In this chapter, we present some results pertaining to certain classes of operators on a Hilbert space $H$; viz. the class $(N;k)$, the class of hyponormal operators and that of operators of ascent and descent $0$ or $1$. We first prove that the class $(N;k)$ and the class of normaloid operators are uniformly closed and that the class of operators of ascent $0$ or $1$ is not closed in $B(H)$. We then discuss the similarity invariance of the class $(N;k)$. Lastly, we prove a result on the sum and product of two hyponormal operators.

**Theorem 25.** The class $(N;k)$ of operators is strongly (hence uniformly) closed on $B(H)$.

**Proof.** Let $\left\{ T_n \right\}_{n=1}^{\infty}$ be a sequence of operators of class $(N;k)$ and let $T \in B(H)$ such that $\|T_n x - T x\| \to 0$. 

as \( n \to \infty \) for each \( x \in \mathcal{H} \). Then

\[
\| T_n^k x - T^k x \| \leq \| T_{n-1}^k + T_{n-2}^k + \ldots + T^k \| \cdot \| T_n x - T x \| \\
\to 0 \text{ as } n \to \infty .
\]

This gives

\[
\| T x \|_k = \| \lim_{n \to \infty} T_n^k x \|_k
\]

\[
= ( \lim_{n \to \infty} \| T_n x \|_k )^k
\]

\[
= \lim_{n \to \infty} \| T_n x \|_k
\]

\[
\leq \lim_{n \to \infty} \| T_n^k x \|
\]

\[
= \| \lim_{n \to \infty} T_n^k x \|
\]

\[
= \| T^k x \|
\]

for all \( x \in \mathcal{H} \) with \( \| x \| = 1 \).

Since uniform convergence implies strong convergence,
the result follows.

We have seen that the class \( (N; k) \) is properly
included in both the class $\mathcal{A}$ and the class of all normaloid operators, which are mutually independent. The class $(N; k)$ has been just shown to be uniformly closed, while as we shall shortly see, the class $\mathcal{A}$ is not closed. In this connection, it is interesting to have the following result for the class of normaloid operators.

**Theorem 26.** The class of normaloid operators is uniformly closed in $B(H)$.

**Proof.** An operator $T$ is normaloid if and only if
\[ \| T^n \| = \| T \|^n \] for each $n = 1, 2, 3, \ldots$. If \( \{ T_k \}_{k=1}^\infty \) is a sequence of normaloid operators such that \( \| T_k - T \| \to 0 \) as $k \to \infty$, then for any $n, n = 1, 2, 3, \ldots$,

\[
\| T^n \| = \| ( \lim_{k \to \infty} T_k )^n \| \\
= \| \lim_{k \to \infty} T_k^n \| \\
= \lim_{k \to \infty} \| T_k^n \| \\
= \lim_{k \to \infty} \| T_k \|^n \\
= ( \lim_{k \to \infty} \| T_k \| )^n
\]
Thus $T$ is normaloid.

Next we ask whether the class $\mathcal{A}$ is closed in $B(H)$. In this connection we prove the following

**Theorem 27.** The class of operators of ascent and descent 0 or 1 is not closed in $B(H)$. Consequently, the class $\mathcal{A}$ is not closed even if $\dim H < \infty$.

**Proof.** We prove the theorem by producing two counter-examples. The first example is that of [20, Problem 85] and was suggested to the author by S.M. Patel.

(i) Let $H$ be the Hilbert space of all the two-way square-summable sequences, with the orthonormal basis

$$\{ \ldots, e_2, e_1, e_0, e_1, e_2, \ldots \}$$

where

$$e_n = \{ \ldots, 0, 0, 1, 0, 0, \ldots \}$$

with 1 at the $n$th place and 0 elsewhere. Define $A_k$ for each $k = 1, 2, 3, \ldots$, to be the weighted shift
\[ A_k e_n = \begin{cases} e_{n+1} & \text{if } n \neq 0 \\ \left( \frac{1}{k} \right) e_{n+1} & \text{if } n = 0 \end{cases} \]

where we put \( \frac{1}{\infty} = 0 \). The sequence of weights of \( A_k \) is

\[ \{ \ldots, 1, 1, (\frac{1}{k}), 1, 1, \ldots \} . \]

It can be seen that for each \( k < \infty \), \( A_k \) is invertible so that \( A_k \) is of ascent and descent 0 or 1. As \( k \to \infty \), \( \| A_k - A_\infty \| \to 0 \) where \( A_\infty \) is the weighted shift whose sequence of weights is

\[ \{ \ldots, 1, 1, (0), 1, 1, \ldots \} . \]

Hence: \( A_\infty^2 e_{-1} = A_\infty (1e_0) = 0 \) whereas \( A_\infty e_{-1} = e_0 \neq 0 \), which shows that \( A_\infty \not\in \mathcal{B} \).

(ii) Let \( H \) be the two-dimensional Hilbert space and define \( T_n \) on \( H \) for each \( n = 1, 2, 3, \ldots \) by the matrix

\[ T_n = \begin{bmatrix} 0 & \frac{1}{n} \\ 1 & 0 \end{bmatrix} . \]
If $T$ is defined by

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

then $T_n \to T$ strongly (or uniformly), since for each $n$,

$$T_n(x,y) = (\frac{1}{n} y, x) \text{ for } (x,y) \in \mathcal{H} \text{ and } T(x,y) = (0, x)$$

so that

$$||T_n(x,y) - T(x,y)|| = ||(\frac{1}{n} y, 0)|| = \frac{1}{n} ||y||$$

which tends to 0 as $n \to \infty$. For each $n$, $T_n$ is one-to-one, for

$$T_n(x,y) = 0 \Rightarrow (\frac{1}{n} y, x) = 0 \Rightarrow x = 0 = y,$$

so that $T_n \in \mathcal{A}$. Since $\dim \mathcal{H} < \infty$, $T_n$'s are also of descent 0 or 1. However $T \notin \mathcal{A}$, since $T^2 = 0$ whereas $T \neq 0$. This example shows that the class $\mathcal{A}$ is not closed even if $\mathcal{H}$ is finite-dimensional.

It was shown in Theorem 4 that $\mathcal{A}$ is similarity invariant. Although we are unable to claim that the class $(\mathcal{N};x)$ is also similarity invariant, we can however show that this class is invariant under some special similarity transformations and that the same is true for the class of normaloid operators.
**Theorem 28.** Let $S$ be an invertible operator such that $||S|| = ||S^{-1}|| = 1$.

(i) If $T$ is an operator of class $(N;k)$, so is $S^{-1}TS$.

(ii) If $T$ is normaloid, then so is $S^{-1}TS$.

**Proof.** (i) Since for any $k$, $(S^{-1}TS)^k = S^{-1}T^kS$, for $x \in X$ such that $||x|| = 1$, we obtain

$$||S^{-1}TS\cdot x||^k \leq ||S^{-1}||^k \cdot ||T(Sx)||^k$$

$$= ||T(Sx)||^k \cdot ||S||^k$$

$$\leq ||T^kSx|| \cdot ||S||^{k-1}$$

$$\leq ||S||^{k-1} \cdot ||x||^{k-1} \cdot ||S^{-1}T^kS\cdot x||$$

$$\leq ||(S^{-1}TS)^k\cdot x||.$$

This proves that $S^{-1}TS$ is of class $(N;k)$.

(ii) Since $||T|| = r(T)$, the spectral radius of $T$, we obtain, in view of the fact that similar operators have the same spectrum,
Thus $S^{-1}TS$ is normaloid.

It is well known that the sum and the product of two normal operators are normal if each commutes with the adjoint of the other \cite[270]{44}. Here we intend to impose conditions on the real and imaginary parts of two normal operators so as to ensure the normality of their sum.

If $T_1 = A + iB$ and $T_2 = C + iD$ are two normal operators in their Cartesian forms then it is easily seen by a simple calculation that $T_1 + T_2$ is normal if and only if $AD - DA = BC - CB$. In particular, if $T_1$ and $T_2$ are normal operators such that the real part of each commutes with the imaginary part of the other, then $T_1 + T_2$ is normal. In the case of hyponormal operators, the situation demands closer examination. We in fact, prove the following

**Theorem 29.** Let $T_1$ and $T_2$ be hyponormal operators on $H$. 
(i) If each of $T_1$ and $T_2$ commutes with the adjoint of the other, then $T_1 + T_2$ and $T_1^*T_2$ are hyponormal.

(ii) If the real part of each of $T_1$ and $T_2$ commutes with the imaginary part of the other, then $T_1 + T_2$ is hyponormal.

**Proof.** (i) Under the given conditions, we have

$$(T_1 + T_2)(T_1 + T_2)^* = (T_1 + T_2)(T_1^* + T_2^*)$$

$$= T_1^*T_1 + T_1^*T_2^* + T_2^*T_1 + T_2^*T_2^*$$

$$= T_1^*T_2 + T_1^*T_2^* + T_2^*T_1 + T_2^*T_2$$

$$= (T_1^* + T_2^*)(T_1 + T_2).$$

This proves that $T_1 + T_2$ is hyponormal.

To prove the hyponormality of $T_1T_2$, observe first that for any two self-adjoint operators $S$ and $T$ such that $S \leq T$, we have $R^*SR \leq R^*TR$ for any operator $R$; see [1, page 150]. In fact,

$$(R^*SRx,x) = (SRx,Rx)$$

$$\leq (TRx,Rx)$$

$$= (R^*TRx,x)$$
for all \( x \in H \). Therefore,

\[
(T_1T_2)(T_1T_2)^* = T_1(T_2T_2^*)T_1^*
\]

\[
\leq (T_1T_2^*)(T_2T_1^*)
\]

\[
= T_2^*(T_1T_1^*)T_2
\]

\[
\leq T_2^*(T_1^*T_1)T_2
\]

\[
= (T_2^*T_1^*)(T_1T_2)
\]

\[
= (T_1T_2)^*(T_1T_2).
\]

Thus \( T_1T_2 \) is hyponormal.

(ii) We can verify that an operator \( T = A + iB \) is hyponormal if and only if the imaginary part of \( AB \) is negative; that is \( (AB - BA)/2i \leq 0 \). [1, page 161]. Indeed the hyponormality of \( T \) is equivalent to the condition

\[
(A + iB)(A - iB) - (A - iB)(A + iB) \leq 0
\]

which is the same as

\[
2i(BA - AB) \leq 0.
\]

Thus

\[
\frac{BA - AB}{2i} \leq 0
\]

as asserted above.
Now

\[(T_1 + T_2)(T_1 + T_2)^* = (A + C)^2 + (B + D)^2 + i(B + D)(A + C) - i(A + C)(B + D)\]

and

\[(T_1 + T_2)^*(T_1 + T_2) = (A + C)^2 + (B + D)^2 + i(A + C)(B + D) - i(B + D)(A + C)\]

Therefore,

\[(T_1 + T_2)(T_1 + T_2)^* - (T_1 + T_2)^*(T_1 + T_2)\]

\[= 21 \left[ (B + D)(A + C) - (A + C)(B + D) \right]\]

\[= 21 \left[ (BA - AB) + (BC - CB) + (DA - AD) + (DC - CD) \right]\]

\[= 4 \left[ P + Q \right]\]

where \(P\) and \(Q\) are the imaginary parts of \(AB\) and \(CD\). Since \(P < 0\) and \(Q < 0\) imply \(P + Q < 0\), the proof is complete.