CHAPTER 4

A K-OUT-OF-N : G SYSTEM - OPTIMAL REPLACEMENT POLICY (N*, n* ) UNDER COMMON CAUSE FAILURES WITH REPAIR PROVISION

4.1 INTRODUCTION

The literature on K - out - of - N systems revealed the fact that the repairable systems which are over looked in the past gained the attention of the researchers in the recent years for which reason that it is much more common than non - repairable system.

Nakagawa T (1980) established a K - out - of - N redundant systems determining the optimal number of units N* and with optimal replacement time T* based on the conventional cost consideration.

Generally the K - out - of - N system models assumes that the redundant units fail independently. The stochastical assumption of independent failures does not hold good for a complex redundant system as this system failure occur due to design deficiencies, abnormal environmental conditions (for example high temperature, humidity, vibration changes, dust, etc..) called as Common Cause Failures (CCF’s). Keeping in view of the importance of these failures, models are established for complex reliability systems. A brief history of these systems are referred here.

P.J. Boland and F. Proschan (1983), considered a system subject to shocks occurring according to a non - homogeneous Poisson process. It is assumed that occurrence of each shock causes additive damage to the system and increases the operating cost by constant value ‘C’ per unit time. For this system the optimal replacement period that minimises the long run expected cost per unit of time is developed.
CHAPTER 4

A K-OUT-OF-N: G SYSTEM-OPTIMAL REPLACEMENT POLICY (N*, m*) UNDER COMMON CAUSE FAILURES WITH REPAIR PROVISION
Considering the common cause failures D.S. Bai, W.Y. Yun and S.W. Chung (1991),
developed redundancy optimization of K-out-of-N systems with common cause failures. 
H.E. Ascher (1983) presented discussions in lawless procedures. H.E. Ascher and Felgond
(1984) proposed Reliability modelling repairable systems with misconceptions and their causes
with its inferences.

For a single unit system. T. Nakagawa and S. Osaki (1974) suggested the optimum
repair replacement policies. L. Rade (1976) considered a parallel reliability system which
consists of N identical components situated in a random environment. For this results mean
time before failures are developed.

S. Osaki (1985) established a stochastic system reliability model and proposed the
mathematical models under difficult conditions in a novel way. L. Yeh (1983) presented a detailed
note on a class of optimal replacement problems. In recent years CCFs have drawn considerable
attention in theory of reliability. In Chapter four and five the author bestowed attention to this
aspect.

However, by nature, to study and solve many of the problems that arise in reliability
theory, the methods of probability theory and mathematical statistics are necessary.

In the present chapter the author incorporates the most important repair cost into the
model and suggests a procedure for obtaining optimal repair stage $n^*$, the optimal redundant unit
$N^*$ and the optimal pair $(N^*, n^*)$ and the proposed model is demonstrated through a case study
problem.

Here an attempt is made practically to suggest a procedure for obtaining
optimal repair stage $n^*$, the optimal redundant unit $N^*$ and the optimal pair $(N^*, n^*)$. Numerical
results are presented in the end.
4.2 **THE MODEL**

Consider a K-out-of-N:G system with repairable units that are subject to both random and common-cause failures i.e., the system failure results from both random failures and common-cause failures. The system consists of N identical units in which at least K-out-of-N units should be good in order to be operating a system successfully. The system is repaired after system failures and is allowed to undergo a prefixed number (say, n) of repairs. The following assumptions are made.

**ASSUMPTIONS**

1. The time span is infinite.
2. Random failures and CCFs are mutually statistically independent.
3. The successive repair times after each system failure \( \{w_i, i=1,2,\ldots\} \) constitute a non-decreasing geometric process. A sequence of random variables \( \{w_i, w_j, \ldots\} \) and for some \( \alpha > 0 \), \( \alpha^{-1} w_i, i=1,2,\ldots \) forms a renewal process. Then \( \{w_i, i=1,2,\ldots\} \) is called a geometric-process (G.P) and \( \alpha \) is the parameter of the geometric process. The G.P is called a non-increasing G.P if \( \alpha \geq 1 \) and a non-decreasing G.P if \( \alpha \leq 1 \).
4. The maintenance cost of the system during \( \mu_k \) (N, \( \beta \)) units of time increases with the number of components \( k \).
5. A failure system after each repair will becomes “as good as new”, i.e. the mean time to failure (MTTF) of the system remains same.
6. Each failure of the system can be identified and monitored continuously.
7. Each unit of the system has the same failure time distribution.
8. Probability distribution function of Common Cause Failures and random failure time are exponential.
NOTATION

C₁  Acquisition cost of each unit.
C₂  Repair cost per unit time for the system.
λₗ  Hazard rate of random failure
λₑ  Common - Cause failure rate
β  \( \frac{λₑ}{λₗ} \)
C(N, n)  Long-run average cost per unit time
Rₑ(t)  Reliability of the system subject to CCFs and random failures
μₖ(N, β)  MTTF of the system subject to CCFs and random failures

The maintenance cost of the system during \( μₖ(N, β) \) units of time, \( a > 0 \), where \( a \) is a constant.

\( wᵢ \)  Random repair time after the \( i^{th} \) system failure
\( i=1,2..., n \) and \( E(wᵢ) = τ \).

n  Number of repairs, \( n=1,2,........ \)

4.3 THE SOLUTION

The total expected cost incurred in a cycle (i.e., up to \( n \) repairs) for the system is

\[ NCᵢ+(n+1)Cᵢ \ e^{-αk} + Cᵢ \ \frac{N}{\sum_{i=1}^{n} (1/α)^{i+1}} \]  ... 4.1

The expected length of a cycle is

\[ (n+1) \ μₖ(N, β) + \frac{N}{\sum_{i=1}^{n} (1/α)^{i+1}} \]  ... 4.2

Where \( μₖ(N, β) = \int_{0}^{α} Rₑ(t) \ dt \)

\[ = \frac{N!}{\lambda \ \Gamma(N+β+1)} \ \frac{\sum_{i=k}^{N} \Gamma(i+β)}{i!} \]  ... 4.3
Since \( R_c(t) = \left\{ \sum_{i=1}^{N} \binom{N}{i} (e^{-t})^i (1-e^{-t})^{N-i} \right\} e^{-t} \)

using the assumption (viii)

The assumption (iii) on the model implies

\[ E(w_i) = E(\alpha w_i) = E(\alpha^2 w_i) \ldots = \tau \]

Since \( \{\alpha^{i-1}w_i, i = 1,2,\ldots\} \) forms renewal process.

Therefore,

\[ E(w_i) = \tau, \ E(w_j) = \tau/\alpha, \ E(w_k) = \tau/\alpha^2, \ldots \]

and \[ E(w_i) \leq E(w_j) \leq E(w_k), \ldots \] 4.4

From the renewal arguments (6), the long-run average cost per unit time is

\[
C(N, n) = \frac{NC_1 + (n+1) C_2 e^{n/k} + C_2 \tau \sum_{i=1}^{n} (1/\alpha)^{i-1}}{(n+1) \mu_k (N, \beta) + \tau \sum_{i=1}^{n} (1/\alpha)^{i-1}} \] 4.5

First we state and prove the following Lemma

**Lemma 1.** \( \{ \mu_k (N+1, \beta) - \mu_k (N, \beta) \} \) is decreasing in \( N \) and

\[
\lim_{N \to \alpha} \left\{ \mu_k (N+1, \beta) - \mu_k (N, \beta) \right\} = 0
\]

**Proof:** Using (4.3), we get

\[
\{ \mu_k (N+1, \beta) - \mu_k (N, \beta) \} = \frac{1}{\lambda_c (N + \beta + 1)} - \beta \frac{\mu_k (N, \beta)}{(N + \beta + 1)}
\]
\[
\begin{align*}
\left[ \mu_x(N+2, \beta) - \mu_x(N+1, \beta) \right] &= \\
&\frac{1}{\lambda_x(N+\beta+2)} - \beta \left[ \frac{1}{\lambda_x(N+\beta+1)} + \frac{(N+1)}{(N+\beta+1)} \mu_x(N, \beta) \right]
\end{align*}
\]

Therefore \( \left[ \mu_x(N+1, \beta) - \mu_x(N, \beta) \right] \geq \left[ \mu_x(N+2, \beta) - \mu_x(N+1, \beta) \right] \)

Thus \( \left\{ \mu_x(N+1, \beta) - \mu_x(N, \beta) \right\} \) is decreasing in \( N \) ... 4.6

Further \( \lim_{N \to \infty} \left[ \mu_x(N+1, \beta) - \mu_x(N, \beta) \right] = 0 \) ... 4.7

Since \( \lim_{N \to \infty} \mu_x(N, \beta) = 1/\lambda_x \)

Thus (4.6) and (4.7) establish the Lemma.

We now obtain the optimal repair stage (ie., optimal number of repairs) \( n^* \) through theorem 1, the optimal number of redundant units \( N^* \) for a given \( n \) through theorem 2 and the optimum pair \( (N^*, n^*) \) using an algorithm on the basis of theorems 1 and 2.

**THEOREM 1**: The \( n^* \) which minimises \( C(N, n) \) in (4.5) satisfies the following pair of inequalities

\[
L(n) > C_1 / C_2 \quad \text{...4.8}
\]

and \( L(n-1) < C_1 / C_2 \quad \text{...4.9} \)

where

\[
L(n) = \frac{\tau \mu_x(N, \beta) \left[ (n+1) (1/\alpha)^n - \sum_{i=1}^n (1/\alpha)^{i-1} \right]}{N \mu_x(N, \beta) + N \tau (1/\alpha)^n + \tau e^{-\tau/k} \left[ (n+1) (1/\alpha)^n - \sum_{i=1}^n (1/\alpha)^{i-1} \right]} \quad \text{...4.10}
\]

Further, \( n^* \) is finite and unique.
\textbf{Proof:} First we note that the domain of the function \( C(N, n) \) consists of non-negative or positive integers i.e., \( C(N, n) \) is discrete in \( n \). Hence to obtain \( n^* \) which minimizes \( C(N, n) \). We form the pair of inequalities.

\[ C(N, n + 1) > C(N, n) \quad \ldots \quad (4.11) \]

and

\[ C(N, n) < C(N, n - 1) \quad \ldots \quad (4.12) \]

using (4.5) in (4.11), we obtain the following inequality

\[
\frac{N C_1 + (n+2) C_2 e^{-s/k} + C_2 \tau \sum_{i=1}^{n+1} (1/\alpha)^{i-1}}{(n+2) \mu_\alpha (N, \beta) + \tau \sum_{i=1}^{n+1} (1/\alpha)^{i-1}} > \frac{N C_1 + (n+1) C_1 e^{-s/k} + C_2 \tau \sum_{i=1}^{n} (1/\alpha)^{i-1}}{(n+1) \mu_\alpha (N, \beta) + \tau \sum_{i=1}^{n} (1/\alpha)^{i-1}}
\]

which after some simplifications gives

\[
\frac{\tau \mu_\alpha (N, \beta) \left\{(n+1) (1/\alpha)^n - \sum_{i=1}^{n} (1/\alpha)^{i-1}\right\}}{N \mu_\alpha (N, \beta) + N \tau (1/\alpha)^n + e^{-s/k} \tau \left\{(n+1) (1/\alpha)^n - \sum_{i=1}^{n} (1/\alpha)^{i-1}\right\}} > \frac{C_1}{C_2},
\]

That is,

\[ L(n) > \frac{C_1}{C_2} \quad \ldots \quad (4.13) \]

A similar procedure substituting (4.5) in (4.12) leads to

\[ L(n - 1) < \frac{C_1}{C_2} \quad \ldots \quad (4.14) \]

The first part of the theorem is thus proved through inequalities (4.13) and (4.14).
To show that $n^*$ is finite and unique, it is sufficient to show that $L(n)$ is strictly increasing in $n$, i.e., $[L(n + 1) - L(n)] > 0$ for every $n$ and further tends to $+\alpha$ as $n \to \alpha$. This is shown in the following.

$$
\begin{align*}
L(n + 1) - L(n) &= \mu^2(N, \beta) \left[ \frac{(n + 2)(1 - \alpha)}{\alpha^{n+1}} \right] \\
&+ \frac{1}{\alpha} \left[ (1 / \alpha)^n + \sum_{i=1}^{n} \left\{ (1 / \alpha)^i \right\} \left\{ (1 - \alpha) \right\} (1 - \alpha) > 0 \right. \quad \ldots \text{4.15}
\end{align*}
$$

using the assumption that $0 < \alpha < 1$. Hence, $L(n)$ is increasing in $n$.

Further, $\lim_{n \to \alpha} L(n) \to \alpha$

Therefore $L(n)$ crosses the finite value $C_1 / C_2$ just once.

This completes the proof of the theorem.

**THEOREM 2:** The $N^*$ which minimises $C(N, n)$ given in (4.5) satisfies the pair of inequalities.

$$
L(N) > C_2 / C_1 \quad \ldots \text{4.16}
$$

and

$$
L(N-1) < C_2 / C_1 \quad \ldots \text{4.17}
$$

$$
L(N) = \frac{\mu^2(N, \beta)}{\tau \sum_{i=1}^{n} (1/\alpha)^i \left\{ \mu(N+1, \beta) - \mu(N, \beta) \right\}} - \frac{N + (n+1)e^{-\alpha}}{\tau \sum_{i=1}^{n} (1/\alpha)^i} \quad \ldots \text{4.18}
$$

Further $N^*$ is finite and unique
**Proof**: Since \( C(N, n) \) is discrete in \( N \), we obtain \( N^* \) which minimizes \( C(N, n) \) through the pair of inequalities.

\[
C(N + 1, n) > C(N, n) \quad \ldots \quad 4.19
\]

and

\[
C(N, n) < C(N - 1, n) \quad \ldots \quad 4.20
\]

using equation (4.5) in equation (4.16), we obtain

\[
(N+1) C_i + (n+1) C_1 e^{-\kappa/k} + C_2 \tau \sum_{i=1}^{n} \left( 1 / \alpha \right)^{i-1} > (n+1) \mu_N (N+1, \beta) + \tau \sum_{i=1}^{n} \left( 1 / \alpha \right)^{i-1}
\]

\[
\frac{NC_i + (n+1) e^{-\kappa/k} + C_1 \tau \sum_{i=1}^{n} (1/\alpha)^{i-1}}{(n+1) \mu_N (N, \beta) + \tau \sum_{i=1}^{n} (1/\alpha)^{i-1}}
\]

After some simplifications,

\[
\frac{\mu_N (N, \beta)}{\tau \sum_{i=1}^{n} (1/\alpha)^{i-1}} - \frac{N + (n + 1) e^{-\kappa/k}}{\tau \sum_{i=1}^{n} (1/\alpha)^{i-1}}
\]

\[
\frac{1}{(n+1) \left\{ \mu_N (N+1, \beta) - \mu_N (N, \beta) \right\}}
\]

i.e., \( L(N) > C_2 / C_1 \) \quad \ldots \quad 4.21

A similar procedure using (4.5) in (4.17) leads to

\[
L(N - 1) < C_2 / C_1 \quad \ldots \quad 4.22
\]

The first part of the theorem is thus established through (4.21) and (4.22)
To prove the later part of the theorem, that $N^*$ is finite and unique, it is sufficient to show that $L(N)$ is strictly increasing in $N$ and further tends to $+\alpha$ as $N \to \alpha$. This is shown in the following.

$$
\begin{bmatrix}
\frac{1}{\mu_k(N+2,\beta) - \mu_k(N+1,\beta)} - \frac{1}{\mu_k(N+1,\beta) - \mu_k(N,\beta)} \\
\frac{\mu_k(N+1,\beta)}{\tau \sum_{i=1}^{n} (1/\alpha)^{i-1}} + \frac{1}{n+1}
\end{bmatrix} > 0
$$

By Lemma '1'.

Hence $L(N)$ is increasing in $N$.

Further, 

$$
\lim_{N \to \alpha} L(N) = \alpha
$$

The proof of the theorem is complete.

To obtain the optimum pair $(N^*, n^*)$, we develop the following algorithm on the basis of theorem 1 and 2.

**ALGORITHM:**

The following steps that forms algorithm leads to the optimal pair $(N^*, n^*)$.

Step 1. Set $N = k + 1$ and find $n_1^*$ using theorem 1

Step 2. Set $n = n_1^*$ in step 1 and find $N_2^*$, if it exits using theorem '2'.
Step 3. Continue until $N_i = N_i + 1$, where $i$ represents $i^{th}$ iteration and at this stage $(N^*, n^*)$ is the optimal pair.

Step 4. In step 2, if $N_i$ does not exist, then set $N = k + 2$ in step 1, if again $N_i$ (for $N = k + 2$) does not exist, set $N_i = k + 3$ and so on and the steps may be continued as given in steps 1 - 3 above.

Thus, when the iterations stop, the $(N^*, n^*)$ at ultimate stage represents the optimal pair.

4.4 NUMERICAL RESULTS:

We compute optimal repair stage i.e., optimal number of repairs $n^*$, optimal number of units $N^*$, and optimal pair $(N^*, n^*)$ including corresponding resulting costs, through theorem 1, theorem 2 and algorithm respectively for $k = 4$, $\tau = 0.3$, $\alpha = 0.8$ and $\beta = 2$ and present these in tables 1, 2 and 3.
| Table 1: $C_1 / C_2$, $n^*$ and $C(N, n^*) / C_1$ |
|---|---|---|---|
| $C_1 / C_2$ | $N = 5$ | $N = 6$ | $N = 7$ |
| | $n^*$ | $C(N, n^*) / C_1$ | $n^*$ | $C(N, n^*) / C_1$ | $n^*$ | $C(N, n^*) / C_1$ |
| $\beta = 0$ | | | | | | |
| 0.1 | 4 | 4.1915 | 4 | 3.6722 | 4 | 3.3782 |
| 0.2 | 7 | 6.7805 | 6 | 5.7196 | 6 | 5.1763 |
| 0.3 | 10 | 10.8074 | 8 | 8.7563 | 8 | 8.4559 |
| 0.4 | 14 | 16.3178 | 11 | 13.3179 | 10 | 11.5937 |
| $\beta = 0.4$ | | | | | | |
| 0.1 | 4 | 4.4965 | 4 | 4.0462 | 4 | 3.8087 |
| 0.2 | 7 | 7.1615 | 7 | 6.0411 | 7 | 5.9437 |
| 0.3 | 11 | 11.6326 | 9 | 9.8439 | 9 | 9.1673 |
| 0.4 | 17 | 17.8614 | 13 | 15.2408 | 12 | 13.8730 |
| $\beta = 0.6$ | | | | | | |
| 0.1 | 4 | 4.6404 | 4 | 4.2251 | 4 | 4.0176 |
| 0.2 | 8 | 7.5174 | 7 | 6.6357 | 7 | 6.2075 |
| 0.3 | 12 | 12.1649 | 10 | 10.5553 | 10 | 9.9579 |
| 0.4 | 19 | 18.5248 | 14 | 16.0477 | 13 | 14.8607 |

| Table 2: $C_2 / C_1$, $N^*$ and $C(N^*, n) / C_1$ |
|---|---|---|---|
| $C_2 / C_1$ | $n = 2$ | $n = 3$ | $n = 4$ |
| | $N^*$ | $C(N^*, n) / C_1$ | $N^*$ | $C(N^*, n) / C_1$ | $N^*$ | $C(N^*, n) / C_1$ |
| $\beta = 0$ | | | | | | |
| 5 | 11 | 3.8230 | 12 | 3.2366 | 14 | 2.9112 |
| 10 | 13 | 4.8092 | 16 | 4.0717 | 19 | 3.8136 |
| 15 | 15 | 5.2523 | 19 | 4.8212 | 23 | 4.6206 |
| $\beta = 0.4$ | | | | | | |
| 5 | 9 | 4.6161 | 10 | 3.9412 | 11 | 3.5716 |
| 10 | 10 | 5.6787 | 12 | 5.1360 | 14 | 4.8796 |
| 15 | 12 | 6.6629 | 14 | 6.2385 | 16 | 6.0813 |
| $\beta = 0.6$ | | | | | | |
| 5 | 8 | 4.9697 | 9 | 4.2481 | 10 | 3.8635 |
| 10 | 9 | 6.1777 | 11 | 5.6099 | 12 | 5.3468 |
| 15 | 11 | 7.3048 | 12 | 6.8762 | 14 | 6.7317 |
Table 3: $C_1 / C_2$, $(N^*, n^*)$ and $C(N^*, n^*) / C_1$

<table>
<thead>
<tr>
<th>$C_1 / C_2$</th>
<th>$N^*$</th>
<th>$n^*$</th>
<th>$C(N^<em>, n^</em>) / C_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta=0.4$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>16</td>
<td>7</td>
<td>3.7576</td>
</tr>
<tr>
<td>0.18</td>
<td>15</td>
<td>7</td>
<td>3.3799</td>
</tr>
<tr>
<td>0.20</td>
<td>15</td>
<td>8</td>
<td>3.1732</td>
</tr>
<tr>
<td>0.22</td>
<td>14</td>
<td>8</td>
<td>2.9680</td>
</tr>
<tr>
<td>$\beta=0.6$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>14</td>
<td>3</td>
<td>8.0850</td>
</tr>
<tr>
<td>0.077</td>
<td>13</td>
<td>4</td>
<td>6.1889</td>
</tr>
<tr>
<td>0.10</td>
<td>13</td>
<td>5</td>
<td>5.2571</td>
</tr>
<tr>
<td>0.12</td>
<td>14</td>
<td>6</td>
<td>5.1802</td>
</tr>
</tbody>
</table>

4.5 DISCUSSION:

From table 1, we note that optimal number of repairs $n^*$ and resulting cost $C(N, n^*) / C_1$ increases with increase in the ratio of $C_1 / C_2$ as well as in $\beta$ (i.e., increase in the common-cause failure rate) which is in accordance with physical experience.

Table 2 reveals that optimal number of units $N^*$ increases with increase in $C_1 / C_1$ values which is coinciding with operational experience that as per unit time repair cost turn out to be more relative to acquisition costs, it is better to see that the system does not fail in a smaller intervals of time (or use) and that is achieved only by increasing $N$. Similar conclusions can be made using table 3.
REFERENCES:


