CHAPTER - IV

Construction of Higher Associate PBIB Designs by Using Partial Geometry with Finite Graphs and Boolean Algebra

Introduction

Two new series of Partially Balanced Incomplete Block (PBIB) designs for three and four associate classes by using the concept of Boolean algebra / regular graphs and partial geometry (r, k, t) with finite graphs respectively are introduced in this chapter. To construct the new series of designs firstly, we study in detail the classification and analysis of PBIB design with two associate classes introduced by Bose and Shimamoto (1952). Bose and Nair (1939) have also used finite geometries and galois fields in the construction of incomplete block designs. Then Clatworthy (1954) represents a geometrical configuration consisting of certain suitably chosen lines of the finite projective geometry PG(3,s) which may also be interpreted as PBIB design with particular parametric combinations. Bruck (1963) introduced finite nets II with a collection of t mutually orthogonal latin squares of side n.

Using these ideas, Bose (1963) introduced the concept of partial geometry (r, k, t) by using undefined points and lines together with a relation of incidence and non-incidence for the construction of two associate class PBIB designs. Then, Benson (1965) introduced a partial geometry (q^3 + 1, q^2 + 1, 1) and constructed corresponding PBIB designs. Similarly, Nourine and Raynaud (1999) present a simple, efficient algorithm to compute covering graph of the lattice and suggest that this algorithm can be used to compute the Galois lattice.

From the above literature, we get an idea to explore the results in partial geometry and lattices for the construction of PBIB designs based on relevant fields and we are able to introduce a class of four associate class PBIB designs using partial geometry (r, k, t) for r = 3 with finite graphs and also introduced method of constructions of three and generalized associate class PBIB designs by using Boolean algebra with some specific restrictions. In the construction of four associate class PBIB designs, we use partial geometry with finite graphs by taking number of replications of treatments is exactly three, but in the construction of PBIB designs by using Boolean algebra with some specific restrictions, number of replications of treatments varies.

In this chapter, section 4.1 of this chapter, the information about regular graph and a concept of partial geometry (r, k, t) introduced by Bose (1963) is given. In section 4.2, we explain the axioms of Boolean algebra and shown practically how a set represent a Boolean algebra. Section 4.3 represents a class of four associate PBIB design by using partial geometry (r, k, t) for r = 3 along with its construction method. Section 4.4 gives the association scheme along with P-matrices of a series of PBIB designs constructed by using partial geometry. We study about Boolean algebra and construction method of PBIB designs by using Boolean algebra in section 4.5. Section 4.6 represent association scheme along with P- matrices of a class of three associate PBIB designs constructed in section 4.5. An extended result of construction of few PBIB designs again by using Boolean algebra are discussed in section 4.7. Illustrations of above mentioned series of PBIB designs are given in section 4.8 with
complete description of the construction methods and association schemes along their
P – matrices. Efficiencies tables for the above constructed series are given in section
4.9. Graphical representations of efficiencies are presented in section 4.10. Summary
and discussion on newly constructed PBIB designs by applying the concept of partial
graphy and Boolean algebra are presented in section 4.11 of this chapter.

4.1 Basic Study of Regular Graphs and Partial Geometry

Here, we discuss the basic definition of graphs and the various types of graphs
according to discrete mathematics and discussed the study of partial geometry
introduced by Bose (1963) which serves to unify and generalize certain theorems on
embedding of nets and uniqueness of association schemes of partially balanced
designs.

A graph is said to be ‘regular’ of degree $n_1$ if each vertex of G is joined to exactly
$n_1$ other vertices. In this case each vertex will be un - joined to $n_2$ other vertices such
that $n_1 + n_2 = v − 1$.

A regular graph G will be said to be ‘strongly regular’ if

a) Any two vertices which are joined in G are both simultaneous joined to
   exactly $p_{11}^1$ other vertices.

b) Any two pairs of vertices which are un - joined in G are both simultaneous
   joined to $p_{11}^2$ vertices.

Bose (1963) introduced the concept of partial geometry $(r, k, t)$ which is a system
of undefined points and lines together with a relation of incidence and non
incidence for the construction of two associate class PBIB designs. The graph $G$
of a partial geometry is defined as a graph whose vertices correspond to the points
of the geometry and in which two vertices are joined or unjoined according as the corresponding points are incident or non-incident with a common line. In other words, a graph is a graph of partial geometry if it satisfies the following axioms:

A1: Any two points are incident with not more than one line.

A2: Each point is incident with r lines.

A3: Each line is incident with k points.

A4: If the point P is not incident with the line \( l \), there pass through P exactly t

\[ (t \geq 1) \] lines intersecting line \( l \).

Hence, Bose (1963) represent that a graph \( G \) of a partial geometry \((r, k, t)\) is strongly regular with parameters

\[ n_1 = r(k-1), \quad n_2 = (r-1)(k-1) / t, \quad p_{11} = (t-1)(r-1) +k-2, \quad p_{11}^2 = rt \]

where \( 1 \leq t \leq r, \quad 1 \leq t \leq k \) and a necessary condition for the existence of a partial geometry \((r, k, t)\) is that the number \( \alpha = rk(r-1)(k-1) / t (k+r-t-1) \) is an integral.

4.2 Brief Study of Boolean Algebra

To clarify the fundamental theory of Boolean algebra and lattice theory, first of all we discuss some particular definitions of Discrete Mathematics present in text books related to discrete mathematics as given below:

**Partially Ordered Relation:** Consider a relation \( R \) on a set \( S \) satisfying

\( R \) is reflexive iff \( xRx \ \forall \ x \in S. \)

\( R \) is anti symmetric iff \( xRy \) and \( yRx \), then \( x = y \)

\( R \) is transitive iff \( xRy \) and \( yRz \), then \( xRz \).
Then R is partially order relation and the set S together with relation R is called partially order set or also called **POSET**.

**Lattice:** A lattice L is a poset in which every pair of elements has a least upper bound and a greatest lower bound. In other words, a lattice is partially ordered set in which

\[ a \Lambda b = \text{infimum} \ (a, b) \quad a \vee b = \text{supremum} \ (a, b) \text{ where infimum is greatest lower bound and supremum is called least upper bound.} \]

**Boolean Algebra:** A complemented distributive lattice is called a Boolean algebra. The Boolean algebra is denoted by \(( B, +, *, ', 0, 1)\). The set \( B = D_a \) contains all divisors (natural numbers) of a is positive integer, ‘+’ and ‘*’ are two binary operations, ‘’ is complement operation, ‘0’ and ‘1’ are called zero element and unit element of set B.

Set B represent a Boolean algebra if it satisfies the following laws for all elements a, b, c \( \in B \).

a) **Commutative Law** :  
   1) \( a + b = b + a \) 
   2) \( a * b = b * a \)

b) **Distributive Law** :  
   1) \( a + (b * c) = (a + b) * (a + c) \) 
   2) \( a * (b + c) = (a * b) + (a * c) \)

c) **Identity Law** :  
   1) \( a + 0 = a \) 
   2) \( a * 1 = a \)

d) **Complement Law** :  
   1) \( a + a' = 1 \) 
   2) \( a * a' = 0 \)

It should be noted that binary operations ‘+’ and ‘*’ are defined as

\[ a + b = \text{l.c.m} \ (a, b) , \ a * b = \text{g.c.d} \ (a, b) \text{ and unitary operation } a' = n/a. \text{ ‘0’ and ‘1’ are least and greatest elements of set B respectively.} \]
For example, let $B = \{1, 2, 3, 6\}$ contains of divisors of +ve integer 6. The binary operations ‘+’ and ‘*’ are defined as

\[
\begin{align*}
\text{a} + \text{b} &= \text{l.c.m (a,b)} & \text{a} \cdot \text{b} &= \text{g.c.d (a,b)} & \text{a}' &= \frac{6}{\text{a}} & \forall \text{ a } \in B.
\end{align*}
\]

Then

\[
\begin{array}{cccccc}
+ & 1 & 2 & 3 & 6 \\
1 & 1 & 2 & 3 & 6 \\
2 & 2 & 2 & 6 & 6 \\
3 & 3 & 6 & 3 & 6 \\
6 & 6 & 6 & 6 & 6 \\
\end{array}
\quad
\begin{array}{cccccc}
\ast & 1 & 2 & 3 & 6 \\
1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 & 2 \\
3 & 1 & 1 & 3 & 3 \\
6 & 1 & 2 & 3 & 6 \\
\end{array}
\quad
\begin{array}{cccccc}
a & 1 & 2 & 3 & 6 \\
a' & 6 & 3 & 2 & 1 \\
\end{array}
\]

From the above composition tables it is clear that $B$ satisfies commutative, distributive, identity and complement laws for all elements. Therefore, $(B, +, \ast, ', 0, 1)$ represents Boolean algebra.

**4.3 Series - 1: Four Associate Class PBIB Designs**

In this section, we introduce a class of PBIB designs with four associates with parameters

\[
v = s^2, \ b = 3s, \ r = 3, \ k = s, \ \lambda_1 = \lambda_2 = \lambda_3 = 1, \ \lambda_4 = 0, \ n_1 = n_2 = n_3 = s-1, \ n_4 = (s-1) (s-2).
\]

The construction method of four associate class PBIB designs is given below in section 4.3.1.
4.3.1 Construction method of four associate class PBIB designs

For the construction of four associate class PBIB designs, we use finite graphs which satisfies the following necessary conditions for the existence of partial geometry introduced by Bose (1963) as

\[
v = k \frac{((r-1) (k-1) + t)}{t} \quad b = r \frac{((r-1) (k-1) + t)}{t}
\]

\[1 \leq t \leq r, \quad 1 \leq t \leq k, \quad \alpha = \frac{rk((r-1) (k-1))}{t(k+r-t-1)} \] for \( r=3 \)

where \( v \) is number of treatments (vertices) and \( b \) is number of blocks (lines).

The construction method uses a finite number of lines and divides these lines equally into 3 classes. Each class has ‘s’ lines. Let us denote the first class as H – group which has ‘s’ parallel lines, second class as V- group which has ‘s’ parallel lines and a third class of ‘s’ curvilinear/ oblique lines called O-group. Lines belongs to same group never intersect each other and lines belongs to different groups intersect each other only once at a point.

So, each line of V- group intersects with each line of H- group only at one intersection point on its extremities or between. In this way, we have ‘s^2’ intersection points which represent s^2 vertices and each line of H-group as well as V-group has ‘s’ vertices. After that ‘s’ lines from O-group intersect the lines of H- group by following “location relationship” which represent a set having ‘s’ elements.

The set is represent as \{ i, i+1, i+2, ..., i+j, ... \} under the condition

\[ i+j = \{ \text{sum of } i \text{ and } j, \text{ if } i+j \leq s \} \quad \text{or} \]

\[ i+j = \{ \text{mod ‘s’ under addition if } i+j > s \} \], where \( i=1, 2, ..., s \).
The “location relationship set” is defined as the set in which first element indicates the location of vertex on 1st line of H-group, second element indicates the location of vertex on 2nd line of H-group and process continue for the remaining elements as well.

In this way, we have 3s total lines of graph of partial geometry (3, k, t) which is isomorphic to PBIB design (3, k, λ₁, λ₂, λ₃, λ₄). From the point of view of PBIB designs, we obtained PBIB designs with four associates with parameters
\[ v= s^2, \quad b= 3s, \quad r=3, \quad k=s, \quad \lambda₁ = \lambda₂ = \lambda₃ = 1, \quad \lambda₄ = 0, \quad n₁ = n₂ = n₃ = s - 1, \quad n₄ = (s - 1)(s - 2). \]

which follows association scheme - 4.4 defined below.

### 4.4 Association Scheme of Four Associate Class PBIB Designs

In this design, it is clear that we have ‘\(s^2\)’ points of intersections called vertices and 3s number of lines which represent number of blocks with \(r=3\) and \(k=s\). Each vertex present in three particular blocks. We select a vertex say ‘\(0\)’ from a line of H-group. This vertex ‘\(0\)’ occurs with vertices \(\theta_1, \theta_2, \ldots, \theta_{s-1}\) in a block and all these vertices lies on same line of H-group. These are first associates of ‘\(0\)’ and the number of first associates are \(n₁ = s - 1\).

This vertex ‘\(0\)’ also occurs on one particular line of V-group. Vertices \(\varphi_1, \varphi_2, \ldots, \varphi_{s-1}\) occurs with ‘\(0\)’ in a block and all these vertices also lies on that particular line of V-group in which ‘\(0\)’ lies. In other words, all these vertices are collinear. These are second associates of ‘\(0\)’ and the number of second associates are \(n₂ = s - 1\).

This vertex ‘\(0\)’ is also a point of intersection of one particular line of H – group with a line of V – group and also occurs with vertices \(\tau_1, \tau_2, \ldots, \tau_{s-1}\) in a block as well as all
these vertices occurs on one particular curve (oblique line) belongs to O – group and these are third associate of ‘0’ and number of third associates are \(n_3 = s-1\).

Lastly, vertices which are not present with ‘0’ in any block are fourth associate of ‘0’ and number of fourth associates are \(n_4 = (s-1) (s-2)\).

### 4.4.1 P – matrices

\[
P_1 = \begin{bmatrix}
    s-2 & 0 & 0 & 0 \\
    0 & 0 & 1 & s-2 \\
    0 & 1 & s-2 & s-2 \\
    0 & s-2 & s-2 & (s-2)(s-3) \\
\end{bmatrix}
\]

\[
P_2 = \begin{bmatrix}
    0 & 0 & 1 & s-2 \\
    0 & s-2 & 0 & 0 \\
    1 & 0 & 0 & s-2 \\
    s-2 & 0 & s-2 & (s-2)(s-3) \\
\end{bmatrix}
\]

\[
P_3 = \begin{bmatrix}
    0 & 1 & 0 & s-2 \\
    1 & 0 & 0 & s-2 \\
    0 & 0 & s-2 & 0 \\
    s-2 & s-2 & 0 & (s-2)(s-3) \\
\end{bmatrix}
\]

\[
P_4 = \begin{bmatrix}
    0 & 1 & 1 & s-3 \\
    1 & 0 & 1 & s-3 \\
    1 & 1 & 0 & s-3 \\
    s-3 & s-3 & s-3 & (s-3)^2 + 1 \\
\end{bmatrix}
\]

### 4.5 Series – П: Three Associate Class PBIB Designs

We construct a class of PBIB designs having three associate class with parameters \(v = 2(p+1) = b, r = p+1 = k, \lambda_1 = 2, \lambda_2 = p-1, \lambda_3 = 0, n_1 = p = n_2 \) and \(n_3 = 1\).

The construction method of three associate class PBIB designs is given below in section 4.5.1.

### 4.5.1 Construction method of three associate class PBIB designs

Now, we introduce a new method for the construction of PBIB designs by using Boolean algebra with some restrictions. Let us consider a set \((B, +, *, ', 0, 1)\) for \(D_n\),
where \( D_n \) is a set of divisors of ‘n’ from 1 to n. Set \((B, +, *, ', 0, 1)\) is also represent Boolean algebra.

As mentioned above our first restrictions is on ‘p’ that is

\[ p \geq 3 \text{ and } n \text{ is a product of first ‘p’ prime numbers only.} \]

The binary operations +, *, ’ defined as

\[ a + b = \text{L.C.M} (a, b), \ a*b = \text{G.C.D} (a, b) \text{ and } a’ = n/a \text{ where } a, b \in D_n. \]

From graphical point of view, Hasse diagram of \( D_n \) satisfies handshaking lemma

i.e. sum of all degrees = twice of number of edges.

Firstly, let us take first ‘p’ prime numbers \((p \geq 3)\). We take its product and let it be ‘n’. Then we consider a divisor set \( D_n \) which has exactly ‘\( 2^p \)’ elements which are divisors of ‘n’. \( D_n \) follows all axioms of Boolean algebra for every \( a, b, c \) elements of \( D_n \). Then, we apply next restriction that is on elements of \( D_n \). From \( 2^p \) divisors of ‘n’, we select the only divisors say \( p’ \) as \( p’ = n/p \), where \( p’ \)s are prime numbers of set \( D_n \) whose product is ‘n’.

In this way, we arise a new set of divisors of ‘n’ which contains

a) ‘1’ Lowest divisor of n (for all n) is called zero element.

b) ‘n’ Highest divisors of n (itself n) is called unit element.

c) ‘p’ Set of atoms (prime numbers).

d) ‘p’ ‘Composite numbers \((p’ = n/p)\) as well as divisors of ‘n’

From graphical point of view, Hasse diagram of \( D_n \) reduces to a \( p – \) regular graph \( G(V, E) \) having \( 2(p+1) \) divisors/elements only with vertices – edges relation as
a) Zero element ‘1’ is incidence with every prime number ‘p’.

b) Unit element ‘n’ is incidence with every composite number ‘p’.

c) (p, p’) pair wise incidence if p’/p = e ∀ e ∈ Dn.

In this way, this graph G (V, E) represents a PBIB designs with parameters

v= 2(p+1) =b, r=p+1=k, λ1 =2, λ2=p-1, λ3=0, n1= p = n2 and n3 = 1.

which follows association scheme along with P – matrices as given in section 4.6 below.

4.6 Association Scheme of Three Associate Class PBIB Designs

Using above construction method, graph G (V, E) has contained 2(p+1) divisors out of 2p divisors of Dn. We recognized as a treatment to each divisor using ascending order of divisors. We construct 2(p+1) blocks by using incidence relations.

It means that we consider those treatments in a block which are incidence to each other by an edge. In this way, we consider all divisors one by one and we get 2(p+1) blocks having plot size ‘p+1’ with replication ‘p+1’.

Let us take a particular treatment ‘θ’ which occurs in ‘p+1’ blocks. Treatments θ1′, θ2′,…, θp′ occurs in a particular block with ‘θ’ as well as these vertices are incidence with ‘θ’ by an edge are first associate of ‘θ’ and are n1 =p.

Treatments φ1′, φ 2′,…, φp′ are present with ‘θ’ in some particular ‘p’ blocks but not incidence with ‘θ’ by an edge are called second associate of ‘θ ‘ and are n2 = p.

The remaining treatment which is not occurs with ‘θ’ in any block is called third associates of ‘θ’ and n3 =1.
4.6.1 P–matrices

\[
P_1 = \begin{bmatrix}
0 & p - 1 & 0 \\
p - 1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
P_2 = \begin{bmatrix}
p - 1 & 0 & 1 \\
0 & p - 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
P_3 = \begin{bmatrix}
0 & p & 0 \\
p & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

4.7 Series of p–Associate Class PBIB Designs

One more result arising from the concept of Boolean Algebra to construct PBIB designs having two and higher associate classes as we omit one restriction on set of \( D_n \). We discuss the construction method, association scheme for two and higher associate class PBIB designs as given below.

4.7.1 Construction methodology

Here, we introduce the concept for the construction of PBIB designs by using Boolean algebra with a restriction. Let us consider a set which is also a Boolean algebra \((B, +, *, ', 0, 1)\) for \( D_n \), where \( D_n \) is a set of divisors of ‘n’ from 1 to n. As we mentioned above, our restriction on \( p \) is that we take an interval of \( p \) as \( 2 \leq p \leq 5 \) and number ‘n’ is a product of first ‘p’ prime numbers only. The binary operations \(+, *, '\) defined as

\( a + b = \text{L.C.M} (a, b) \), \( a * b = \text{G.C.D} (a, b) \) and \( a' = n / a \) where \( a, b \in D_n \).
From graphical point of view, **Hasse diagram** of \( D_n \) satisfies **handshaking lemma** i.e. sum of all degrees = twice of number of edges.

Let us take first ‘p’ prime numbers (\( p \geq 2 \)). We take its product and let it be ‘n’. Then we consider Boolean algebra for \( D_n \) which has exactly ‘2\(^p\)’ elements which are divisors of ‘n’. We draw the Hasse diagram of \( D_n \) which also follows all axioms of Boolean algebra for all \( a, b, c \) elements of \( D_n \) contains

a) Lowest divisor of n (it is 1 for all n) is called zero element.

b) Highest divisors of n (itself n) is called unit element.

c) Composite numbers as well as divisors of ‘n’

d) Set of atoms (prime numbers).

This Hasse diagram represents a PBIB designs with parameters

\[
\begin{align*}
&v = 2^p = b, \quad r = p+1 = k, \quad \lambda_1 = \lambda_2 = 2, \quad n_1 = p, \quad n_2 = p(p-1)/2 \text{ and } \lambda_p = 0 \text{ and } n_p \text{ varies } \forall \ p \geq 2
\end{align*}
\]

which follows an association scheme 4.7.2 defined below.

**4.7.2 Association scheme**

Using above construction methodology, hasse diagram of \( D_n \) contain \( 2^p \) divisors. We assign a treatment to each divisor in ascending order of divisors. We construct \( 2^p \) blocks by using incidence relations. It means that we consider treatments in a block which are incidence to each other by an edge. In this way, we consider all divisors one by one and we get \( 2^p \) blocks having plot size ‘\( p+1 \)’ and replication ‘\( p+1 \)’. Let us take a particular treatment say ‘0’. The following steps represent the associates categories of among treatments with respect to ‘0’ as.
a) This particular treatment ‘0’ occurs in ‘p+1’ blocks. Treatments $\theta_1^\prime$, $\theta_2^\prime$, …, $\theta_p^\prime$ occurs in a particular block with ‘0’ as well as these vertices are incident with ‘0’ by an edge are first associate of ‘0’ and are $n_1 = p$. Treatments $\varphi_1^\prime$, $\varphi_2^\prime$, …, are present with ‘0’ in some particular ‘p’ blocks but not incidence with ‘0’ by an edge are called second associate of ‘0’ and $n_2 = p(p-1)/2$.

The association of remaining treatments which do not occurs with ‘0’ in any block are sub divided into following categories with respect to ‘0’ as.

b) The product of a particular treatment says ‘$\alpha$’ with a treatment ‘0’ is equal to ‘n’. That is $p^{th}$ associate of ‘0’ and $n_p = 1$.

c) The product of some other treatments say $\beta_1$, $\beta_2$, …,$\beta_p$ with some particular first associates of ‘0’ is equal to ‘n’. These are $(p-1)^{th}$ associates of ‘0’.

d) The product of remaining treatments with second associates of ‘0’ is equal to ‘n’. These are $(p-2)^{th}$ associates of ‘0’.

4.7.3 Remark

For $p=2$, only step a) exists and it has a two associate class association scheme.

For $p=3$, step a) and b) exists and it has a three associate class association scheme.

For $p = 4$ and $p = 5$, step c) and step d) are also valid and above association scheme read as a four associate class association scheme and five associate class association scheme respectively.

4.8 Illustrations of Construction

We illustrate below the method of constructions of new series of PBIB designs along with their association schemes and P- matrices.
4.8.1 Illustration of four associate class PBIB designs using partial geometry

*Construction method*

For $s=4$: We have $3s=12$ lines and divide these lines into three groups [H, V, O]. Each group has $s=4$ lines. From the point of view for existence of partial geometry $(r, k, t)$

\[ v = 4[2.3 +2]/2 = 16, \quad b = 3[2.3 +2] /2 = 12, \quad 1 \leq t \leq 3, \quad 1 \leq t \leq 4, \]

\[ \alpha = 3.4 (2)(3) / 2(4+3-2-1) = 9 \text{ is an integer.} \]

In the sense of PBIB design, we get a design having $v=16$, $b=12$, $r=3$, $k=4$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = 0$.

Now we explain the construction of PBIB design. We have 4 lines in each H, V, O groups. Lines in H- group as well as in V – group are parallel to each other. Each line of V – group intersect with each line of H – group at one and only one point at extremities or betweens. In this way, we have $s^2$ points of intersections. Let intersection points

1, 2, 3, 4 (also called vertices) lies on first line of H – group.

5, 6, 7, 8 (also called vertices) lies on second line of H – group.

9, 10, 11, 12 (also called vertices) lies on third line of H – group.

13, 14, 15, 16 (also called vertices) lies on fourth line of H – group.

Intersection points of V – group’s lines with H-group’s lines as

1, 5, 9, 13(also called vertices) lies on first line of V – group.
2, 6, 10, 14 (also called vertices) lies on second line of V – group.

3, 7, 11, 15 (also called vertices) lies on third line of V – group.

4, 8, 12, 16 (also called vertices) lies on fourth line of V – group.

By using “location relationship set” which has ‘s’ elements, each belongs to different line of H - group, we get 1, 6, 11, 16 incidence on first line of O- group. 2, 7, 12, 13 incidence on second line of O – group. 3, 8, 9, 14 incidence on third line and 4, 5, 10, 15 incidence on fourth line of O – group. In this way, points of these 12 lines represent 12 blocks as

( 1, 2, 3, 4 ) (5, 6, 7, 8 ) (9,10,11,12) (13,14,15,16)
(1,5,9,13) ( 2,6,10,14) ( 3, 7, 11, 15) ( 4, 8, 12, 16 )
(1,6,11,16) (2,7,12,13) ( 3, 8, 9, 14 ) (4, 5, 10 , 15)

**Association scheme**

Let us consider a particular vertex, say ’0’ =1. Vertices 2, 3, 4 occur with ‘1’ in first block as well as all these vertices lies on one ( first) line of H- group. These are first associate of ‘1’ and n_1=3. Vertices 5,9,13 occur with ‘1’ in a block as well as all these vertices lies on one (first) line of V- group. These are second associates of ‘1’ and n_2= 3. Similarly, vertices 1, 6, 11, 16 lies in same block as well as all these vertices incident on first line of O – group are third associate of ‘1’ and n_3= 3. Remaining vertices 7, 8, 10, 12, 14, 15 which are neither occurs with ‘1’ in any block nor collinear with ‘1’ on any line of H, V, O groups are called fourth associates of ‘1’ and n_4= 6.
For simplicity lines parallel to X-axis represent H-group’s lines, lines parallel to Y-axis represent V-group’s lines and oblique lines / curves represent O-group’s lines.

In graphical form we represent the above relation as in figure [1].

**Figure [1]**

Graphical representation of illustration of four associate class PBIB design(s=4)

\[
P_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 2 \end{bmatrix}
\]

\[
P_3 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 0 & 2 \end{bmatrix} \quad P_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}
\]

First, Second, Third, Fourth and Fifth associates of each treatment are given in the following table.
4.8.2 Illustration of three associate class PBIB designs using boolean algebra

Construction method

As per our restriction (p ≥ 3), let us consider first four prime number. For p = 4, we select first four prime numbers 2,3,5,7 and their product is 210 = n.

Then $D_{210} = \{1,2,3,5,6,7,10,14,15,21,30,35,42,70,105,210\}$

Clearly, we have $2^p = 16$ divisors. $D_{210}$ is also a Boolean algebra $(B, +, *, ', 0, 1)$ with zero element is 1 and unit element is 210. We draw its Hasse diagram. Each divisor consider as a vertex and each vertex has ‘p=4’ degree from graphical point of view.

<table>
<thead>
<tr>
<th>Treatment</th>
<th>1&lt;sup&gt;st&lt;/sup&gt; Associates</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; Associates</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt; Associates</th>
<th>4&lt;sup&gt;th&lt;/sup&gt; Associates</th>
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<tbody>
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<td>7,12,13</td>
<td>5,8,9,11,15,16</td>
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<td>4,8,12</td>
<td>1,6,11</td>
<td>2,3,5,7,9,10</td>
</tr>
</tbody>
</table>
view. According to next restriction, we select the divisors say \( p' \) such that \( p' = n/p \).

Clearly for \( n=210 \) and \( p' \)'s are 2,3,5,7 the possible divisors say \( p' \)'s are 30, 42, 70, 105 respectively. So, set \( D_{210} \) into a new set which contains

a) Lowest divisor ‘1’ of 210 is called zero element.

b) Highest divisors ‘210’ (itself 210) is called unit element.

c) Composite numbers 30,42,70,105 as well as divisors of ‘210’

d) Set of atoms 2,3,5,7 (prime numbers).

From graphical point of view, Hasse diagram of \( D_n \) reduces to a 4 – regular graph \( G(V,E) \) having \( 2(p+1) = 10 \) divisors/elements only with vertices – edges relation as

a) Zero element ‘1’ is incidence with every prime number ‘2, 3, 5, 7’.

b) Unit element ‘210’ is incidence with every composite number ‘30, 42, 70, 105’.

c) \((p, p')\) pair wise incidence if \( p' / p = e \ \forall \ e \in D_n \).

Using incidence relations given above of 4 - regular graph \( G(V, E) \), we construct blocks using incidence relation of every vertex one by one and get blocks as

\[
(1, 2, 3, 5, 7)
\]
\[
(2, 1, 30, 42, 70) (3, 1, 30, 42, 105) (5, 1, 30, 70, 105) (7, 1, 42, 70, 105)
\]
\[
(30, 2, 3, 5, 210) (42, 2, 3, 7, 210) (70, 2, 5, 7, 210) (105, 3, 5, 7, 210)
\]
\[
(210, 30, 42, 70, 105)
\]

In this way, this graph \( G(V, E) \) represents a PBIB design with parameters \( v = 10 = b \), \( r =5 = k \), \( \lambda_1 =2 \), \( \lambda_2=3 \), \( \lambda_3=0 \), \( n_1= 4 = n_2 \) and \( n_3 = 1 \) follows an association scheme as given below.
*Association scheme*

We recognized each divisor as a treatment. Let us consider a particular divisor ‘1’.

The divisors 2, 3, 5, 7 occur with ‘1’ in a block as well as all are incidence with ‘1’ by an edge. These divisors are first associate of ‘1’ and \( n_1 = 4 \).

The divisors 30, 42, 70, 105 occur with ‘1’ in four different blocks, but not incident with ‘1’ by an edge are second associates of ‘1’ and \( n_2 = 4 \).

Remaining divisor which is not occurring with ‘1’ in any block as well as not incidence is 210 and a treatment belong to this divisor is third associates of ‘1’ and \( n_3 = 1 \).

In graphical form, we represent the above relation as in figure [2].

**Figure [2]**

*Graphical representation of illustration of four associate class PBIB design(s=4)*
P- matrices are

\[
P_1= \begin{bmatrix}
0 & 3 & 0 \\
3 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

\[
P_2= \begin{bmatrix}
3 & 0 & 1 \\
0 & 3 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
P_3= \begin{bmatrix}
0 & 4 & 0 \\
4 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

First, Second and Third associates of each treatment are given in the following table.

<table>
<thead>
<tr>
<th>Treatment (Divisors)</th>
<th>1st Associates</th>
<th>2nd Associates</th>
<th>3rd Associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2,3,5,7</td>
<td>30,42,70,105</td>
<td>210</td>
</tr>
<tr>
<td>2</td>
<td>1,30,42,70</td>
<td>3,5,7,210</td>
<td>105</td>
</tr>
<tr>
<td>3</td>
<td>1,30,42,105</td>
<td>2,5,7,210</td>
<td>70</td>
</tr>
<tr>
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<td>2,3,7,210</td>
<td>42</td>
</tr>
<tr>
<td>7</td>
<td>1,42,70,105</td>
<td>2,3,5,210</td>
<td>30</td>
</tr>
<tr>
<td>30</td>
<td>2,3,5,210</td>
<td>1,42,70,105</td>
<td>7</td>
</tr>
<tr>
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</tr>
<tr>
<td>70</td>
<td>2,5,7,210</td>
<td>1,30,42,105</td>
<td>3</td>
</tr>
<tr>
<td>105</td>
<td>3,5,7,210</td>
<td>1,30,42,70</td>
<td>2</td>
</tr>
<tr>
<td>210</td>
<td>30,42,70,105</td>
<td>2,3,5,7</td>
<td>1</td>
</tr>
</tbody>
</table>
4.8.3 Illustration of p- associate class PBIB designs using boolean algebra

Construction method

For \( p = 4 \), the first four prime numbers 2,3,5,7 and their product is 210 = n.

Then \( D_{210} = \{ 1,2,3,5,6,7,10,14,15,21,30,35,42,70,105,210 \} \)

Clearly, we have \( 2^p = 16 \) divisors. \( D_{210} \) is also a Boolean algebra \((B, +, *, ', 0, 1)\) with zero element is 1 and unit element is 210. We consider each divisor as a vertex and assign a treatment to each divisor in ascending order. Now, we draw its Hasse diagram. Each divisor consider as a vertex from graphical point of view and each vertex has ‘p=4’ degree. We construct blocks using incidence relation of every vertex one by one and get blocks as

\[
\begin{align*}
(1, 2, 3, 5, 7) & \quad (2, 1, 6, 10, 14) & \quad (3, 1, 6, 15, 21) & \quad (5, 1, 10, 15, 35) \\
(7, 1, 14, 21, 35) & \quad (6, 2, 3, 30, 42) & \quad (10, 2, 5, 30, 70) & \quad (14, 2, 42, 7, 70) \\
(15, 3, 5, 30, 105) & \quad (21, 3, 7, 42, 105) & \quad (35, 5, 7, 70, 105) & \quad (30, 6, 10, 15, 210) \\
(42, 6, 14, 21, 210) & \quad (70, 10, 14, 35, 210) & \quad (105, 15, 21, 35, 210) & \quad (210, 30, 42, 70, 105)
\end{align*}
\]

This Hasse diagram represents a PBIB design with parameters \( v=16 = b, r=5=k, \lambda_1 = \lambda_2=2, \lambda_3=0, \lambda_4=0, n_1= 4, n_2= 6, n_3= 4, n_4= 1 \) follows association scheme defined below.

Association scheme

Association scheme of p associate class PBIB designs represent a four associate class association scheme for \( p = 4 \) explained as.
Let us consider a particular vertex (divisor) ‘1’. The vertices 2, 3, 5, 7 occur with ‘1’ in a block as well as all are incidence with ‘1’ by an edge. The treatments assigns to these divisors are first associate of ‘1’ and \( n_1 = 4 \). The vertices 6, 10, 14, 15, 21, 35 occur with ‘1’ in four different blocks, but not incident with ‘1’ by an edge. The treatments assigns to these divisors are second associates of ‘1’ and \( n_2 = 6 \). Remaining vertices which are not occurs with ‘1’ in any block as well as not incidences are 30, 42, 70, 105, 210 gives third and fourth associates of ‘1’ explained as.

a) The product of divisor ‘210’ with a particular divisor ‘1’ is equal to ‘210 = n’.

The treatment belongs to this divisor is \( 4^{th} \) associate of ‘1’ and \( n_4 = 1 \).

b) The product of divisors ‘30, 42, 70, 105’ with ‘7, 5, 3, 2’ respectively are equal to ‘210 = n’. The treatment belongs to these divisors are \( 3^{rd} \) associates of ‘1’ and \( n_3 = 4 \).

In graphical form, we represent the above relation as in figure [3].

**Figure [3]**

*Hasse Diagram of illustration of three associate class PBIB design (p=4)*
\[ P_1 = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]

\[ P_3 = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad P_4 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 6 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

First, Second, Third and Fourth associates of each treatment are given in following table – 4.3.

**Table - 4.3**

<table>
<thead>
<tr>
<th>Treatment (Divisor)</th>
<th>1&lt;sup&gt;st&lt;/sup&gt; Associates</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; Associates</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt; Associates</th>
<th>4&lt;sup&gt;th&lt;/sup&gt; Associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2,3,5,7</td>
<td>6,10,14,15,21,35</td>
<td>30,42,70,105</td>
<td>210</td>
</tr>
<tr>
<td>2</td>
<td>1,6,10,14</td>
<td>3,5,7,30,42,70</td>
<td>15,21,35,210</td>
<td>105</td>
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<tr>
<td>3</td>
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4.9 List of Designs of Series – I, Series - II and Extended Case

Now, we have given various kinds of efficiencies and average efficiency factor (A.E.F) of newly constructed PBIB designs of series - I in table - 4.4. In table – 4.5, we represent the efficiencies $E_1, E_2, E_3$ and $E$ for series - II and table 4.6 represents the efficiencies of $p$ - associate class PBIB designs by using Boolean algebra for suitable values of $v, b, r, k, \lambda_i$’s and $n_i$’s.

### Table.- 4.4

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### Table - 4.5

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<td>14</td>
<td>14</td>
<td>7</td>
<td>7</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>1</td>
<td>.847</td>
<td>.957</td>
<td>.816</td>
<td>.891</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>16</td>
<td>16</td>
<td>8</td>
<td>8</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>7</td>
<td>7</td>
<td>1</td>
<td>.843</td>
<td>.967</td>
<td>.820</td>
<td>.895</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table - 4.6

**Series of p - associate class PBIB designs**

<table>
<thead>
<tr>
<th>p</th>
<th>Associate design</th>
<th>v</th>
<th>b</th>
<th>r</th>
<th>k</th>
<th>$\lambda_i$</th>
<th>$n_i$</th>
<th>E factors</th>
<th>Average E</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Two</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>$\lambda_1 = 2, \lambda_2 = 2$</td>
<td>$n_1 = 2, n_2 = 1$</td>
<td>$E = .88$</td>
<td>.888</td>
</tr>
<tr>
<td>3</td>
<td>Three</td>
<td>8</td>
<td>8</td>
<td>4</td>
<td>4</td>
<td>$\lambda_1 = 2, \lambda_2 = 2$</td>
<td>$n_1 = 3, n_2 = 3$</td>
<td>$E = .86$</td>
<td>.840</td>
</tr>
<tr>
<td>4</td>
<td>Four</td>
<td>16</td>
<td>16</td>
<td>5</td>
<td>5</td>
<td>$\lambda_1 = 2, \lambda_2 = 2$</td>
<td>$n_1 = 4, n_2 = 6$</td>
<td>$E = .85$</td>
<td>.822</td>
</tr>
<tr>
<td>5</td>
<td>Five</td>
<td>32</td>
<td>32</td>
<td>6</td>
<td>6</td>
<td>$\lambda_1 = 2, \lambda_2 = 2$</td>
<td>$n_1 = 5, n_2 = 10$</td>
<td>$E = .84$</td>
<td>.818</td>
</tr>
</tbody>
</table>

**4.10 Graphical Representation of Constructed Designs**

We have read the efficiencies of table of some two associate class PBIB designs listed by Clatworthy (1973) and also compare listed designs in table - 4.4, 4.5 and in table 4.6 with some other higher associate class PBIB designs given by various authors by using various techniques. Designs listed in table – 4.4 have wider than designs listed in table - 4.5 and in table - 4.6. Now, we represent all kinds of efficiencies in graphical form by using column charts for all series for all listed designs in table - 4.4, table – 4.5 and table - 4.6 as.
GRAPH – [1] represent all kinds of efficiencies as well as average efficiency factor (A.E.F) for listed designs of table - 4.4

GRAPH – [2] represent all kinds of efficiencies and average efficiency factor (A.E.F) for listed designs of table - 4.5.

4.11 Summary and Discussion

In this chapter, in series - I we represent a new class of four associate PBIB designs by using partial geometry with finite graphs. However, in series – II, we develop a new class of three associate PBIB designs by introducing the concept of Boolean algebra along with some particular restrictions. Both series - I and II exhibit much efficient results in their respective associate class PBIB designs. Series-I is wider than series - II and its extended case which gives few p – associate class PBIB designs, because in series - II and series of p – associate class PBIB designs, number of treatments and blocks are increased very rapidly due to the product of prime numbers. Designs generated from all the series are comparable with existing designs and also found to be more efficient and more applicable in practical situations.
The important factor in the present research work is that, we introduce a new link between field of mathematics with a part of statistics by developing PBIB designs using Boolean algebra. As number of replications in series -1 is three, but in PBIB designs developing from this link, there is no restriction to number of replications. However, we also update the connection of graph theory with PBIB designs by doing some modification in graph of partial geometry. On the whole, we hope our construction methods along with their association schemes also give some new ideas to introduce and construct some more new series of PBIB designs from various other mathematical fields.