# CHAPTER 3

AN EFFICIENCY COMPARISON OF DUAL RATIO AND PRODUCT ESTIMATORS UNDER A SUPERPOPULATION MODEL

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3.1 INTRODUCTION

Srivenkateramana (1980) has suggested a dual ratio estimator for estimating the population total of a study character which uses auxiliary information on non-sampling units in place of sampling units as in the case of ratio estimator. He exhibited the fact that the expressions for bias and mean square error (MSE) of the dual ratio estimator are exact and established the superiority of his estimator over the usual ratio estimator under certain conditions with respect to the design based comparison.

In this chapter, since the conventional product estimator also is a dual to ratio estimator, the efficiency of the dual ratio estimator is compared with the conventional product estimator with respect to the superpopulation model introduced by Durbin (1959). The bias and MSE of the dual ratio estimator are obtained for finite population and are compared with those of the product estimator for an infinite population. It has been found that the dual ratio estimator performs better than the product estimator when auxiliary character has a gamma distribution with parameter greater than or equal to one, in the case when the regression is through origin or when the intercept x slope is positive.

3.2 THE ESTIMATORS AND THE SUPERPOPULATION MODEL

Consider a finite population \{ (y_i, x_i), i = 1, 2, \ldots, N \} of N units with population means (\bar{y}, \bar{x}), variances (S_y^2, S_x^2)
A simple random sample \( \{ (y^*_i, x^*_i) \} \) of \( n \) units is drawn from a population of \( N \) units, then we have sample means \((\bar{y}, \bar{x})\), variances \((s_y^2, s_x^2)\) and covariance \(s_{yx}\). Let \( \hat{Y} = N\bar{y} \) be an unbiased estimator of the population total \( Y = NY \). In simple random sampling, an unbiased estimator of the population total \( X = NX \) is \( \hat{X} = N\bar{x} \). Also, \( \hat{X}' = N\bar{x}' \) is an unbiased estimator of \( X \), where \( \bar{x}' = (N\bar{x} - nx)/(N-n) \) is the mean of \( x \)-values of \( (N-n) \) non-sampling units. From variance point of view, \( \hat{X}' \) is better than \( \hat{X} \) for estimating \( X \) wherever \( N-2n \geq 0 \) i.e. \( f = n/N \leq 0.5 \), a situation usually met in practice. Using this fact, for estimating the population total \( Y \), Srivenkataramana (1980) has suggested a dual ratio estimator as

\[
\hat{y}_{DR} = \hat{y} \frac{\hat{X}}{\bar{x}} \tag{3.2.1}
\]

From (3.2.1), the dual ratio estimator for estimating the population mean \( \bar{Y} \) is defined by

\[
\hat{y}_{DR} = \hat{y} \frac{\bar{x}'}{\bar{x}} \tag{3.2.2}
\]
The usual product estimator for $\bar{Y}$ is given by

$$\hat{Y}_p = \frac{y \bar{x}}{\bar{x}}$$  \hspace{1cm} (3.2.3)

The unbiased product estimator obtained by correcting $\hat{Y}_p$ for its bias (Robson (1957)) is defined as

$$\hat{Y}_{pu} = \hat{Y}_p - \frac{n-n}{nN} \frac{s_{yx}}{\bar{x}}$$  \hspace{1cm} (3.2.4)

where $s_{yx} = \frac{1}{n-1} \sum_{i} (y_i - \bar{y})(x_i - \bar{x})$.

We consider the following superpopulation model of Durbin (1959).

$$y_i = \alpha + \beta x_i + e_i, \quad i = 1, 2, \ldots, N; \hspace{1cm} (3.2.5)$$

where $\alpha, \beta$ are unknown real constants, $e_i (i = 1, 2, \ldots, N)$ are random errors distributed with $E_c(e_i/x_i) = 0$, $E_c(e_i e_j/x_i x_j) = 0$ for every $i \neq j$ and $E_c(e_i^2/x_i) = \delta x_i^t$, $0 < \delta < \infty$, $0 \leq t \leq 2$ and $x_i (i = 1, 2, \ldots, N)$ are independently identically distributed with a common gamma density,
\[ G(g) = \frac{1}{|g|}e^{-gx}x^{g-1}, \quad 0 < x < \infty, \quad g \geq 1. \] 

(3.2.6)

Here, \( E_C \) denote the conditional expectation, given \( x_1(i = 1, 2, \ldots, N) \). We write \( E_X \) for the expectation with respect to the common distribution of \( x_1(i = 1, 2, \ldots, N) \).

Let \( E_n = E_X E_C \) denote the model expectation and the design expectation will be denoted by \( E_D \).

### 3.3 THE BIASES AND MEAN SQUARE ERRORS OF THE ESTIMATORS

The average bias (ABIAS) and average MSE (AMSE) of the product estimator \( \hat{Y}_p \) defined in (3.2.3) are obtained by Chaubey, Dwivedi and Singh (1984) and are given respectively as

\[
\text{ABIAS}(\hat{Y}_p) = \beta \frac{(N-n)\theta}{n(N^0+1)} \tag{3.3.1}
\]

and

\[
\text{AMSE}(\hat{Y}_p) = \sigma^2 A_p + \beta^2 B_p + 2\alpha\beta C_p + \delta D_p, \tag{3.3.2}
\]

where

\[
A_p = \frac{N-n}{n(N^0+1)},
\]
\[ B_p = \frac{N^2 \theta(n\theta+1)(n\theta+2)(n\theta+3)}{n^3(N\theta+2)(N\theta+3)} - \frac{2\theta(n\theta+1)}{n} \]

\[ + \frac{\theta(n\theta+1)}{N} , \]

\[ C_p = \frac{N^2 \theta(n\theta+1)(n\theta+2)}{n^2(N\theta+1)(N\theta+2)} - \frac{N\theta(n\theta+1)}{n(N\theta+1)} \]

and

\[ D_p = \left[ \frac{N^2(n\theta+t)(n\theta+t+1)}{n^2(N\theta+t)(N\theta+t+1)} - \frac{2(n\theta+t)}{n(N\theta+t)} + \frac{1}{N} \right] \frac{\theta+t}{\theta} . \]

To obtain the average bias and average MSE of the dual ratio estimator \( \hat{y}_{DR} \), we take the expectations in two steps: First we operate with the design expectation \( E_D \) and then with the model expectation \( E_M \). The design expectation can be evaluated using the results in Sukhatme and Sukhatme (1970, p. 190) and the model expectation can be evaluated using the lemma given in the appendix A of this chapter. Now, the dual ratio estimator \( \hat{y}_{DR} \) defined in (3.2.2) can be written as

\[ \hat{y}_{DR} = ay + b\hat{y}_p , \quad (3.3.3) \]
where \( a = \frac{N}{N-n} \), \( b = -\frac{n}{N-n} \), \( a + b = 1 \).

The average bias (ABIAS) of \( \hat{Y}_{DR} \) is obtained as follows.

\[
\text{ABIAS}(\hat{Y}_{DR}) = E_{ME}^L(\hat{Y}_{DR} - \bar{Y}) \tag{3.3.4}
\]

\[
= E_{ME}^L(a\bar{y} + b\hat{Y}_P - \bar{Y})
\]

\[
= E_{ME}^L[a(\bar{y} - \bar{Y}) + b(\hat{Y}_P - \bar{Y})]
\]

\[
= b \ E_{ME}^L(\hat{Y}_P - \bar{Y})
\]

\[
= \frac{n}{N-n} \cdot \frac{\theta(N-n)\theta}{n(N\theta+1)} \quad \text{(using (3.3.1))}
\]

\[
= -\frac{\theta}{N\theta+1}
\]

Next, the average MSE (AMSE) of \( \hat{Y}_{DR} \) can be obtained as under

\[
\text{AMSE}(\hat{Y}_{DR}) = E_{ME}^L(\hat{Y}_{DR} - \bar{Y})^2
\]

\[
= E_{ME}^L[a(\bar{y} - \bar{Y}) + b(\hat{Y}_P - \bar{Y})]^2
\]

\[
= a^2 E_{ME}^L(\bar{y} - \bar{Y})^2 + b^2 E_{ME}^L(\hat{Y}_P - \bar{Y})^2
\]

\[
+ 2ab E_{ME}^L(\bar{y} - \bar{Y})(\hat{Y}_P - \bar{Y}).
\]
\[
= a^2 E_M E_D [\beta (\bar{x} - \bar{x}) + (\bar{e} - \bar{E})]^2 + b^2 A : SE\left(\bar{Y}_p\right) \\
+ 2ab E_M E_D [\beta (x^* - \bar{x}) + (\bar{e} - \bar{E})] \left[ a\left(\frac{\bar{x}}{x}\right) - 1\right] \\
+ \beta \left(\frac{\bar{x}}{x} - \bar{x}\right) + (\bar{e} - \bar{E})] \\
= a^2 E_M E_D [\beta^2 (\bar{x} - \bar{x})^2 + (\bar{e} - \bar{E})^2] + b^2 \\
\left[ a^2 A_p + \beta^2 B_p + 2\alpha\beta C_p + \delta D_p \right] + 2ab \\
E_M E_D [\beta^2 (\bar{x} - \bar{x}) (\frac{\bar{x}}{x} - \bar{x}) + \alpha\beta (\bar{x} - \bar{x}) (\frac{\bar{x}}{x} - 1) \\
+ (\bar{e} - \bar{E}) (\frac{\bar{e}x}{x} - \bar{E})] + \text{terms whose model expectation will vanish.} \\
= a^2 b^2 A_p + \beta^2 \left[ b^2 B_p + a^2 E_M E_D (\bar{x} - \bar{x})^2 \\
+ 2ab E_M E_D (\bar{x} - \bar{x}) (\frac{\bar{x}}{x} - \bar{x})\right] + 2\alpha\beta \left[ b^2 C_p + ab E_M E_D (\bar{x} - \bar{x}) (\frac{\bar{x}}{x} - 1)\right]
\]
\[ + \left[ b^2 D_p + a^2 E_M E_D (\bar{\varepsilon} - \bar{\theta})^2 + 2ab \right] \frac{E_M E_D (\bar{\varepsilon} - \bar{\theta}) (\frac{\varepsilon x}{X} - \bar{\theta})}{X} \]

\[ = a^2 A_{dr} + \beta^2 B_{dr} + 2a\beta C_{dr} + \delta D_{dr}, \]

where

\[ A_{dr} = b^2 A_p, \]
\[ B_{dr} = b^2 B_p + a^2 E_M E_D (\bar{x} - \bar{X})^2 + 2ab \]
\[ E_M E_D (\bar{x} - \bar{X}) (\frac{\varepsilon x}{X} - \bar{X}), \]
\[ C_{dr} = b^2 C_p + ab E_M E_D (\bar{x} - \bar{X}) (\frac{\varepsilon x}{X} - 1) \]

and

\[ D_{dr} = b^2 D_p + \frac{1}{6} \left[ a^2 E_M E_D (\bar{\varepsilon} - \bar{\theta})^2 + 2ab \right] \frac{E_M E_D (\bar{\varepsilon} - \bar{\theta}) (\frac{\varepsilon x}{X} - \bar{\theta})}{X}. \]
Here, \( A_p, B_p, C_p \) and \( D_p \) are the coefficients of \( a^2, \beta^2, 2a\beta \) and \( \delta \) respectively in the expression of \( \text{AMSE}(\hat{Y}_p) \) given in (3.3.2).

We below derive the results which will be used to arrive at the expression of \( \text{AMSE}(\hat{Y}_{DR}) \).

\[
E_M E_D [\bar{x}] = E_M [\bar{x}] = 0 \quad (3.3.6)
\]

\[
E_M E_D[\bar{x}^2] = \frac{1}{n^2} E_M E_D \left[ \sum_i x_i \right]^2
\]

\[
= \frac{1}{n^2} E_M E_D \left[ \sum_i x_i^2 + \sum_{i \neq j} x_i x_j \right]
\]

\[
= \frac{1}{n^2} E_M \left[ \frac{N}{n} \sum_{i=1}^N x_i^2 + \frac{n(n-1)}{N(N-1)} \sum_{i \neq j} x_i x_j \right]
\]

\[
= \frac{1}{n^2} \left[ N \frac{\theta + 2}{\theta} + \frac{n-1}{N-1} N(N-1) \frac{\theta^2}{\theta^2} \right]
\]

\[
= \frac{1}{n} \left[ \theta(\theta+1) + (n-1) \theta^2 \right]
\]

\[
= \frac{\theta(n\theta+1)}{n} \quad (3.3.7)
\]
\[ E_M E_D [\bar{X}^2] = E_M [\bar{X}^2] \]

\[ = \frac{1}{N} E_M \left[ \sum_{i=1}^{N} x_i \right]^2 \]

\[ = \frac{1}{N^2} E_M \left[ \sum_{i=1}^{N} x_i^2 + \sum_{i \neq j=1}^{N} x_i x_j \right] \]

\[ = \frac{1}{N^2} \left[ N \frac{\Theta + 2}{\Theta} + N(N-1) \frac{(\Theta + 1)^2}{(\Theta)^2} \right] \]

\[ = \frac{1}{N} \left[ (\Theta + 1) + (N-1)\Theta^2 \right] \]

\[ = \frac{\Theta(N\Theta + 1)}{N} \] (3.3.8)

\[ E_M E_D [\bar{X} \bar{X}] = E_M [\bar{X} E_D(\bar{X})] \]

\[ = E_M [\bar{X}^2] \]

\[ = \frac{\Theta(N\Theta + 1)}{N} \] (using (3.3.8)) (3.3.9)

\[ E_M E_D \left[ \frac{\bar{X}^2}{\bar{X}} \right] = E_M \left[ \frac{1}{\bar{X}} E_D(\bar{X}^2) \right] \]
\[ E_M \left[ \frac{X^3}{X} \right] = E_M \left[ \frac{1}{X} E_D (X^3) \right] \]
\[
= \frac{1}{n^3} E_M \left[ \frac{1}{X} E_D \left( \frac{1}{n} \sum_{i=1}^{N} x_i^3 + 3 \sum_{i \neq j}^{n} x_i^2 x_j \right) \right.
+ \sum_{i \neq j \neq k}^{n} x_i x_j x_k \left. \right] \right]
= \frac{1}{n^3} E_M \left[ \frac{1}{X} \left( \frac{1}{N} \sum_{i=1}^{N} x_i^3 + 3 \frac{n(n-1)}{N(N-1)} \right) \right]
= \frac{1}{n^2} E_M \left[ \sum_{i=1}^{N} x_i^3 \right]
+ \frac{3(n-1)}{N-1} \sum_{i \neq j}^{N} x_i x_j
+ \frac{(n-1)(n-2)}{N(N-1)(N-2)} \sum_{i \neq j \neq k}^{N} x_i x_j x_k \right]
= \frac{1}{n^2} \left[ N \frac{\Theta + 3}{\Theta} \frac{1}{N^{\Theta+2}} + 3 \frac{(n-1)}{N-1} \frac{N(N-1)}{N(N-1)} \right]
\[
\frac{\sigma^2}{(\sigma)^2} \frac{\sigma + 1}{N\sigma + 2} \frac{1}{(N-1)(N-2)} + \frac{(n-1)(n-2)}{N(N-1)(N-2)}
\]

\[
N(N-1)(N-2) \left( \frac{\sigma + 1}{(\sigma)^3} \frac{1}{N\sigma + 2} \right)
\]

\[
= \frac{N}{n^2(N\sigma + 2)} \left[ \sigma (\sigma + 1)(\sigma + 2) + 3(n-1) \sigma^2 (\sigma + 1) + (n-1)(n-2)\sigma^3 \right]
\]

\[
= \frac{N\sigma(n\sigma + 1)(n\sigma + 2)}{n^2(N\sigma + 2)} \quad (3.3.11)
\]

\[
E_{1:1}E_D \left[ \bar{e}^2 \right] = \frac{1}{n^2} E_{1:1}E_D \left[ \sum_i e_i \right]^2
\]

\[
= \frac{1}{n^2} E_{1:1}E_D \left[ \sum_i e_i^2 + \sum_i e_i e_j \right] \quad i \neq j
\]

\[
= \frac{1}{n^2} E_{1:1} \left[ \sum_i N e_i^2 + \frac{n(n-1)}{N(N-1)} \sum_{i \neq j=1}^N e_i e_j \right]
\]
\[ \begin{align*}
\text{EMED} [ \mathbb{E}^2 ] &= \text{EM} [ \mathbb{E}^2 ] \\
\quad &= \frac{1}{N^2} \text{EX} \left[ \sum_{i=1}^{N} e_i \right]^2 \\
\quad &= \frac{1}{N^2} \text{EX} \left[ \sum_{i=1}^{N} e_i^2 + \sum_{i \neq j=1}^{N} e_i e_j \right] \\
\quad &= \frac{1}{N^2} \text{EX} \left[ \sum_{i=1}^{N} E_C \left( \frac{e_i^2}{x_i} \right) \right] \\
\quad &\quad + \sum_{i \neq j=1}^{N} E_C \left( \frac{e_i e_j}{x_i x_j} \right) \\
\quad &= \frac{1}{nN} \sum_{i=1}^{N} E M \left[ \sum_{i=1}^{N} E_C \left( \frac{e_i^2}{x_i} \right) \right] \\
\quad &\quad + \frac{n-1}{N} \sum_{i \neq j=1}^{N} E C \left( \frac{e_i e_j}{x_i x_j} \right) \\
\quad &= \frac{\delta}{nN} \left[ \sum_{i=1}^{N} x_i^t + 0 \right] \\
\quad &= \frac{\delta \theta + t}{\theta} \\
\quad &\quad (3.3.12)
\end{align*} \]
\[
\frac{1}{N^2} E \left[ \sum_{i=1}^{N} x_i + O \right] = \frac{\delta[Q+t]}{N!} (3.3.13)
\]

\[
E_nE_D \left[ \bar{e}E \right] = E_M \left[ \bar{E} E_D (\bar{e}) \right] = E_M \left[ \bar{E}^2 \right]
\]

\[
\frac{\delta[Q+t]}{N!} \text{ (using (3.3.13))} (3.3.14)
\]

\[
E_nE_D \left[ \frac{eEx}{X} \right] = E_M \left[ \frac{E}{X} E_D (\bar{e}x) \right] = \frac{1}{n^2} E_M \left[ \frac{E}{X} E_D \left( \sum_{i}^{n} e_i \left( \sum_{i}^{n} x_i \right) \right) \right]
\]

\[
= \frac{1}{n^2} E_M \left[ \frac{E}{X} E_D \left( \sum_{i}^{n} e_i x_i + \sum_{i \neq j}^{n} e_i x_j \right) \right]
\]

\[
= \frac{1}{n^2} E_M \left[ \frac{E}{X} \left( \sum_{i=1}^{N} e_i x_i + \frac{n(n-1)}{N(N-1)} \sum_{i \neq j=1}^{N} e_i x_j \right) \right]
\]
\[
\begin{align*}
\text{M} & = \frac{1}{nN} \sum_{i=1}^{N} E_{M} \left[ \frac{1}{N} \sum_{i=1}^{N} e_{i} \left( x_{i} + \frac{n-1}{N-1} \sum_{i \neq j=1}^{N} e_{i}x_{j} \right) \right] \\
& = \frac{1}{nN} \sum_{i=1}^{N} E_{M} \left[ \frac{1}{N} \sum_{i=1}^{N} e_{i}^{2}x_{i} + \frac{n-1}{N-1} \sum_{i \neq j=1}^{N} e_{i}e_{j}x_{i} \right] \\
& \quad + \frac{n-1}{N-1} \left( \frac{1}{N} \sum_{i \neq j=1}^{N} e_{i}^{2}x_{j} + \frac{n-1}{N-1} \sum_{i \neq j=1}^{N} e_{i}e_{j}x_{i} \right) \\
& = \frac{1}{nN} \sum_{i=1}^{N} E_{M} \left[ \frac{1}{N} \sum_{i=1}^{N} e_{i}^{2}x_{i} + \frac{n-1}{N-1} \sum_{i \neq j=1}^{N} e_{i}e_{j}x_{i} \right] \\
& \quad + \text{terms whose model expectation will vanish.}
\end{align*}
\]

\[
\begin{align*}
\text{X} & = \frac{1}{nN} \sum_{i=1}^{N} E_{X} \left[ \frac{1}{N} \sum_{i=1}^{N} e_{i}e_{i} \left( x_{i}/x_{i} \right) \right] \\
& \quad + \frac{n-1}{N-1} \sum_{i \neq j=1}^{N} \frac{1}{N} \sum_{i=1}^{N} e_{i}e_{j} \left( x_{i}/x_{i} \right) \\
& = \frac{1}{nN} \sum_{i=1}^{N} E_{X} \left[ \frac{1}{N} \sum_{i=1}^{N} e_{i}e_{i} \left( x_{i}/x_{i} \right) \right] \\
& \quad + \frac{n-1}{N-1} \sum_{i \neq j=1}^{N} \frac{1}{N} \sum_{i=1}^{N} e_{i}e_{j} \left( x_{i}/x_{i} \right)
\end{align*}
\]
\[
\begin{align*}
&= \frac{1}{nN} \sum_{i=1}^{N} \left( \frac{x_{i}^{t+1}}{N} + \frac{1}{N} \sum_{j \neq i}^{n-1} \frac{x_{i}^{t} x_{j}}{N} \right) \\
&= \frac{\delta}{nN} \left[ N \left( \frac{\theta + t}{\theta} \right)^{1/N} + \frac{1}{N} \sum_{i,j=1}^{n-1} \frac{x_{i} x_{j}}{x_{i}} \right] \\
&= \frac{\delta (n \theta + t)}{n (n \theta + t) \theta} \left[ \theta (\theta + t) + (n-1) \theta \right] \\
&= \frac{\delta (n \theta + t)}{n (n \theta + t) \theta} \tag{3.3.15}
\end{align*}
\]
\[
\begin{align*}
&= \frac{1}{n^3} E_M \left[ \frac{1}{N} E_D \left( \sum_{i=1}^{n} \frac{e_i^2 x_i}{N} + \sum_{i \neq j}^{n} \frac{e_i^2 x_j}{n(n-1)} \right) \right] \\
&+ \text{terms whose model expectation will vanish.} \\
&= \frac{1}{n^3} E_M \left[ \frac{1}{N} \sum_{i=1}^{N} e_i^2 x_i + \frac{n(n-1)}{N(N-1)} \sum_{i \neq j=1}^{n} e_i^2 x_j \right] \\
&= \frac{1}{n^2} E_M \left[ \frac{1}{N} \sum_{i=1}^{N} e_i^2 x_i + \frac{n-1}{N-1} \sum_{i \neq j=1}^{n} e_i^2 x_j \right] \\
&= \frac{1}{n^2} E_M \left[ \frac{N}{N} \sum_{i=1}^{N} e_i^2 x_i - \sum_{i=1}^{N} x_i + \frac{n-1}{N-1} \sum_{i \neq j=1}^{n} e_i^2 x_j \right] \\
&= \frac{1}{n^2} E_M \left[ \frac{N}{N} \sum_{i=1}^{N} e_i^2 x_i + \frac{n-1}{N-1} \sum_{i \neq j=1}^{n} e_i^2 x_j \right] \\
&= \frac{1}{n^2} E_M \left[ \frac{N}{N} \sum_{i=1}^{N} e_i^2 x_i + \frac{n-1}{N-1} \sum_{i \neq j=1}^{n} e_i^2 x_j \right] \\
&= \frac{1}{n^2} E_M \left[ \frac{N}{N} \sum_{i=1}^{N} x_i E \left( \frac{e_i^2}{x_i} \right) + \frac{n-1}{N-1} \sum_{i \neq j=1}^{n} e_i^2 x_j \right] \\
&= \frac{1}{n^2} E_M \left[ \frac{N}{N} \sum_{i=1}^{N} x_i E \left( \frac{e_i^2}{x_i} \right) + \frac{n-1}{N-1} \sum_{i \neq j=1}^{n} e_i^2 x_j \right] \\
&= \frac{1}{n^2} E_M \left[ \frac{N}{N} \sum_{i=1}^{N} x_i E \left( \frac{e_i^2}{x_i} \right) + \frac{n-1}{N-1} \sum_{i \neq j=1}^{n} e_i^2 x_j \right]
\end{align*}
\]
where,

\[ A_{dr} = \frac{n}{(N-n)(N+1)} \]

\[ B_{dr} = \frac{\theta}{nN(N-n)(N+2)(N+3)} \left[ \frac{N^2 \theta^2 (N-2n)^2}{N(N-2n)^2} + N\theta(N^2 - nN + n^2) + 6n(N-n) \right] \]

\[ C_{dr} = -\frac{N(N-2n)\theta^2}{(N-n)(N+1)(N+2)} \]

and

\[ D_{dr} = \frac{1}{(N+1)(N+2)} \left[ \frac{\theta(N\theta - n\theta - 1)}{n} + \frac{n\theta}{N-n} + \frac{t(t+1)}{N} \right] \frac{t+1}{\theta} \]

### 3.4 EFFICIENCY COMPARISON

In this section, the efficiency of \( \hat{\gamma}_{DR} \) is compared with \( \hat{\gamma}_P \) under the superpopulation model (3.2.5) for an infinite population and for \( t = 0, 1 \) and 2. In this case,
Next, from (3.3.2) and (3.3.17), we obtain

\[
\text{AMSE}(\hat{Y}_p) - \text{AMSE}(\hat{Y}_{DR}) = \alpha^2 (A_p - A_{dr}) + \beta^2 (B_p - B_{dr}) + 2\alpha\beta (C_p - C_{dr}) + \delta (D_p - D_{dr}),
\]

(3.4.1)

where we find that

\[
A_p - A_{dr} = \frac{1}{n\theta} > 0
\]

\[
B_p - B_{dr} = \frac{(n\theta+3)(3n\theta+2)}{n^2\theta} > 0
\]

\[
C_p - C_{dr} = \frac{2(n\theta+1)}{n^2\theta} > 0
\]

and

\[
D_p - D_{dr} = \frac{1}{n^2\theta} > 0 \quad \text{(when } t = 0)\]

\[
= \frac{3n\theta+2}{n^3\theta} > 0 \quad \text{(when } t = 1)\]

\[
= \frac{(\theta+1)(5n\theta+6)}{n^3\theta} > 0 \quad \text{(when } t = 2)\]
Now, we consider the two cases: \( a = 0 \) and \( a \neq 0 \).

**Case 1.** \( a = 0 \)

In this case, we see that the difference in (3.4.1) is positive for \( t = 0, 1 \) and 2 and for \( \theta \geq 1 \) and \( n \geq 2 \).

**Case 2.** \( a \neq 0 \).

In this case, from (3.4.1), we observe that the difference \( \text{AMSE}(\widehat{Y}_p) - \text{AMSE}(\widehat{Y}_{DR}) \) is positive for \( t = 0, 1 \) and 2 and for \( \theta \geq 1 \) and \( n \geq 2 \) when \( a \) and \( \beta \) are positive or both of them are negative. Thus, \( \widehat{Y}_{DR} \) is better than \( \widehat{Y}_p \) if \( a \times \beta > 0 \).

It is interesting to note that, for an infinite population, the average bias of \( \widehat{Y}_{DR} \) vanishes whereas it is \( \beta/n \) for \( \widehat{Y}_p \). Thus, considering the two main criteria of efficiency comparison i.e. bias and MSE, we see that the dual ratio estimator \( \widehat{Y}_{DR} \) is superior to the usual product estimator \( \widehat{Y}_p \) when regression is through origin \((a=0)\) or when intercept \( \times \) slope.
is positive \((\alpha \times \beta > 0)\).

We further note that the average variance of the unbiased product estimator \(\hat{y}_{Pu}\) as defined in (3.2.4) turns out to be same as \(\text{AMSE}(\hat{y}_p)\) as \(N \rightarrow \infty\) and for large samples (see Chaubey, Singh and Dwivedi (1981)). Thus, the dual ratio estimator \(\hat{y}_{DR}\) is also more efficient than \(\hat{y}_{Pu}\) under the same conditions for which it is more efficient than \(\hat{y}_p\).
Lemma 1

Let $X_1, X_2, \ldots, X_n$ be i.i.d. gamma variates with parameter $\theta$, then for given non-negative integers $a$ and $b$, we have

$$
E \left[ x_i^a \left( \sum_{i=1}^{n} x_i \right)^b \right] = \frac{\theta^{a+b} \prod_{i=1}^{b} (n\theta + a + b - i)}{\Gamma(a+1) \Gamma(b+1)}
$$

The proof is easy and is omitted.

Lemma 2

Let $X_1, X_2, \ldots, X_n$ be i.i.d. gamma variates with parameter $\theta$, then for given non-negative integers $a$, $b$ and $c$, we have

$$
E \left[ \frac{x_i^a x_j^b}{(\sum_{i=1}^{n} x_i)^c} \right] = \frac{\theta^{a+b} \prod_{i=1}^{c} (n\theta + a + b - i)}{(\Gamma(\theta))^c \prod_{i=1}^{c} (n\theta + a + b - i)}
$$

The proof is given by Rao and Webster (1966).
Lemma 3

Let $X_1, X_2, \ldots, X_n$ be i.i.d. gamma variates with parameter $\Theta$, the for given non-negative numbers $m_1, m_2, \ldots, m_p$ and $k$, we have

$$E\left[ \frac{x_1^{m_1} x_2^{m_2} \cdots x_p^{m_p}}{x_k} \right] = \frac{E[ x_1^{m_1} x_2^{m_2} \cdots x_p^{m_p} ] E[ T^{s-k} ] n^k}{E[ T^s ]}$$

where $\{ i_1, i_2, \ldots, i_p \}$ is a subset of $p$ distinct elements from $\{ 1, 2, \ldots, n \}$, $s = \sum_{j=1}^{p} m_j$ and

$$T = \sum_{i=1}^{n} X_i.$$ 

The proof is given by Chaubey, Dwivedi and Singh (1984). This lemma is a generalization of lemma 2.