \( t_1^*, t_2^* \) and \( t_3^* \) as defined in (5.2.4) to (5.2.6) for \( \sigma_0^2 \) can be obtained by replacing \( s_0^2 \) and \( \sigma_0^2 \) for \( \bar{y} \) and \( \bar{Y} \) respectively and putting '* ' over the constants in the corresponding results of the estimators \( t_1, t_2 \) and \( t_3 \), e.g. in two variate case, the bias and MSE of \( t_1^* \) as defined in (5.2.4) are given by

\[
B(t_1^*) = \frac{\sigma_0^2}{2} \alpha^*[ (\alpha-1)(W_1^* b_1^2 + W_2^* b_2^2 + 2W_1^* W_2^* b_{12})
- 2(\bar{w}_1^* k_1 + \bar{w}_2^* k_2) + 2(\bar{w}_1 m_1 + \bar{w}_2 m_2)]
\]

(5.6.1)

and

\[
M(t_1^*) = V(s_0^2) + \sigma_0^4 \left[ \alpha^2 (W_1^* b_1^2 + W_2^* b_2^2 + 2W_1^* W_2^* b_{12})
- 2\alpha^* (\bar{w}_1^* k_1^* + \bar{w}_2^* k_2^*) \right]
\]

(5.6.2)

where

\[
V(s_0^2) = \frac{\sigma_0^4}{n} [\beta_2(y) - 1]
\]

(5.6.3)
Similarly, the biases, MSE's, the optimum values of parameters and the resulting (optimum) biases and MSE's of the estimators $t^{**}_1 (i = 1, 2, 3)$ as defined in (5.2.7), (5.2.8) and (5.2.9) for estimating $C_o$ can be obtained by replacing $c_o$ and $C_o$ for $\bar{y}$ and $\bar{Y}$ respectively and putting '***' over the parameters in the corresponding expressions of the estimators $t_1, t_2$ and $t_3$. However, since $c_o$ is biased for $C_o$, the bias of $t^{**}_i (i = 1, 2, 3)$ will contain an additional term as the bias of $c_o$ and is given by

$$B(c_o) = C_o \left[ \frac{V(\bar{y})}{\bar{y}^2} - \frac{1}{2} \frac{\text{Cov}(\bar{y}, s^2_o)}{\bar{y}^2} - \frac{1}{8} \frac{V(s^2_o)}{\sigma^4_o} \right] (5.6.5)$$

For example, in two variate case, the bias and MSE of $t^{**}_1$ defined in (5.2.7) are given as
\[ B(t^*) = B(c_0) + \frac{c_0}{2} a^* \left[ (a^* - 1)(w_1^* b_1^2 \right. \\
\left. + w_2^* b_2^2 + 2w_1^* w_2^* b_{12}) - 2(w_1^* k_1^* + w_2^* k_2^* \right) \\
+ 2(w_1^* m_1 + w_2^* m_2) \right] \quad (5.6.6) \]

and

\[ M(t^*) = M(c_0) + c_0^2 \left[ a^* \left( w_1^* b_1^2 + w_2^* b_2^2 + 2w_1^* w_2^* b_{12} \right) \\
- 2a^* (w_1^* k_1^* + w_2^* k_2^* \right) \right] \quad (5.6.7) \]

where

\[ M(c_0) = c_0^2 \left[ \frac{V(\bar{y})}{\bar{y}^2} - \frac{Cov(\bar{y}, s_0^2)}{\bar{y} \sigma_o^2} + \frac{1}{4} \frac{V(s_0^2)}{\sigma_o^4} \right] \quad (5.6.8) \]

and

\[ k_1^* = \frac{Cov(c_0, c_1^2)}{c_0 c_1^2} \]

\[ = 1 \frac{Cov(s_0^2, c_1^2)}{\sigma_o^2 c_1^2} - \frac{Cov(\bar{y}, c_1^2)}{\bar{y} c_1^2} \]
\[
\frac{1}{2} \frac{\text{Cov}(s_o^2, s_1^2)}{\sigma_o^2 \sigma_1^2} - \frac{\text{Cov}(s_o^2, \bar{x}_i)}{\bar{x}_1 \sigma_o^2} - \frac{\text{Cov}(\bar{y}, s_1^2)}{\bar{y} \sigma_1^2} \\
+ 2 \frac{\text{Cov}(\bar{y}, \bar{x}_i)}{\bar{y} \bar{x}_i}; \quad i = 1, 2. \quad (5.6.9)
\]

and \( V(s_o^2) \) and \( B(c_o) \) are respectively given in (5.6.3) and (5.6.5).
APPENDIX B

Variances and covariances of certain statistics in terms of moments of $y$ and $x_i; i = 1, 2, \ldots, p$ as used in Chapter 4 and Chapter 5.

\[ V(\bar{y}) = \frac{1}{n} \mu_2(y) \]

\[ V(\bar{x}_i) = \frac{1}{n} \mu_2(x_i) \]

\[ V(s_o^2) = \frac{\sigma_o^{-4}}{n} \left[ \frac{\mu_4(y)}{\sigma_o^{-4}} - 1 \right] \]

\[ V(s_i^2) = \frac{\sigma_i^{-4}}{n} \left[ \frac{\mu_4(x_i)}{\sigma_i^{-4}} - 1 \right] \]

\[ \text{Cov}(\bar{y}, \bar{x}_i) = \frac{1}{n} \mu_{11}(y, x_i) \]

\[ \text{Cov}(\bar{y}, s_i^2) = \frac{1}{n} \mu_{12}(y, x_i) \]

\[ \text{cov}(\bar{y}, s_o^2) = \frac{1}{n} \mu_3(y) \]
\[
\text{Cov}(\bar{x}_1, s_1^2) = \frac{1}{n} \mu_3(x_1)
\]

\[
\text{Cov}(\bar{x}_1, s_0^2) = \frac{1}{n} \mu_{21}(y, x_1)
\]

\[
\text{Cov}(s_0^2, s_1^2) = \frac{1}{n} [\mu_{22}(y, x_1) - \sigma_o^2 \sigma_1^2]
\]

where

\[
\mu_2(y) = \frac{1}{N} \sum_{j=1}^{N} (y_j - \bar{y})^2
\]

\[
\mu_2(x_1) = \frac{1}{N} \sum_{j=1}^{N} (x_{1j} - \bar{x}_1)^2
\]

\[
\mu_3(y) = \frac{1}{N} \sum_{j=1}^{N} (y_j - \bar{y})^3
\]

\[
\mu_3(x_1) = \frac{1}{N} \sum_{j=1}^{N} (x_{1j} - \bar{x}_1)^3
\]

\[
\mu_4(y) = \frac{1}{N} \sum_{j=1}^{N} (y_j - \bar{y})^4
\]

\[
\mu_4(x_1) = \frac{1}{N} \sum_{j=1}^{N} (x_{1j} - \bar{x}_1)^4
\]

\[
\mu_{11}(y, x_1) = \frac{1}{N} \sum_{j=1}^{N} (y_j - \bar{y})(x_{1j} - \bar{x}_1)
\]
\[
\mu_{12}(y, x_1) = \frac{1}{N} \sum_{j=1}^{N} (y_j - \bar{Y})(x_{1j} - \bar{x}_1)^2
\]

\[
\mu_{21}(y, x_1) = \frac{1}{N} \sum_{j=1}^{N} (y_j - \bar{Y})^2(x_{1j} - \bar{x}_1)
\]

\[
\mu_{22}(y, x_1) = \frac{1}{N} \sum_{j=1}^{N} (y_j - \bar{Y})^2(x_{1j} - \bar{x}_1)^2
\]