In this chapter, we study operators in the class whose descent is also 0 or 1. Since the condition that an operator should be of ascent 0 or 1 is rather very general, it is natural to impose certain further restrictions on operators in the class $A$, so as to obtain stronger results. An operator may belong to $A$ without having finite descent. An example is provided by the unilateral shift $U$ defined on the space $l^2$ of square-summable complex sequences by $U(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots)$. Since $U$ is isometric, $N(U) = N(U^2) = \{0\}$, so that $U \in A$. But since $R(U^2)$ is a proper subspace of $R(U)$, the descent of $U$ is not 0 or 1, and hence by Theorem 5.41-B of Taylor [47], the descent is not finite.

Although the condition for an operator in $A$ to have
Theorem 12. Let \( \dim X < \infty \), then \( T \in \mathcal{A} \) if and only if \( T \) is of descent 0 or 1.

Proof. It is clear that every operator \( T \) on a finite-dimensional space \( X \) is a Fredholm operator of index zero. Indeed, \( R(T) \) being finite-dimensional, is closed and \( n(T) \) and \( d(T) \) are both finite. Also the relations \( \dim X = \dim N(T) + \dim R(T) \) and \( \dim X = \dim R(T) + \dim R(T)^\perp \) yield \( n(T) = d(T) \). It now follows from Theorem 4.5(c) and (d) of Taylor [48] that \( T \in \mathcal{A} \) if and only if \( s(T) = 0 \) or 1.

The proof of Theorem 12 depends on the fact that in a finite-dimensional space, every operator is a Fredholm operator of index zero. Since not every operator is Fredholm if \( X \) is not finite-dimensional, it is natural to ask for conditions under which an operator is a Fredholm operator of index zero. One such condition is given in the following

Theorem 13. Let \( T \in \mathcal{A} \) and \( s(T) \) be finite. Then \( T \) is a Fredholm operator of index zero if and only if there exists an operator \( A \) such that \( AT - I \) is compact.
We shall show that this theorem follows as a consequence of the following lemma which is contained in [13, Theorem 1.1].

**Lemma 13.1.** The following conditions on an operator $T$ are equivalent:

(a) Either $R(T)$ is not closed or $N(T)$ is not finite-dimensional.

(b) There cannot exist any operator $A$ such that $AT - I$ is compact.

**Proof of Theorem 13.** It will be shown in Theorem 17 that if $T \in \mathcal{A}$ and $\delta(T)$ is finite, then $R(T)$ is closed. Thus if $AT - I$ is compact for some operator $A$, then by Lemma 13.1, it follows that $n(T)$ is finite. But then by Corollary 4.4 of Taylor [48], $d(T)$ is also finite and in fact, $n(T) = d(T)$. Conversely, if $T$ is a Fredholm operator of index zero, then again in virtue of the Lemma, the finiteness of $n(T)$ and the closedness of $R(T)$ yields an operator $A$ such that $AT - I$ is compact.

As was seen at the beginning of this chapter, an operator may belong to $\mathcal{A}$ without having finite descent. In our next three theorems, we discuss conditions under which an operator in $\mathcal{A}$ should be of descent 0 or 1.
**THEOREM 14.** Let $X$ be reflexive. If $T$ is an operator of ascent 0 or 1, then $R(T^*) = R(T^{*2})$.

Consequently, if $X = H$, a Hilbert space, then $T \in \mathcal{A}$ if and only if $R(T^*) = R(T^{*2})$.

**Proof.** Since $X$ is reflexive, the relation $(\perp N^*)^\perp = N^*$ holds for any subspace $N^*$ of $X^*$.

[18, Theorem II.3.5]. Also $R(T^*) = N(T)^\perp$ for any operator $T$. [18, Theorem II.3.8]. Thus

$$R(T^*) = N(T)^\perp = N(T^{*2})^\perp = (\perp R(T^{*2}))^\perp = R(T^{*2}).$$

Since a Hilbert space is reflexive, it follows in virtue of Theorem 1 (Chapter I) that $T \in \mathcal{A}$ if and only if $R(T^*) = R(T^{*2})$.

For any operator $T$, $R(T)$ is closed if and only if $R(T^*)$ is closed [18, Theorem IV.1.2]. Also $T^2$ is Fredholm whenever $T$ is [18, Theorem IV.2.7]. This leads to the following

**COROLLARY 14.1.** Let $T$ be a Fredholm operator. Then $T \in \mathcal{A}$ if and only if $T^*$ is of descent 0 or 1.

**THEOREM 15.** (1) If $T$ is an operator such that $T^*$ is of ascent 0 or 1, then $R(T) = R(T^2)$. If, in particular, $T$ is Fredholm, then $\delta(T) = 0$ or 1.
(ii) Let $T$ be a Fredholm operator of index zero. Then $T \in \mathcal{A}$ if and only if $\delta(T) = 0$ or 1.

**Proof.** (i) Since $R(T) = R(T^*) = \mathbb{N}(T)$ by [18, Theorem II.3.7], we obtain $R(T) = \overline{R(T)} = \mathbb{N}(T^*) = \overline{\mathbb{N}(T^*)} = \overline{\mathbb{N}(T^{*2})} = \overline{\mathbb{N}(T^{2})} = R(T^2)$.

If, now, $T$ is Fredholm, then $R(T)$ and $R(T^2)$ are both closed and we are done.

(ii) Since $n(T)$ and $d(T)$ are finite and equal, invoking Theorem 4.5(c) and (d) of [48], it follows that $T \in \mathcal{A}$ if and only if $\delta(T)$ is 0 or 1.

As an immediate consequence we obtain the following

**Corollary 15.1.** If $T$ and $T^*$ have both ascent (or descent) equal to 0 or 1, then $n(T) = n(T^2)$ and $d(T) = d(T^2)$. If, in addition, $T$ is a Fredholm operator of index zero, then all the four quantities $n(T)$, $n(T^2)$, $d(T)$ and $d(T^2)$ are equal.

Our next theorem gives a necessary and sufficient condition that a normal operator should be of descent 0 or 1.

**Theorem 16.** Let $T$ be a normal operator on $H$. $T$ is of descent 0 or 1 if and only if $T^*T$ is of descent 0 or 1.
PROOF. It has been shown in [12, Theorem 2.2] that for any operator $A$, $R(A) = R((AA^*)^{1/2})$. This fact, together with our hypothesis yields $R(T^2) = R(T^2 T^2)^{1/2} = R(T^4) = R((T^4 T^4)^{1/2}) = R(T^8)$. Since $R(T^4) \subseteq R(T^3) \subseteq R(T^2)$, it follows that $R(T^2) = R(T^3)$. Thus $\delta(T) \leq 2$. From Theorem 5.41-E of Taylor [47] it follows that $\delta(T) = \alpha(T) = 0$ or 1. Conversely if $\delta(T) = 0$ or 1, then clearly $R(T^2) = R(T^4)$. By another application of [12, Theorem 2.2] we obtain $R(T^6) = R((T^6 T^6)^{1/2})$. This completes the proof.

Remark. The above theorem does not remain true for a non-normal operator even if it is hyponormal. In fact, if $U$ is the unilateral shift on the space $l^2$, then $U$ is isometric so that $\alpha(U) = 0$. Also $U^*U = I$ so that $\delta(U^*U) = \delta(I) = 0$; but the descent of $U$ is not finite.

THEOREM 17. Let $T \in \mathcal{A}$ and $\delta(T)$ be finite. Then $R(T)$ is closed.

PROOF. By our hypothesis and Theorem 5.41-G of Taylor [47], it follows that $\alpha(T) = \delta(T) = 0$ or 1 and that $X = N(T) \oplus R(T)$. It has been proved in
that if $N$ is a closed subspace of $X$ such that $R(T) \oplus N$ is closed, then $R(T)$ is closed. Since $N(T)$ is always closed, the closedness of $R(T)$ follows.

**Corollary 17.1.** Let $T$ be an operator such that $T - \lambda I \in \mathcal{A}$ for all scalars and $\delta(T - \lambda I)$ be finite for all $\lambda \neq 0$. If one of the quantities $n(T - \lambda I)$ and $d(T - \lambda I)$ is finite, then $T$ is a Riesz operator.

**Proof.** Caradus has characterized Riesz operators $T$ by the condition that for each non-zero scalar $\lambda$, $T - \lambda I$ should have finite ascent, descent, nullity, defect and closed defect. By Theorem 17, $R(T - \lambda I)$ is closed, so that $d(T - \lambda I) = \delta(T - \lambda I)$ for each $\lambda \neq 0$. The corollary follows now by the above mentioned characterization of Caradus.

Ch. Constantin has shown that any restriction-normaloid Riesz operator is normal. This leads to the following

**Corollary 17.2.** If $T$ is a restriction-normaloid operator satisfying the conditions of Corollary 17.1, then $T$ is normal.

In the following theorem we obtain a relation between
operators in the class $c^1$ and commutators of operators.

**THEOREM 18.** Let $T$ be an operator of ascent 0 or 1 on $H$. If $\delta(T)$ is not finite, then $T$ is a commutator.

**PROOF.** Assume, to the contrary, that $T$ is not a commutator. Then by the well known Brown-Pearcy characterization of commutators $[6]$, $T$ can be expressed in the form $\lambda + C$ where $\lambda$ is a non-zero scalar and $C$ is compact. But then by Theorem 5.5-2 of $[47]$, we obtain $\sigma(T) = \delta(T)$. This contradiction proves the theorem.

It was proved in $[37, \text{Theorem 3}]$ that if $T \in c^1$ and $R(T)$ is not closed, then $T$ is a commutator. This result is immediate in virtue of the following stronger result.

**THEOREM 19.** If $T$ is an operator on $H$ whose range is not closed then $T$ is a commutator.

**PROOF.** Berberian $[3]$ has shown that if $0 \in w(T)$, $w(T)$ being the Weyl's spectrum of $T$, then $T$ is a commutator. Thus it suffices to show that if $R(T)$ is not closed then $0 \in w(T)$. In fact, if $R(T)$ is not closed then clearly $T$ is not a Fredholm operator of index zero. The result now follows from Schechter’s characterization of $w(T)$
as the set $w(T) = \{ \lambda | T - \lambda I \text{ is not a Fredholm operator of index zero}\}$.

We obtain a corresponding result for self-commutators in the following

**Theorem 20.** If $T$ is a self-adjoint operator on a separable Hilbert space $H$ such that $R(T)$ is not closed, then $T$ is a self-commutator.

**Proof.** Radjavi [33] has characterized self-commutators on a separable Hilbert space $H$ as follows: An operator $T$ on $H$ is a self-commutator if and only if $T$ cannot be expressed as $\lambda + C$ where $\lambda$ is a non-zero positive scalar and $C$, a compact operator. If, now, $T$ is not a self-commutator, then by Radjavi's characterization, there exists a non-zero positive number $\lambda$ and a compact operator $C$ such that $T = \lambda + C$. This implies in virtue of [47, Theorem 55, E] and Theorem 17 that $R(T)$ is closed, a contradiction.

In the following theorems we discuss for an operator $T$ of descent (ascent) 0 or 1, on $H$, conditions under which the compactness of some power of $T$ implies that of $T$.

**Theorem 21.** Let $T \in \mathcal{A}$ and $\delta(T)$ be finite. If
$T^k$ is compact for some $k$, then $T$ is finite-dimensional.

**Proof.** It is clear from the hypothesis that $T^k$ has ascent and descent equal to 0 or 1 and that $R(T^k) = R(T)$. Further, if the range of a compact operator is closed, then it is finite-dimensional [18, Theorem III.1.12]. Thus it only remains to observe that, in virtue of Theorem 17 and the compactness of $T^k$, $R(T^k)$ is finite-dimensional.

We obtain a more general result in the following

**Theorem 22.** Let $T \in \mathcal{A}$ and $R(T^k)$ be closed for some $k$. If $T^k$ is compact then $T$ is finite-dimensional.

**Proof.** We first observe that an operator $A$ is finite-dimensional (compact) if and only if $A^*$ is finite-dimensional (compact). [47, Theorem 5.5-B]. Further, if $A$ is a compact operator with closed range, then $A$ is finite-dimensional. Since $N(T^j)$ are equal for all $j \geq 1$ and $R(T^k)$ is closed it follows that

$$R(T^*) = R(T^{*2}) = \ldots = R(T^{*k}) = R(T^k).$$

Also $T^{*k}$ is finite-dimensional, being a compact operator with closed range; so that
This proves that $T^*$, and hence $T$, is finite-dimensional.

Remarks. 1. As was already observed, for the unilateral shift $U$, the range of every power of $U$ is closed, but the descent of $U$ is not finite. Thus Theorem 22 is a proper generalization of Theorem 21.

2. It has been proved in [14, Theorem 2.5] that if $V$ is a linear subspace of $H$, then any operator $A$ on $H$ with $R(A) \subseteq V$ is compact if and only if $V$ contains no closed infinite-dimensional subspace of $H$. From this result we deduce that if $T$ is an operator of descent 0 or 1 and $T^k$ is compact, then $T$ is finite-dimensional. In fact

$$R(T) = R(T^2) = \ldots = R(T^k) \subseteq R(T^k) = V,$$

and $T^k$ is compact we obtain $\dim R(T) = \dim R(T^k) \leq \dim R(T^k) < \infty$. 

\[ \dim R(T^*) \leq \dim R(T^*) \leq \dim R(T^k) < \infty. \]