Chapter 3

MAXIMAL DOMINATION NUMBER
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In the earlier chapter we have discussed about the domination number and a related concept, upper domination number. In this chapter we have discussed another related concept of the domination number called the maximal domination number of a graph. This concept was introduced by Kulli and Janakiram [19]. It is defined as follows.

**Definition 3.1:** A dominating set $D$ of a graph $G = (V, E)$ is a maximal dominating set if $V \setminus D$ is not a dominating set of $G$. The maximal domination number $\nu(G)$ is the minimum cardinality of a maximal dominating set.

Kulli and Janakiram [19] investigated several interesting properties of the maximal domination number of a graph. They have proved that if $G$ is a graph with no isolated vertices then for any minimal dominating set $D$ of $G \setminus D$ is also a dominating set of $G$. In fact they have proved that a minimal dominating set $D$ of a graph $G$ is a maximal dominating set if and only if $G$ contains an isolated vertex. They have obtained the upper and lower bounds for the maximal domination number. They have proved that in any graph $G$ with no isolated vertex

$$\delta(G) + 1 \leq \nu(G) \leq \alpha_0(G) + 1 \quad \text{(Theorem 3 of [19])}$$

where $\delta(G)$ is the minimum degree of $G$ and $\alpha_0(G)$ is the vertex covering number of $G$. They have also given the bounds of the maximal domination number in terms
of the number of vertices of the graph and edges of the graph and the connectivity of the graph. They have proved that

(1) $k(G) + 1 \leq \gamma(G)$ where $k(G)$ is the connectivity of $G$.

(2) $\frac{1}{2}[2q - p(p-3)] \leq \gamma(G)$ and

(3) $p-q + \delta(G) \leq \gamma(G)$.

They have obtained the exact maximal domination number of some standard graphs. They have also proved that

(4) $\gamma(G) = p$ if and only if $G = K_p$ or $G = \overline{K}_p$

(5) $\gamma(K_{m,n}) = m+1$ where $m = \min (m,n)$.

(6) $\gamma(C_p) = \lceil p/3 \rceil + 1$, if $p \equiv 1,2 \pmod{3}$.

(7) $\gamma(C_p) = \lceil p/3 \rceil + 2$, if $p \equiv 0 \pmod{3}$, where $C_p$ is a cycle on $p$ vertices.

(8) $\gamma(P_p) = \lceil p/3 \rceil + 1$, where $P_p$ is the path on $p$ vertices.

(9) $\gamma(W_p) = 4$, where $W_p (= K_1 + C_{p-1})$ is the wheel on $p$ vertices.

Here $\lceil x \rceil$ denotes the least positive integer $\geq x$.

They have also given an interesting relationship between the maximal domination number, domination number. They have proved that for any graph $G$, $\gamma(G) \leq \gamma(G) + \delta(G)$. In the case of some special graphs they have obtained interesting upper bounds. They have proved that for any tree $T$, $\gamma(T) \leq n + 1$, where $n$ is the number
of cut vertices of $T$. They have also given several interesting results of the maximal domination number in relation with other domination parameters like block domination number, global domination number, connected domination number etc.

In this chapter we have discussed the maximal domination number of an arithmetic graph $V_m$ defined in chapter 2. We have obtained the following result.

**Theorem 3.2**: Let $m$ be any positive integer $> 1$ and $m = \prod_{i=1}^{r} p_i^{a_i}$ be the canonical representation of $m$ (where $p_i$'s are primes and $a_i$'s $> 0$). Then the maximal domination number of arithmetic graph $V_m$ is

\[
\nu(V_m) = \begin{cases} 
  r+1 & \text{if } \mu(m) = 0 \\
  r & \text{if } \mu(m) \neq 0
\end{cases}
\]

when $r = 1$

\[
\nu(V_m) = \begin{cases} 
  r+1 & \text{if } \mu(m) = 0 \\
  r & \text{if } \mu(m) \neq 0
\end{cases}
\]

when $r = 2$

\[
\nu(V_m) = r+1 \quad \text{if } r \geq 3
\]

where $\mu$ is the Möbius function.

**Proof**: (1) If $r = 1$, $\mu(m) = 0$, then $m = p_1^{a_1}$ where $a_1 > 1$
The vertex set of $V_m$ is \{ $p_1, p_1^2, \ldots, p_1^{a_1}$ \} and $p_1$ is adjacent with all the vertices and all the other vertices except $p_1$ are mutually nonadjacent. Hence \{ $p_1, p_1^{b_1}$ \} where $1 < b_1 \leq a_1$ constitutes a maximal dominating set of minimum cardinality.

Thus $v(V_m) = r + 1$ in this case.

![Figure 3.1: $V_m$ graph where $m = p_1^6$, $v(V_m) = 2$](image)

When $r = 1, \mu(m) \neq 0$, $m$ becomes $p_1$ and the case is trivial.

(2) If $r = 2, \mu(m) = 0$ then $m = p_1^{a_1} \cdot p_2^{a_2}$

If either $a_1 = 1$ or $a_2 = 1$ then $m = p_1^{a_1} \cdot p_2$ or $m = p_1 \cdot p_2^{a_2}$

Suppose $m = p_1^{a_1} \cdot p_2$

In the vertex set of $V_m$, there are several vertices of degree 2. Any vertex of degree 2 will be adjacent to two of the vertices in the set $S = \{ p_1, p_2, p_1, p_2 \}$. Any two vertices of $S$ is a minimal dominating set and the domination number of $V_m$ is 2.
Thus any vertex of degree 2 along with two adjacent vertices in $S$ constitutes a maximal dominating set of minimum cardinality. Hence $\nu(V_m) = r + 1$.

Figure 3.2: $V_m$ graph where $m = P_1 \cdot P_2$, $\nu(V_m) = 3$

If $m = P_1 \cdot P_2^{\alpha^2}$, the proof is similar to the above case.

Figure 3.3: $V_m$ graph where $m = P_1 \cdot P_2^{\alpha^1}$, $\nu(V_m) = 3$
If neither \( a_1 = 1 \) nor \( a_2 = 1 \) then \( m = p_1^{a_1} p_2^{a_2} \). The vertices of the form \( p_1^{b_1}, p_2^{b_2} \) (where \( b_1, b_2 \) are both \( > 1 \)) are all adjacent to the vertices \( p_1, p_2 \) only.

Since \( \{ p_1, p_2 \} \) is a minimal dominating set, any vertex \( p_1^{b_1}, p_2^{b_2} \) (\( b_1 > 1, b_2 > 1 \)) together with the vertices \( p_1, p_2 \) will be the maximal dominating set of minimum cardinality. This follows from theorem 3 of [19]. Hence \( \nu(V_m) = r + 1 \).

![Figure 3.4: \( V_m \) graph where \( m = p_1 p_2^{-1}, \nu(V_m) = 3 \)](image)

If \( r = 2 \), \( \mu(m) \neq 0 \) then \( m = p_1 p_2 \) and the \( V_m \) graph is

![Figure 3.5: \( V_m \) graph where \( m = p_1 p_2, \nu(V_m) = 2 \)](image)

clearly \( \{ p_1, p_1 p_2 \}, \{ p_2, p_1 p_2 \} \) are the maximal dominating sets.
Hence $v(V_m) = r$.

If $r \geq 3$, then $m = p_1^{q_1} p_2^{q_2} \ldots p_r^{q_r}$ where $q_i$'s are $\geq 1$.

Since $\delta(V_m)$ is $r$, when $r \geq 3$, using the result of theorem 3 of Kulli and Janakiram [19] we have $v(V_m) \geq \delta(V_m) + 1$.

i.e. $v(V_m) \geq r + 1$ \hspace{1cm} (1)

This set $D$ is a dominating set of $V_m$ and $V \setminus D$ is not a dominating set for, the vertex $m$ is adjacent only to vertices $p_1$, $p_2$, ..., $p_r$ and therefore, this is a maximal dominating set and its cardinality is minimum in view of (1). Hence $v(V_m) = r + 1$.

Hence the theorem.

We now consider the question of constructing a graph with a given number $n$ as its maximal domination number. For this we first prove the following theorem.
Theorem 3.3: Given a positive integer \( n \), there exists a graph for which \( n \) is the maximal domination number.

Proof: If \( n = 1 \), taking \( m = p_1 \), the \( V_m \) graph is the graph for which 1 is the maximal domination number.

If \( n = 2 \), choosing any two primes \( p_1 \) and \( p_2 \) and by setting \( m = p_1 p_2 \) we have \( v(V_m) = 2 \) from theorem 3.2.

If \( n = 3 \), choosing any two primes \( p_1 \), \( p_2 \) and two positive integers \( a_1 \), \( a_2 \geq 2 \) and setting \( m = p_1^{a_1} p_2^{a_2} \), we have \( v(V_m) = 3 \) from Theorem 3.1.

If \( n > 3 \), choosing \( n - 1 \) distinct primes \( p_1, p_2, \ldots, p_{n-1} \) and \( n - 1 \) positive integers \( a_1, a_2, \ldots, a_{n-1} \) and setting \( m = p_1^{a_1} p_2^{a_2} \ldots p_{n-1}^{a_{n-1}} \), we have \( v(V_m) = n \) from Theorem 3.1 case 3.

Hence the theorem.

Using the above theorem we can now construct a \( V_m \) graph with a given positive integer \( n \) as its maximal domination number. Construction of the \( V_m \) graph is quite simple and requires very elementary concepts of number theory.

Algorithm:

We now present an algorithm for the construction of this \( V_m \) graph whose maximal domination number is a given positive integer \( n \).
Input: Maximal domination number

Output: The graphs with a given maximal domination number.

Step 1: Enter the maximal domination number

Step 2: If $n = 1$, enter the prime number $p_1$, and set $m = p_1$

If $n = 2$, enter the prime numbers $p_1$, $p_2$ and set $m = p_1p_2$

If $n = 3$, enter the prime numbers $p_1$, $p_2$ and two positive integers $a_1, a_2 \geq 2$ and set $m = p_1^{a_1} p_2^{a_2}$

If $n \geq 4$, choosing primes $p_1, p_2, \ldots, p_{n-1}$ and $a_1, a_2, \ldots, a_{n-1}$ (n-1) positive integers and set $m = p_1^{a_1} p_2^{a_2} \ldots p_{n-1}^{a_{n-1}}$

Step 3: The vertex set $V = \{d/d$ is a divisor of $m\}$

Step 4: Construct a $V_m$ graph with vertex set $V$ in which every two distinct vertices $a$ and $b$ are adjacent if $(a,b) = p_i$, where $p_i$ is a prime number

Illustration: We have given illustration of the $V_m$ graphs whose maximal domination numbers has been given as $n = 2, n = 3, n = 4, n = 5$ in figures 3.7, 3.8, 3.9 and 3.10 respectively.
(1) $n = 2$

Figure 3.7: $V_m$ graph where $m = p_1p_2$

(2) $n = 3$

Figure 3.8: $V_m$ graph where $m = p_1^2p_2^2$
(3) $n = 4$

Figure 3.9: $V_n$ graph where $m = p_1^3, p_2^3, p_3$. 

(4) $n = 5$

Figure 3.10: $V_m$ graph where $m = p_1, p_2, p_3, p_4$. 
Applications:

The theory of domination have various applications and the most interesting application frequently discussed is regarding a communication network. This application can be extended to the construction of a graph with a given number as the maximal domination number.

Suppose there is a outbreak of a disease in an epidemic form and the health department has identified certain areas of the region from which the disease is spreading in an epidemic form. If the resources of the department to wage a battle with this disease are limited what is the minimum number of these areas at which vigorous efforts can be made to combat with the disease so that some areas of the region are not affected by this disease. The entire region serves as the graph and identified areas from which the disease is spreading to the entire region serves as the dominating set. Finding the minimum number of these centers where the control of the disease will result the remaining areas of the region unaffected by this epidemic disease is finding a maximal domination number of the given graph. There can be many applications of this nature in communication network etc.

We now discuss an interesting situation of constructing a graph with two given numbers as two different domination parameters. We first make the following observations.

1. In any non trivial graph $\gamma = v$ can happen only in a disconnected graph.
2. In a connected graph $\gamma < \nu$.

The problem that we have discussed is: Given two distinct positive integers $s$ and $t$ ($s < t$) how to construct a graph whose domination number is $s$ and maximal domination number is $t$. For this we first prove the following theorem.

**Theorem 3.4:** Given two positive integers $s$ and $t$ ($s < t$) there exist a connected graph whose domination number is $s$ and maximal domination number is $t$.

**Proof:** We divide the proof into three cases viz., when $s = 1$, $s = 2$ and $s \geq 3$.

**Case 1:** If $s = 1$, we define arithmetic graph $S_m$ where $m = p^t$, $p$ is a prime and $t$ is a positive integer. The vertex set of this graph is the set of all divisors of $m$ (excluding 1). Two distinct vertices $a$ and $b$ in this graph are adjacent if $(a, b) = p^\alpha$, $1 \leq \alpha \leq t - 1$. Here each vertex acts as a minimal dominating set.

![Figure 3.11: S_m graph where m = p^6](image)

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Hence the domination number is 1 and the maximal domination number is the set of all vertices of the graph i.e. t.

Case 2: If s = 2, then in view of the observation (2) t is ≥ 3.

Choosing \( m = p_1^{t-1} \cdot p_2 \) where \( p_1, p_2 \) are any two primes, we construct an arithmetic graph \( S_m \) defined as follows. The vertex set of \( S_m \) graph is the set of all divisors of \( m \) (excluding 1) and two distinct vertices \( a \) and \( b \) are said to be adjacent if \( (a, b) = p_1^{\alpha_1} \) or \( (a, b) = p_2 \) where \( 1 \leq \alpha_1 < t-1 \). The minimal dominating set of this \( S_m \) graph is \( \{ p_1, p_2 \} \) and the vertex set

\[
D = \{ p_1, p_1^2, p_1^3 \ldots p_1^{t-2}, p_2, p_1^{t-1} \cdot p_2 \}
\]

is a maximal dominating set of minimum cardinality. Since the vertices \( \{ p_1, p_2 \} \) are in the set \( D \), this set \( D \) is clearly a dominating set. \( \forall \setminus D \) is not a dominating set, since no vertex in \( \forall \setminus D \) is adjacent with \( m \). Hence \( D \) is a maximal dominating set. The minimum degree of this graph is the degree of the vertices \( m \) and \( p_2 \) and their degree is \( t - 1 \). We assert this for the following reasons. The vertex set of this arithmetic graph \( S_m \) is \( m, p_2 \) and the other vertices are of the type \( p_1^{\beta}, p_1^\alpha \cdot p_2 \) where \( \alpha < t - 1, 1 \leq \beta \leq t - 1 \). The vertex \( m \) is adjacent with \( p_1, p_1^2, \ldots p_1^{t-2}, p_2 \) and hence is of degree \( t - 1 \). The vertex \( p_2 \) is adjacent with \( p_1, p_2, p_1^2 p_2, \ldots, p_1^{t-1} p_2 \) and thus is of degree \( (t - 1) \),
where as the vertices of the type $p_i^\beta (1 \leq \beta \leq t - 1)$ are adjacent with $p_1, p_1^2, \ldots, p_1^{t-1}$ which are different from $p_1^\alpha$ and $p_1 p_2, p_1^2 p_2, \ldots, p_1^{t-1} p_2$. Thus these vertices are of degree $2t - 3$ and the vertices of the type $p_1^\alpha, p_2 (\alpha < t - 1)$ are adjacent with $p_1, p_1^2, \ldots, p_1^{t-1}, p_2$ and thus is of the degree each. Hence $\delta(S_m) = t - 1$. Using theorem 3 of [19], we have $v(S_m) \geq \delta(S_m) + 1 = t - 1 + 1 = t$, it follows that $D$ is a maximal dominating set of minimum cardinality. Hence the arithmetic graph $S_m$ is such that $\gamma(S_m) = s$ and $v(S_m) = t$.

![Figure 3.12: $S_m$ graph where $m = p_1^4, p_2, s = 2, t = 6$](image)

Case 3: If $s \geq 3$ we divide the proof into two parts.

Part a: $t - s = 1$, Part b: $t - s > 1$
**Part a:** when \( t - s = 1 \), choosing \( m = p_1^2 \cdot p_2^2 \cdot p_3^2 \cdots p_{s+1}^2 \cdot p_s \) where \( p_1, p_2, \ldots, p_s \) are primes, we have the arithmetic graph \( V_m \) which is such that its domination number is \( s \) \([29]\) and maximal domination number is \( s + 1 \) (by Theorem 3.2).

![Figure 3.13: V_m graph where \( m = p_1^2 \cdot p_2^2 \cdot p_3 \), s = 3, t = 4](image)

**Part b:** If \( t - s > 1 \), choosing \( m = p_1^{t-s-1} \cdot p_2 \cdot p_3 \cdots p_{s+1} \) where \( p_i \)'s are primes and we construct an arithmetic graph \( S_m \) defined as follows. The vertex set of \( S_m \) is the set of all divisors of \( m \) (excluding 1) and two distinct vertices \( a, b \) are defined to be adjacent if \((a,b) = p_i \) or \((a,b) = p_i^\alpha \) where \( 1 \leq \alpha \leq (t - s) - 1 \). Clearly \( \{p_1, p_2, \ldots, p_{s+1}, p_s \} \) is a dominating set. If we
Figure 3.14: $S_m$ graph where $m = p_1, p_2, p_3, p_4, s = 3$

delete any vertex, the resultant set is not a dominating set and hence this set is a minimal dominating set. Further it is of minimum cardinality for the following reasons.

(i) The structure of the vertex set as such involves all the primes $p_1, p_2, ..., p_{s+1}$ and hence the dominating set should include all these primes.

(ii) If we have some products of primes as elements of the dominating set say for the example $(p_1p_2, p_3p_4, ...)$ then the vertex $p_1p_2p_3p_4$ of the graph is not adjacent with any vertex of this dominating set.
(or) for example \( p_1 p_2 p_3 p_4, p_5 p_6 p_7 p_8, \ldots \) then the multiples of 
\( p_1 p_2 p_3 p_4 \) which are vertices of the graph will not be adjacent with any of the vertices in the dominating set.

Hence \( \{ p_1, p_2, \ldots, p_{s-1}, p_s, p_{s+1} \} \) is a dominating set with minimum cardinality. Hence the domination number is \( s \).

Now we claim that the set \( D = \{ p_1, p_1^2, \ldots, p_1^{t-s-1}, p_2, p_3, \ldots, p_{s+1}, m \} \) is a maximal dominating set of minimum cardinality. As the vertices \( \{ p_1, p_2, \ldots, p_{s+1} \} \) are in this set and as they dominate the graph, \( D \) is a dominating set of \( S_m \). But \( V \setminus D \) is not a dominating set as no vertex of \( V \setminus D \) is adjacent with \( m \). Thus \( D \) is a maximal dominating set. To prove that this \( D \) is a maximal dominating set of minimum cardinality, we observe that the vertices can be classified into four types.

1. Vertices of the form \( p_1^\alpha \) (1 \( \leq \alpha \leq t-s-1 \))
2. Vertices of the form \( p_1 \) (2 \( \leq i \leq s+1 \))
3. Vertices of the form \( p_1^\alpha p_i p_j \ldots p_k \ldots \) Where \( i < j < k \leq s+1 \)
4. Vertex \( m \).

We note that the vertices of type 1 i.e., vertices of the form \( p_1^\alpha \) are adjacent with \( p_1, p_1^2, \ldots, p_1^{t-s-1}, p_1^{s+1}, \ldots, p_1^{t-s-1}, \) multiples of each of these numbers with \( s_3 \).
products of the primes \( p_2, p_3, ..., p_{s+1} \) taken one, two, three ... all at a time.

Hence the degree of \( p_1^\alpha \) is

\[
(t - s - 2) + (t - s - 1)(s_{c_1} + s_{c_2} + ... + s_{c_s})
\]

\[
= (t - s - 2) + (t - s - 1)(2^s - 1)
\]

\[
= 2^s(t - s - 1)
\]

The vertices of type (2) i.e., the vertices of the form \( p_i \) will be adjacent with all the vertices which are multiples of \( p_i \) (\( 2 \leq i \leq s + 1 \)) with the divisors of \( p_1^{t-s-1} \) and/or with the divisors of \( p_2 p_3 ... p_{i-1} p_{i+1} ... p_{s+1} \) taken one, two, three, ..., all at a time.

Hence the degree of \( p_i \) is

\[
(t - s - 1) + (2^{s-1} - 1) + (t - s - 1)(2^{s-1} - 1)
\]

\[
= (t - s - 1)(1 + 2^{s-1} - 1) + (2^{s-1} - 1)
\]

\[
= 2^{s-1}(t - s - 1 + 1) - 1
\]

\[
= 2^{s-1}(t - s) - 1
\]

The vertices of the type (3), i.e., the vertices of the form \( p_1^\alpha p_i p_j p_k ... \) = \( v \) (say) will be adjacent with the set of all divisors of \( p_1^{t-s-1} \) (excluding 1), \( p_i, p_j, p_k ... \) and also with the vertices which are multiples of each of these
numbers with the remaining primes which are missing in the product \( v \). If there are \( r \) primes in the product \( v \) then the degree of \( v \) is

\[
(t - s - 1) + (r - 1) + (t - s - 1)(2^{s+1-r} - 1) + (r - 1)(2^{s+1-r} - 1)
\]

\[
= (t - s - 1)(1 + 2^{s+1-r} - 1) + (r - 1)(1 + 2^{s+1-r} - 1)
\]

\[
= 2^{s+1-r}(t - s + r - 2)
\]

Finally, the vertices of type (4), i.e., the vertex \( m \) is adjacent with \( p_1, p_1^2, \ldots, p_1^{t-s-1}, p_2, p_3, \ldots, p_{s+1} \) and thus the degree of \( m \) is

\[
t - s - 1 + s = t - 1
\]

From (1), (2), (3), (4) and in view of the facts \( t > s \geq 3 \), \( r \leq s + 1 \) and \( t - s > 1 \) it follows that the degree of the vertex \( m \) is minimum. Hence \( \delta(S_m) = t - 1 \). It follows from theorem 3 of [19] that \( v(S_m) = t \). Thus there exists an arithmetic graph whose domination number is the given number \( s \) and the maximal domination number is \( t \).

We illustrate this with few examples.

**Illustration:** We have given illustrations of each of these cases.
Figure 3.15: $S_m$ graph where $m = p^8, s = 1, t = 8$

Figure 3.16: $S_m$ graph where $m = p^4, p_2, s = 2, t = 5$
Figure 3.17: $V_n$ graph where $m = p_1^1.p_2^2.p_3$, $s = 3$, $t = 4$. 
Figure 3.18: $S_m$ graph where $m = p_1^7 \cdot p_2 \cdot p_3 \cdot p_4, s = 3, t = 6$
Using this theorem we can now solve the following problem.

Problem: Given two positive integers $s$ and $t$ we can construct a graph whose domination number is $s$ and maximal domination number is $t$. We now give an algorithm based on this theorem to construct this graph.

Algorithm:

**Input**: Domination number $s$ and maximal domination number $t$

**Output**: Graphs with given domination number $s$ and maximal domination number $t$

**Step 1**: If $s = 1$, choose a prime $p$ and set $m = p^1$. Construct the $S_m$ graph with its vertex set as the set of all divisors of $m$ (excluding $1$) and two distinct vertices $a$, $b$ are adjacent if $(a, b) = p^\alpha \ (1 \leq \alpha \leq t-1)$

**Step 2**: If $s = 2$, choose two primes $p_1$, $p_2$ and set $m = p_1^{t-1} p_2$. Construct the $S_m$ graph with its vertex set as the set of all divisors of $m$ (excluding $1$) and two distinct vertices $a$, $b$ are adjacent if $(a, b) = p_1^\alpha$ or $p_2$ where $1 \leq \alpha < t-1$
Step 3: If \( s \geq 3 \) and

If \( (t - s = 1) \)

\{

choose \( s \) primes \( p_1, p_2, \ldots, p_s \) and set \( m = p_1^2 \cdot p_2^2 \cdot \ldots \cdot p_s^2 \cdot p_{s-1} \cdot p_s \). Then construct the \( V_m \) graph with its vertex set as the set of all divisors of \( m \) (excluding 1) and two distinct vertices \( a, b \) are adjacent if \( (a, b) = p_i \)

\}

else (if \( t - s > 1 \))

\{

choose \( s + 1 \) primes \( p_1, p_2, p_3, \ldots, p_{s+1} \) and set \( m = p_1^{t-s-1} \cdot p_2 \cdot \ldots \cdot p_{s+1} \). Construct the \( S_m \) graph whose vertex set is the set of all divisors of \( m \) (excluding 1) and two distinct vertices \( a, b \) are adjacent if \( (a, b) = p_i^2 \) or \( p_i \), where \( 1 \leq \alpha \leq t - s - 1 \) and \( 1 \leq i \leq s + 1 \)

\}.