# CHAPTER 4

**USE OF MULTIAUXILIARY INFORMATION IN ESTIMATING MEAN, VARIANCE AND COEFFICIENT OF VARIATION IN FINITE POPULATION-I**

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4.1 INTRODUCTION AND NOTATIONS

Let \( U = (1, 2, \ldots, N) \) be a finite population of \( N \) given units. Let \( y \) and \( x \) denote the study character and the auxiliary character taking the values \( y_i \) and \( x_i \) respectively on the unit \( i \) of \( U \) \((1 \leq i \leq N)\). For a finite population, mean, variance and coefficient of variation of a study character \( y \) are defined by

\[
\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} y_i, \quad \sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{Y})^2 \quad \text{and} \quad C_o = \frac{\sigma}{\bar{Y}}
\]
respectively. The problems of estimating total $Y$, mean $\bar{Y}$, variance $\sigma_o^2$ and coefficient of variation $C_o$ of a finite population in the case when auxiliary information is available have been discussed by various authors. Das and Tripathi (1978a, b; 1979) have suggested the classes of estimators

\[ t_o = \bar{Y} \left( \frac{\sigma^2}{s_1^2} \right) \]  \hspace{1cm} (4.1.1)

\[ t_o^* = s_o^2 \left( \frac{\sigma^2}{s_1^2} \right) \]  \hspace{1cm} (4.1.2)

and

\[ t_o^{**} = c_o \left( \frac{\sigma^2}{s_1^2} \right) Y \]  \hspace{1cm} (4.1.3)

respectively for estimating mean $\bar{Y}$, variance $\sigma_o^2$ and coefficient of variation $C_o$ of a finite population using the knowledge on the variance of an auxiliary character when sampling is done by the method of simple random sampling with replacement. Singh (1967, 1969) and also Shah and Shah (1978) have suggested various ratio-cum-product estimators for estimating ratio (product) of two population parameters.
In this chapter, we consider the problem of estimating mean, variance and coefficient of variation of a study character for a finite population using the knowledge on the variances of auxiliary characters. We propose the classes of ratio-type, product-type and ratio-cum-product type estimators for estimating mean $\bar{Y}$, variance $\sigma_o^2$ and coefficient of variation $C_o$ utilizing the knowledge on the variances of two auxiliary characters. We obtain the biases and mean square errors (MSE's) of the proposed estimators to the order of $n^{-1}$ and also the minimum MSE. The efficiency of the proposed estimators compared to the corresponding conventional estimators is investigated. Multivariate extension is also considered. We confine ourselves to the method of simple random sampling with replacement (SRSWR).

The following notations due to Isaki (1983) will be used in the sequel.

$C_i = \text{Coefficient of variation of } s_i^2.$

$= \text{C.V.} (s_i^2), i = 1, 2.$
\( p_{ij} = \text{Coefficient of correlation between } s_i^2 \text{ and } s_j^2. \)

\( = \text{Corr. } (s_i^2, s_j^2), \ i \neq j = 1, 2. \)

\( C_{ij} = \text{Coefficient of variation of } s_i^2 \text{ and } s_j^2. \)

\( = p_{ij} \cdot C_i \cdot C_j, \ i \neq j = 1, 2. \)

Further, we note that the suffix \( o \) stands for the study character \( y \) and suffixes 1 and 2 stand for the auxiliary characters \( x_1 \) and \( x_2 \) respectively.

4.2 CLASSES OF ESTIMATORS AND THEIR PROPERTIES

The proposed classes of estimators for population mean \( \bar{Y} \), variance \( \sigma_o^2 \) and coefficient of variation \( C_o \) are defined respectively by \( t_1, t_1^* \) and \( t_1^{**} \) \((i = 1, 2, 3)\) as follows.

\[
t_1 = \bar{y} \left[ w_1 \left( \frac{\sigma_1^2}{s_1^2} \right) + w_2 \left( \frac{\sigma_2^2}{s_2^2} \right) \right] \alpha;
\]

\[
w_1 + w_2 = 1 \quad (4.2.1)
\]

\[
t_2 = \bar{y} \left( \frac{\sigma_1^2}{s_1^2} \right)^\beta_1 \left( \frac{\sigma_2^2}{s_2^2} \right)^\beta_2 \quad (4.2.2)
\]
\[ t_3 = \bar{y} \left[ \varrho_1 \left( \frac{\sigma_1^{-2}}{s_1^2} \right) \frac{y_1}{s_1^2} + \varrho_2 \left( \frac{\sigma_2^{-2}}{s_2^2} \right) \frac{y_2}{s_2^2} \right] \]

\[ \varrho_1 + \varrho_2 = 1. \quad (4.2.3) \]

\[ t_1^* = s_0^2 \left[ w_1^* \left( \frac{\sigma_1^{-2}}{s_1^2} \right) + w_2^* \left( \frac{\sigma_2^{-2}}{s_2^2} \right) \right] \]

\[ w_1^* + w_2^* = 1. \quad (4.2.4) \]

\[ t_2^* = s_0^2 \left( \frac{\sigma_1^{-2}}{s_1^2} \right) \beta_1^* \left( \frac{\sigma_2^{-2}}{s_2^2} \right) \beta_2^* \]

\[ (4.2.5) \]

\[ t_3^* = s_0^2 \left[ \Theta_1^* \left( \frac{\sigma_1^{-2}}{s_1^2} \right) \frac{\gamma_1^*}{s_1^2} + \Theta_2^* \left( \frac{\sigma_2^{-2}}{s_2^2} \right) \frac{\gamma_2^*}{s_2^2} \right] \]

\[ \Theta_1^* + \Theta_2^* = 1. \quad (4.2.6) \]

\[ t_1^{**} = c_0 \left[ w_1^{**} \left( \frac{\sigma_1^{-2}}{s_1^2} \right) + w_2^{**} \left( \frac{\sigma_2^{-2}}{s_2^2} \right) \right] \]

\[ w_1^{**} + w_2^{**} = 1. \quad (4.2.7) \]

\[ t_2^{**} = c_0 \left( \frac{\sigma_1^{-2}}{s_1^2} \right) \beta_1^{**} \left( \frac{\sigma_2^{-2}}{s_2^2} \right) \beta_2^{**} \]

\[ (4.2.8) \]
where the weights $W$'s and $\Theta$'s and the constants $\alpha$, $\beta$'s and $\gamma$'s are to be determined by minimizing the MSE's of the estimators.

We present below the results for the estimators $t_1$, $t_2$ and $t_3$ of the population mean $\bar{Y}$ and in the end indicate as to how the results for the estimators $t_1^*$ and $t_2^{**}$ $(i = 1, 2, 3)$ for population variance $\sigma_o^2$ and coefficient of variation $C_o$ respectively follow by some suitable adjustment.

Letting $\bar{Y} = \bar{Y}(1 + e_o)$, $\bar{x}_i = \bar{x}_i(1 + e_i)$, $i = 1, 2$ and $s_1^2 = \sigma_1^2(1 + e_i^2)$; $i = 0, 1, 2$ in (4.2.1) to (4.2.3) and expanding by Taylor's series and taking expectations, the biases and MSE's of the suggested estimators $t_1$, $t_2$ and $t_3$, valid to the first degree of approximation, are obtained as under.
\[ B(t_1) = \frac{\tilde{V}}{\tau} \left[ (\alpha - 1)(w_1^2c_1^2 + w_2^2c_2^2 + 2w_1w_2c_{12}) \right. \\
\left. - 2(w_1k_1 + w_2k_2) \right] \tag{4.2.10} \]

\[ B(t_2) = \frac{\tilde{V}}{\tau} \left[ \beta_1^2c_1^2 + \frac{\beta_2^2c_2^2}{w_2c_2} + 2\beta_1\beta_2c_{12} + \beta_1(c_1^2 - 2k_1) \right. \\
\left. + \beta_2(c_2^2 - 2k_2) \right] \tag{4.2.11} \]

\[ B(t_3) = \frac{\tilde{V}}{\tau} \left[ y_1(\frac{y_1}{\tilde{\theta}_1} + 1)c_1^2 + y_2(\frac{y_2}{\tilde{\theta}_2} + 1)c_2^2 - 2(y_1k_1 + y_2k_2) \right] \tag{4.2.12} \]

\[ M(t_1) = V(\tilde{\gamma}) + \tilde{V}^2 \left[ \alpha^2(w_1^2c_1^2 + w_2^2c_2^2 + 2w_1w_2c_{12}) \right. \\
\left. - 2\alpha(w_1k_1 + w_2k_2) \right] \tag{4.2.13} \]

\[ M(t_2) = V(\tilde{\gamma}) + \tilde{V}^2 \left[ \beta_1^2c_1^2 + \frac{\beta_2^2c_2^2}{w_2c_2} + 2\beta_1\beta_2c_{12} \right. \\
\left. - 2(\beta_1k_1 + \beta_2k_2) \right] \tag{4.2.14} \]
\[
M(t_3) = V(\bar{y}) + \gamma^2 \left[ \gamma_1^2 c_1^2 + \gamma_2^2 c_2^2 + 2\gamma_1 \gamma_2 c_{12} \right] - 2(\gamma_1 k_1 + \gamma_2 k_2)
\]  
(4.2.15)

where we have

\[
V(\bar{y}) = \frac{\sigma_0^2}{n}
\]  
(4.2.16)

\[
k_i = \frac{\text{Cov}(\bar{y}, s_i^2)}{\bar{y} \sigma_i^2}; i = 1, 2.
\]  
(4.2.17)

The optimum values of the parameters \(\alpha, \beta's, \gamma's\) and \(\omega's\) are obtained by minimizing the MSE's of the estimators \(t_1, t_2\) and \(t_3\) and are given as follows.

\[
\hat{\alpha} = \frac{A_1 + A_2}{B} \quad \text{and} \quad \hat{\omega}_i = \frac{A_i}{A_1 + A_2}; i = 1, 2.
\]  
(4.2.18)

\[
\hat{\beta}_i = \hat{\gamma}_i = \frac{A_i}{B}; i = 1, 2.
\]  
(4.2.19)

where
When we substitute the optimum values of parameters from (4.2.18) and (4.2.19) in (4.2.10) to (4.2.15), we obtain the biases and the minimum MSE's of the proposed estimators $t_1$, $t_2$ and $t_3$ as

$$B_0(t_1) = \frac{\bar{y}}{2B} \left[ 2(A_1C_1^2 + A_2C_2^2) - \left( \frac{1 + \hat{\sigma}^2}{\sigma} \right) A \right] \quad (4.2.20)$$

$$B_0(t_2) = \frac{\bar{y}}{2B} \left[ A_1C_1^2 + A_2C_2^2 - A \right] \quad (4.2.21)$$

$$B_0(t_3) = \frac{\bar{y}}{2} \left[ \hat{\gamma}_1 \left( \frac{\hat{\gamma}_1}{\hat{\theta}_1} + 1 \right) C_1^2 + \hat{\gamma}_2 \left( \frac{\hat{\gamma}_2}{\hat{\theta}_2} + 1 \right) C_2^2 - 2A \right] \quad (4.2.22)$$

$$M_0(t_i) = V(\bar{y}) - \frac{\bar{y}^2A}{B} \quad ; \quad i = 1, 2, 3. \quad (4.2.23)$$
where

\[ A = k_1A_1 + k_2A_2. \]

Thus, from (4.2.23), we observe that the minimum MSE’s of the estimators \( t_1, t_2 \) and \( t_3 \) of the population mean \( \bar{Y} \) are identically equal.

In practice, the population parameters occurring in the optimum values can be replaced by their sample counterparts.

### 4.3 THEORETICAL COMPARISON

From (4.2.23), we find that

\[
V(\bar{Y}) - M_0(t) = \bar{Y}^2 \frac{A}{B}
\]

\[
= \bar{Y}^2 \frac{k_1^2C_2 + k_2^2C_1 - 2k_1k_2C_{12}}{C_1^2C_2 - C_{12}^2}; \\
\]

\[ i = 1, 2, 3. \quad (4.3.1) \]

Now, since \( C_{12} = \bar{f}_{12}C_1C_2, |C_{12}| < C_1C_2 \) whenever
\[ | \theta_{12} | < 1. \] This shows that the numerator and denominator of R.H.S. of (4.3.1) are positive provided \( | \theta_{12} | < 1 \), this condition being generally satisfied for sample values, the proposed estimators \( t_1, t_2, \) and \( t_3 \) for \( \bar{Y} \) are always more efficient than mean per unit estimator \( \bar{y} \).

Next, Das and Tripathi (1979) have suggested the estimator \( t_o \) as defined in (4.1.1) for population mean \( \bar{Y} \) using auxiliary information on a single character with minimum MSE as

\[
M_o(t_o) = V(\bar{y}) - \bar{y}^2 \frac{k_1^2}{C_1^2} \quad (4.3.2)
\]

Now, from (4.2.23) and (4.3.2), we get that

\[
M_o(t_o) - M_o(t_i) = \bar{y}^2 \frac{(k_2C_1^2 - k_iC_{12})^2}{C_1^2(C_1C_2 - C_{12})} \quad ; \quad i = 1,2,3.
\]

\[ \geq 0, \text{ whenever } | \theta_{12} | < 1. \quad (4.3.3) \]

Hence, we conclude that when optimum weights are used, the use of auxiliary information on more and more
characters helps in increasing the efficiency of the estimator.

4.4 USE OF SIMPLE WEIGHTS

In many cases, due to complicated expressions of the optimum weights, it is advisable to use simple weights if the loss in efficiency is not of high degree. When we take \( W_1 = W_2 = \frac{1}{2} \) in (4.2.1), the estimator turns out to be

\[
t_4 = \bar{y} \left[ \frac{1}{2} \left( \frac{\sigma_1^2}{s_1^2} \right) + \frac{1}{2} \left( \frac{\sigma_2^2}{s_2^2} \right) \right] \delta.
\] (4.4.1)

Following the same procedure as adopted earlier, the bias and MSE of the estimator \( t_4 \), to the order of \( n^{-1} \), are respectively given by

\[
B(t_4) = \frac{\bar{y}}{2} \delta \left[ (\delta - 1)(C_1^2 + C_2^2 + 2C_{12}) + (C_1^2 + C_2^2) \right. \\
- \left. (k_1 + k_2) \right] \] (4.4.2)
Now, minimization of $M(t_{4})$ given in (4.4.3) with respect to $\delta$ yields the optimum value of $\delta$ as

$$\hat{\delta} = \frac{2(k_{1} + k_{2})}{C_{1}^{2} + C_{2}^{2} + 2C_{12}}$$  \hfill (4.4.4)$$

The substitution of the optimum value of $\delta$ in (4.4.2) and (4.4.3) yield the bias and minimum MSE of $t_{4}$ as

$$B_{0}(t_{4}) = \frac{\bar{y}}{2} \delta [ (C_{1}^{2} + C_{2}^{2}) - \left( \frac{k_{1} + k_{2}}{2} \right)(1 + \delta)] \hfill (4.4.5)$$

and

$$M_{0}(t_{4}) = \nu(\bar{y}) - \frac{\bar{y}^{2}(k_{1} + k_{2})^{2}}{C_{1}^{2} + C_{2}^{2} + 2C_{12}} \hfill (4.4.6)$$

From (4.4.6), we see that the estimator $t_{4}$ is better
than $\tilde{y}$ whenever $|\theta_{12}| < 1$.

Next, from (4.2.23) and (4.4.6), we obtain that

$$M_0(t_4) - M_0(t_1) = \gamma^2 \frac{[c_1(c_1^2 + c_{12}) - k_2(c_2^2 + c_{12})]^2}{(c_1^2 + c_{12}^2 + 2c_{12})(c_1^2 c_{12}^2 - c_{12}^2)};$$

$$i = 1, 2, 3. \quad (4.4.7)$$

$$\geq 0; \text{ if } |\theta_{12}| < 1 \quad (4.4.7)$$

which exhibits the fact that the use of simple weights is advisable if the loss in efficiency is not large.

Further, note that, the minimum MSE of $t_3$ turns out to be same if either optimum or simple weights are used and hence, in this case, the use of simple weights is advisable.

4.5 MULTIVARIATE EXTENSIONS

Since the result in (4.2.23) is interesting one would naturally like to enquire as to whether it holds true for multivariate case also i.e. for $p(> 2)$
auxiliary characters. We give below only the estimators and the expressions for biases and MSE's when the optimum values of parameters are substituted. We use the following notations.

\[ u_i = \frac{\sigma_i^2}{s_i^2}; \quad i = 1, 2, \ldots, p. \]

\[ \mathbf{e}' = (1, 1, \ldots, 1) : 1 \times p. \]

\[ \mathbf{c}' = (c_1^2, c_2^2, \ldots, c_p^2) : 1 \times p. \]

\[ k' = (k_1, k_2, \ldots, k_p) : 1 \times p, \]

where

\[ k_i = \frac{\text{Cov}(\bar{y}, s_i^2)}{\bar{y} \sigma_i^2}; \quad i = 1, 2, \ldots, p. \]

\[ G = \text{diag} \left( \frac{c_1^2}{\theta_1}, \frac{c_2^2}{\theta_2}, \ldots, \frac{c_p^2}{\theta_p} \right) : p \times p. \]

\[ B = \begin{bmatrix} c_1^2 & c_{12} & \cdots & c_{1p} \\ c_{12} & c_2^2 & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \cdots & c_p^2 \end{bmatrix} \]
The multivariate analogues of the estimators $t_1$, $t_2$, $t_3$ and $t_4$ are given by

\[
t_{1m} = \bar{y} \left[ \sum_{i=1}^{p} w_i u_i \right]^a ; \quad \sum_{i=1}^{p} w_i = 1 \quad (4.5.1)
\]

\[
t_{2m} = \bar{y} \prod_{i=1}^{p} u_i \beta_i \quad (4.5.2)
\]

\[
t_{3m} = \bar{y} \left[ \sum_{i=1}^{p} \frac{y_i}{q_i} \right] \quad (4.5.3)
\]

\[
t_{4m} = \bar{y} \left[ \sum_{i=1}^{p} \frac{u_i}{p} \right]^6 \quad (4.5.4)
\]

The MSE's of the estimators defined in (4.5.1) to (4.5.4), when minimized, yield the optimum values of parameters as

\[
\hat{\alpha} = e' B^{-1} k \quad (4.5.5)
\]

\[
\hat{w} = \frac{B^{-1} k}{e' B^{-1} k} \quad (4.5.6)
\]

\[
\hat{\beta} = \hat{\gamma} = B^{-1} k \quad (4.5.7)
\]
Substitution of the optimum values of parameters from (4.5.5.) to (4.5.8), yield the resulting biases and MSE's of the estimators \( t_{im} \) \((i = 1, 2, 3, 4)\) as follows.

\[
\hat{\sigma} = p \frac{e' k}{e' B e} \quad (4.5.8)
\]

Substitution of the optimum values of parameters from (4.5.5.) to (4.5.8), yield the resulting biases and MSE's of the estimators \( t_{im} \) \((i = 1, 2, 3, 4)\) as follows.

\[
B_0(t_{1m}) = \frac{\bar{Y}}{2} \left[ k'B^{-1}(2C - \frac{1 + \hat{\sigma}}{\hat{\sigma}}) \right] \quad (4.5.9)
\]

\[
B_0(t_{2m}) = \frac{\bar{Y}}{2} \left[ k'B^{-1}(C - k) \right] \quad (4.5.10)
\]

\[
B_0(t_{3m}) = \frac{\bar{Y}}{2} \left[ k'B^{-1}(GB^{-1}k + C - 2k) \right] \quad (4.5.11)
\]

\[
B_0(t_{4m}) = \frac{\bar{Y}}{2} \frac{\hat{\delta}}{p} \left[ e' \left[ 2C + \left( \frac{1 + \hat{\delta}}{\hat{\delta}} \right) k \right] \right] \quad (4.5.12)
\]

\[
M_0(t_{1m}) = V(\bar{Y}) - \frac{\bar{Y}^2}{2} k'B^{-1}k \quad ; \quad i = 1, 2, 3. \quad (4.5.13)
\]

\[
M_0(t_{4m}) = V(\bar{Y}) - \frac{\bar{Y}^2}{2} \frac{e'k k' e}{e' B e} \quad (4.5.14)
\]

Now, we consider the problem of efficiency
comparison of the estimators $t_{im}$ ($i = 1, 2, 3, 4$) given in (4.5.1) to (4.5.4). We note that the matrix $B$ is a variance-covariance matrix based on auxiliary information only and is positive definite provided $|\phi_{ij}| < 1$, for all $i \neq j = 1, 2, \ldots, p$. This condition being generally satisfied for sample values, it can be easily verified from (4.5.13) and (4.5.14) that the estimators defined in (4.5.1) to (4.5.4) are better than simple mean estimator $\bar{y}$.

Next, from (4.5.13) and (4.5.14), we find that

$$M_o(t_{4m}) - M_o(t_{im}) = \bar{y}^2 \left[ k'B^{-1}k - \frac{e'k}{e'B^{-1}e} \right];$$

$$i = 1, 2, 3. \quad (4.5.15)$$

and using Cauchy-Schwarz inequality (see Rao (1973), p. 54) it is easily seen that

$$M_o(t_{4m}) \geq M_o(t_{im}); i = 1, 2, 3 \quad (4.5.16)$$

which, as in two variate case, again exhibits the fact that in multivariate case also the estimator using optimum
weights is superior to that using simple weights.

4.6 ESTIMATION OF VARIANCE AND COEFFICIENT OF VARIATION

Now, consider the problem of estimation of variance and coefficient of variation. We have obtained the expressions for biases, MSE's, the optimum values of parameters and the resulting (optimum) biases and MSE's of the estimators $t_1$, $t_2$ and $t_3$ of $\bar{Y}$. The biases, MSE's, the optimum values of parameters and the resulting (optimum) biases and MSE's of the estimators $t^*_1$, $t^*_2$ and $t^*_3$ as defined in (4.2.4), (4.2.5) and (4.2.6) for $\sigma_o^2$ can be obtained by replacing $s^2$ and $\sigma_o^2$ for $\bar{y}$ and $\bar{\bar{Y}}$ respectively and putting '$_*$' over the parameters in the corresponding results of the estimators $t_1$, $t_2$ and $t_3$. For example, in two variate case, the bias and MSE of $t^*_1$ as defined in (4.2.4) are given by

$$B(t^*_1) = \frac{\sigma^2}{2} \alpha^*[((\alpha^* - 1)(w^*_1 c^2_1 + w^*_2 c^2_2 + 2w^*_1 w^*_2 c_{12})$$
$$- 2(w^*_1 k^*_1 + w^*_2 k^*_2)]$$ (4.6.1)
and

\[ M(t^*) = V(s_0^2) + \sigma_o^{-4} \left[ \alpha^2 (w_1^* c_1^2 + w_2^* c_2^2 + 2w_1^* w_2^* c_{12}) \right. \]
\[ \left. \quad - 2\alpha^* (w_1^* k_1^* + w_2^* k_2^*) \right] \quad (4.6.2) \]

where

\[ V(s_0^2) = \frac{\sigma_o^{-4}}{n} [ g_2(y) - 1 ] \quad (4.6.3) \]

\[ k_i^* = \frac{\text{Cov}(s_i^2, s_0^2)}{\sigma_0^2 \sigma_1^2} ; \quad i = 1, 2. \quad (4.6.4) \]

Similarly, the biases, MSE's, the optimum values of parameters and the resulting (optimum) biases and MSE's of the estimators \( t_1^{**}, t_2^{**} \) and \( t_3^{**} \) as defined in (4.2.7), (4.2.8) and (4.2.9) can be obtained by replacing \( c_0 \) and \( C_0 \) for \( \bar{y} \) and \( \bar{Y} \) respectively and putting '***' over the parameters in the corresponding expressions of the estimators \( t_1, t_2 \) and \( t_3 \). However, since \( c_0 \) is not unbiased for \( C_0 \), the biases of \( t_i^{**} (i = 1, 2, 3) \) will contain an additional term as the
bias of \( c_0 \) and is given by

\[
B(c_0) = C_0 \left[ \frac{V(\bar{y})}{\bar{y}^2} - \frac{1}{2} \frac{\text{Cov}(\bar{y}, s^2)}{\bar{y} \sigma^2_o} - \frac{1}{8} \frac{V(s^2)}{\sigma^4_o} \right] \quad (4.6.5)
\]

For example, in two variate case, the bias and MSE of \( t^{**} \) defined in (4.2.7) are given

\[
B(t^{**}) = B(c_0) + \frac{C_0}{2} \alpha^{**} \left[ (\alpha^{**} - 1)(w^{**}_1 c_1^2 + w^{**}_2 c_2^2 + 2 w^{**}_1 w^{**}_2 c_{12}) - 2(w^{**}_1 k_1 + w^{**}_2 k_2) \right] \quad (4.6.6)
\]

and

\[
M(t^{**}) = M(c_0) + C_0^2 \left[ \alpha^{**2}(w^{**}_1 c_1^2 + w^{**}_2 c_2^2 + 2 w^{**}_1 w^{**}_2 c_{12}) - 2 \alpha^{**} (w^{**}_1 k_1 + w^{**}_2 k_2) \right],
\]

\[(4.6.7)\]
\[ M(c_0) = c_0^2 \left[ \frac{V(\bar{y})}{\bar{y}^2} - \frac{\text{Cov}(\bar{y}, s_0^2)}{\bar{y} \sigma_o^2} + \frac{1}{4} \frac{V(s_0^2)}{\sigma_o^4} \right] \quad (4.6.8) \]

and

\[ k_{**} = \frac{\text{Cov}(c_o, s_i^2)}{c_o \sigma_i^2} \]

\[ = \frac{1}{2} \frac{\text{Cov}(s_o^2, s_i^2)}{\sigma_o^2 \sigma_i^2} - \frac{\text{Cov}(\bar{y}, s_i^2)}{\bar{y} \sigma_i^2} \quad ; \quad i = 1, 2. \]

\[ (4.6.9) \]

and \( V(s_o^2) \) and \( B(c_o) \) are respectively given in (4.6.3) and (4.6.5).