CHAPTER - VII

Analysis of PBIB Designs and General Solutions of Normal Equations for \( m (m \geq 2) \) Associate Class PBIB Designs

Introduction

With the construction of new PBIB designs having higher associate classes in the previous chapters, the need arises to analysis these designs with more ease. The general analysis of PBIB designs with \( m (m \geq 2) \) associate classes is available in literature as discussed in Dey (1986). Also, he has given the general solution of the normal equations in the intra – block analysis of two associate class PBIB designs. As far as the general solution of the normal equations in the intra – block analysis of three and four associate class PBIB designs are concerned these were introduced by Rao (1947b) and Garg and Mishra (2013) respectively.

With the availability of the general solution of the normal equations in the intra – block analysis of PBIB designs, it becomes very easy to get the estimates of various treatment effects and also to get the estimates of different elementary treatment contrasts. Since in this thesis, we have constructed higher associate class PBIB designs, so we have attempted to find the general solution of the normal equations in the intra – block analysis of PBIB designs which were not available in literature. We have successfully obtained the general solution of the normal equations in the intra – block analysis of PBIB designs for general (any) associate class PBIB designs. We have verified it for \( m = 2, 3, 4 \) and also obtained general solution for five associate class PBIB designs. For \( m = 2 \), solution obtained is equivalent to Dey (1986), for \( m = 3 \) and \( m = 4 \), solution obtained are equivalent to Rao (1947b) and Garg and Mishra (2013) respectively. So, clearly Dey (1986), Rao (1947b) and Garg and Mishra (2013)
are the particular cases of our general solution of normal equations in the intra – block analysis of PBIB designs.

In section 7.1, we reproduce the general analysis of a PBIB design having \( m \) (\( m \geq 2 \)) associate class as given by Dey (1986). In section 7.2, we have expressed the normal equation for intra – block analysis of five associate class PBIB design. We solve reduced normal equations simultaneously to get the variances of estimated elementary treatment contrast among these treatments and give the five kinds of efficiencies and average efficiency factor (A.E.F) in section 7.3. Generalized structure of reduced normal equations in the intra – block analysis of PBIB designs having higher associate class PBIB designs along with their respective kinds of efficiencies as well as average efficiency factor (A.E.F) are introduced in section 7.4. We verify the results of general solution of reduced normal equations for the analysis of three and four associate class PBIB designs originally introduced by Rao (1947b) and Garg and Mishra (2013) in section 7.5. A brief discussion on the merits of general structure of analysis of higher associate class PBIB designs is given in section 7.6.

### 7.1 General Analysis of Higher Associate Class PBIB Design

The analysis of higher associate class PBIB designs given by Dey (1986) is reproducing as:

Let us a \( m \) associate class PBIB designs having \( v \) treatments in \( b \) blocks with plot size \( k \) and replicates \( r \). Then, reduced normal equations of intra block model for treatment effects are

\[
\begin{align*}
 r(k-1)\tau_i - \lambda_1S_1(\tau_i) - \lambda_2S_2(\tau_i) - \ldots - \lambda_mS_m(\tau_i) &= k Q_i \\
\end{align*}
\]  

--- (7.1.1)

where \( i = 1,2,\ldots,m \).
$S_j(\tau_i)$ represent the sum of treatment effects those $n_j$ treatments which are $j$-th associates of treatment ‘$i$’ for $j = 1, 2, \ldots, m$.

Suppose, $n_j$ be the $j$-th associates of $i$-th treatment effects $\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_{n_j}$.

Now, rewriting the normal equations in respect of $\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_{n_j}$, we get

\[ r(k-1) \alpha^i_1 - \lambda_1 S_1(\alpha^i_1) - \lambda_2 S_2(\alpha^i_1) - \lambda_3 S_3(\alpha^i_1) - \ldots - \lambda_m S_m(\alpha^i_1) = k Q(\alpha^i_1) \]

\[ r(k-1) \alpha^i_2 - \lambda_1 S_1(\alpha^i_2) - \lambda_2 S_2(\alpha^i_2) - \lambda_3 S_3(\alpha^i_2) - \ldots - \lambda_m S_m(\alpha^i_2) = k Q(\alpha^i_2) \]

\[ r(k-1) \alpha^i_3 - \lambda_1 S_1(\alpha^i_3) - \lambda_2 S_2(\alpha^i_3) - \lambda_3 S_3(\alpha^i_3) - \ldots - \lambda_m S_m(\alpha^i_3) = k Q(\alpha^i_3) \]

\[ \vdots \]

\[ r(k-1) \alpha^i_{n_j} - \lambda_1 S_1(\alpha^i_{n_j}) - \lambda_2 S_2(\alpha^i_{n_j}) - \lambda_3 S_3(\alpha^i_{n_j}) - \ldots - \lambda_m S_m(\alpha^i_{n_j}) = k Q(\alpha^i_{n_j}) \]

--- (7.1.2)

Summing over $n_j$ equations as given above, we get an equation given below as

\[ r(k-1) S_j(\tau_i) - \lambda_1 P_1 - \lambda_2 P_2 - \lambda_3 P_3 - \ldots - \lambda_m P_m = k S_j(Q_i) \]

--- (7.1.3)

where $P_1, P_2, \ldots, P_m$ are linear functions of treatment effects and $S_j(Q_i)$. Now, from the definition of a PBIB design, we write

\[ P_u = \sum_{t=1}^{m} p_{ju}^f S^t(\tau_u) \quad \text{for } u \neq j, \quad u = 1, 2, \ldots, m. \]

\[ P_j = \sum_{t=1}^{m} \tau_t + \sum_{t=1}^{m} p_{ju}^f S^t(\tau_i). \]

Now, substituting for $P_u$’s in above equation (7.1.3) and rearrange, we get the equation as given below

\[ r(k-1) S_j(\tau_i) - S_i(\tau_i) \sum_{t=1}^{m} \lambda_\tau p_{j\tau}^1 - S_2(\tau_i) \sum_{t=1}^{m} \lambda_\tau p_{j\tau}^2 - S_3(\tau_i) \sum_{t=1}^{m} \lambda_\tau p_{j\tau}^3 - \ldots - S_m(\tau_i) \]
\[ \sum_{t=1}^{m} \lambda_t p_{jlt}^m - n_j \lambda j \tau_i = k S_j (Q_i) \]  

--- (7.1.4)

and solved these equations to get a solution for \( \tau_i \) by taking the restriction for conveniently is

\[ \tau_i + S_1(\tau_i) + S_2(\tau_i) + S_3(\tau_i) + \ldots + S_m(\tau_i) = 0 \]  

---(7.1.5)

### 7.2 Reduced Normal Equations for Five Associate Class PBIB Design

The linear model assuming various effects to be additive for a two way classified data with one observation per cell is

\[ y_{ij} = \mu + \tau_i + b_j + \varepsilon_{ij} \]

where \( y_{ij} \) is the yield of the experimental unit from \( i^{th} \) treatment and \( j^{th} \) block ; \( \mu \) is the general mean effect ; \( \tau_i \) is the effect due to \( i^{th} \) treatment; \( b_j \) is the effect due to \( j^{th} \) block and \( \varepsilon_{ij} \) is the error effect due to random component which are independently normally distributed with mean zero and variance \( \sigma_e^2 \).

The model used for the analysis of five associate class PBIB designs is no different from the model used for the analysis of two associate class PBIB designs as given above. The only difference is that we get five types of treatment contrast among treatments due to the existence of five associate classes.

We are interested only for estimating linear combinations of treatment effects for \( m \) (\( m \geq 2 \)) associate class PBIB design with parameters \( v, b, r, k, \lambda_i, n_i, p_{jk}^i ; i, j, k = 1, 2, \ldots, m \).

Let us consider a five associate class PBIB design (\( m = 5 \)) having ‘\( v \)’ treatments, ‘\( b \)’ blocks, ‘\( r \)’ replications with ‘\( k \)’ block size.
Applying the least square method of estimation, the reduced normal equation of linear model are given by

\[ r(k-1)\tau_i - [\lambda_1 S_1(\tau_i) + \lambda_2 S_2(\tau_i) + \lambda_3 S_3(\tau_i) + \lambda_4 S_4(\tau_i) + \lambda_5 S_5(\tau_i)] = kQ_i \quad ---(7.2.1) \]

where \( i = 1, 2, 3, \ldots, v \) and \( S_j(\tau_i) \) denoted the sum of those \( n_j \) treatment effects which are the \( j^{th} \) associates of \( i \), for \( j = 1, 2, 3, 4, 5 \).

Now, let the \( n_j \), \( j^{th} \) associates of \( i^{th} \) treatment effects \( \alpha_1, \alpha_2, \ldots, \alpha_{n_j} \). Rewriting the normal equations in respect of \( \alpha_1, \alpha_2, \ldots, \alpha_{n_j} \), we get

\[ r(k-1) \alpha_1^i - \lambda_1 S_1(\alpha_1^i) - \lambda_2 S_2(\alpha_1^i) - \lambda_3 S_3(\alpha_1^i) - \lambda_4 S_4(\alpha_1^i) - \lambda_5 S_5(\alpha_1^i) = kQ(\alpha_1^i) \]

\[ r(k-1) \alpha_2^i - \lambda_1 S_1(\alpha_2^i) - \lambda_2 S_2(\alpha_2^i) - \lambda_3 S_3(\alpha_2^i) - \lambda_4 S_4(\alpha_2^i) - \lambda_5 S_5(\alpha_2^i) = kQ(\alpha_2^i) \]

\[ r(k-1) \alpha_3^i - \lambda_1 S_1(\alpha_3^i) - \lambda_2 S_2(\alpha_3^i) - \lambda_3 S_3(\alpha_3^i) - \lambda_4 S_4(\alpha_3^i) - \lambda_5 S_5(\alpha_3^i) = kQ(\alpha_3^i) \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ r(k-1) \alpha_{n_j}^i - \lambda_1 S_1(\alpha_{n_j}^i) - \lambda_2 S_2(\alpha_{n_j}^i) - \lambda_3 S_3(\alpha_{n_j}^i) - \lambda_4 S_4(\alpha_{n_j}^i) - \lambda_5 S_5(\alpha_{n_j}^i) = kQ(\alpha_{n_j}^i) \quad ---(7.2.2) \]

Summing over \( n_j \) equations in set (7.2.2) for \( j = 1, 2, 3, 4, 5 \) we get an equation as

\[ r(k-1) S_j(\tau_i) - \lambda_1 P_1 - \lambda_2 P_2 - \lambda_3 P_3 - \lambda_4 P_4 - \lambda_5 P_5 = kS_j(Q_i) \quad ---(7.2.3) \]

where \( P_1, P_2, P_3, P_4 \) and \( P_5 \) are linear functions of treatment effects and \( S_j(Q_i) \). From the definition of a PBIB design, we write

\[ p_u = \sum_{i=1}^{5} p_{ju}^i S_i(\tau_i) \quad \text{for} \quad u \neq j, \quad u = 1, 2, 3, 4, 5, \]

\[ p_j = n_j \tau_i + \sum_{i=1}^{5} p_{ju}^i S_i(\tau_i). \]

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Now, substituting for \( P_u \)'s in equation (7.2.3) and rearranging, we get equations as given below

\[
\begin{align*}
\text{r(k-1) } S_1(\tau_i) - S_1(\tau_i) \sum_{t=1}^{5} \lambda_t p^1_{jt} - S_2(\tau_i) \sum_{t=1}^{5} \lambda_t p^2_{jt} - S_3(\tau_i) \sum_{t=1}^{5} \lambda_t p^3_{jt} & - S_4(\tau_i) \sum_{t=1}^{5} \lambda_t p^4_{jt} - S_5(\tau_i) \sum_{t=1}^{5} \lambda_t p^5_{jt} - n_j \lambda_j \tau_i = k S_j (Q_i) \quad (7.2.4) \\
\text{Now, we expand equation (7.2.4) for } j = 1, 2, 3, 4, 5 \text{ as given below respectively} \\
\text{r(k-1) } S_1(\tau_i) - S_1(\tau_i) [ \lambda_1 p^1_{11} + \lambda_2 p^1_{12} + \lambda_3 p^1_{13} + \lambda_4 p^1_{14} + \lambda_5 p^1_{15} ] - S_2(\tau_i) [ \lambda_1 p^2_{11} + \lambda_2 p^2_{12} + \lambda_3 p^2_{13} + \lambda_4 p^2_{14} + \lambda_5 p^2_{15} ] - S_3(\tau_i) [ \lambda_1 p^3_{11} + \lambda_2 p^3_{12} + \lambda_3 p^3_{13} + \lambda_4 p^3_{14} + \lambda_5 p^3_{15} ] - S_4(\tau_i) [ \lambda_1 p^4_{11} + \lambda_2 p^4_{12} + \lambda_3 p^4_{13} + \lambda_4 p^4_{14} + \lambda_5 p^4_{15} ] - S_5(\tau_i) [ \lambda_1 p^5_{11} + \lambda_2 p^5_{12} + \lambda_3 p^5_{13} + \lambda_4 p^5_{14} + \lambda_5 p^5_{15} ] - n_1 \lambda_1 \tau_i = k S_1 (Q_i) \\
\text{r(k-1) } S_2(\tau_i) - S_1(\tau_i) [ \lambda_1 p^1_{21} + \lambda_2 p^1_{22} + \lambda_3 p^1_{23} + \lambda_4 p^1_{24} + \lambda_5 p^1_{25} ] - S_2(\tau_i) [ \lambda_1 p^2_{21} + \lambda_2 p^2_{22} + \lambda_3 p^2_{23} + \lambda_4 p^2_{24} + \lambda_5 p^2_{25} ] - S_3(\tau_i) [ \lambda_1 p^3_{21} + \lambda_2 p^3_{22} + \lambda_3 p^3_{23} + \lambda_4 p^3_{24} + \lambda_5 p^3_{25} ] - S_4(\tau_i) [ \lambda_1 p^4_{21} + \lambda_2 p^4_{22} + \lambda_3 p^4_{23} + \lambda_4 p^4_{24} + \lambda_5 p^4_{25} ] - S_5(\tau_i) [ \lambda_1 p^5_{21} + \lambda_2 p^5_{22} + \lambda_3 p^5_{23} + \lambda_4 p^5_{24} + \lambda_5 p^5_{25} ] - n_2 \lambda_2 \tau_i = k S_2 (Q_i) \\
\text{r(k-1) } S_3(\tau_i) - S_1(\tau_i) [ \lambda_1 p^1_{31} + \lambda_2 p^1_{32} + \lambda_3 p^1_{33} + \lambda_4 p^1_{34} + \lambda_5 p^1_{35} ] - S_2(\tau_i) [ \lambda_1 p^2_{31} + \lambda_2 p^2_{32} + \lambda_3 p^2_{33} + \lambda_4 p^2_{34} + \lambda_5 p^2_{35} ] - S_3(\tau_i) [ \lambda_1 p^3_{31} + \lambda_2 p^3_{32} + \lambda_3 p^3_{33} + \lambda_4 p^3_{34} + \lambda_5 p^3_{35} ] - S_4(\tau_i) [ \lambda_1 p^4_{31} + \lambda_2 p^4_{32} + \lambda_3 p^4_{33} + \lambda_4 p^4_{34} + \lambda_5 p^4_{35} ] - S_5(\tau_i) [ \lambda_1 p^5_{31} + \lambda_2 p^5_{32} + \lambda_3 p^5_{33} + \lambda_4 p^5_{34} + \lambda_5 p^5_{35} ] - n_3 \lambda_3 \tau_i = k S_3 (Q_i) \\
\text{r(k-1) } S_4(\tau_i) - S_1(\tau_i) [ \lambda_1 p^1_{41} + \lambda_2 p^1_{42} + \lambda_3 p^1_{43} + \lambda_4 p^1_{44} + \lambda_5 p^1_{45} ] - S_2(\tau_i) [ \lambda_1 p^2_{41} + \lambda_2 p^2_{42} + \lambda_3 p^2_{43} + \lambda_4 p^2_{44} + \lambda_5 p^2_{45} ] - S_3(\tau_i) [ \lambda_1 p^3_{41} + \lambda_2 p^3_{42} + \lambda_3 p^3_{43} + \lambda_4 p^3_{44} + \lambda_5 p^3_{45} ] - S_4(\tau_i) [ \lambda_1 p^4_{41} + \lambda_2 p^4_{42} + \lambda_3 p^4_{43} + \lambda_4 p^4_{44} + \lambda_5 p^4_{45} ] - S_5(\tau_i) [ \lambda_1 p^5_{41} + \lambda_2 p^5_{42} + \lambda_3 p^5_{43} + \lambda_4 p^5_{44} + \lambda_5 p^5_{45} ] - n_4 \lambda_4 \tau_i = k S_4 (Q_i) \\
\text{r(k-1) } S_5(\tau_i) - S_1(\tau_i) [ \lambda_1 p^1_{51} + \lambda_2 p^1_{52} + \lambda_3 p^1_{53} + \lambda_4 p^1_{54} + \lambda_5 p^1_{55} ] - S_2(\tau_i) [ \lambda_1 p^2_{51} + \lambda_2 p^2_{52} + \lambda_3 p^2_{53} + \lambda_4 p^2_{54} + \lambda_5 p^2_{55} ] - S_3(\tau_i) [ \lambda_1 p^3_{51} + \lambda_2 p^3_{52} + \lambda_3 p^3_{53} + \lambda_4 p^3_{54} + \lambda_5 p^3_{55} ] - S_4(\tau_i) [ \lambda_1 p^4_{51} + \lambda_2 p^4_{52} + \lambda_3 p^4_{53} + \lambda_4 p^4_{54} + \lambda_5 p^4_{55} ] - S_5(\tau_i) [ \lambda_1 p^5_{51} + \lambda_2 p^5_{52} + \lambda_3 p^5_{53} + \lambda_4 p^5_{54} + \lambda_5 p^5_{55} ] - n_5 \lambda_5 \tau_i = k S_5 (Q_i) \\

\text{(7.2.8)}
\end{align*}
\]
\[ \lambda_2 p_{52}^2 + \lambda_3 p_{53}^2 + \lambda_4 p_{54}^2 + \lambda_5 p_{55}^2 \] – \[ S_5(\tau_i) [ \lambda_1 p_{51}^3 + \lambda_2 p_{52}^3 + \lambda_3 p_{53}^3 + \lambda_4 p_{54}^3 + \lambda_5 p_{55}^3 \] – \[ S_4(\tau_i) [ \lambda_1 p_{41}^4 + \lambda_2 p_{42}^4 + \lambda_3 p_{43}^4 + \lambda_4 p_{44}^4 + \lambda_5 p_{45}^4 \] – \[ S_3(\tau_i) [ \lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 p_{35}^5 \] \]

\[ \tau_i + S_1(\tau_i) + S_2(\tau_i) + S_3(\tau_i) + S_4(\tau_i) + S_5(\tau_i) = 0 \] 

---(7.2.10)

**7.3 Solution of Normal Equations for Five Associate Class PBIB Design**

To get the solution of \( \tau_i \) for \( m=5 \), firstly consider an equation (7.2.1) written as below

\[ r(k-1) \tau_i - \lambda_1 S_1(\tau_i) - \lambda_2 S_2(\tau_i) - \lambda_3 S_3(\tau_i) - \lambda_4 S_4(\tau_i) - \lambda_5 S_5(\tau_i) = k Q_i \]

Substitute ‘\( S_5(\tau_i) \)’ from equation (7.2.10) in above equation and we get

\[ r(k-1) \tau_i - \lambda_1 S_1(\tau_i) - \lambda_2 S_2(\tau_i) - \lambda_3 S_3(\tau_i) - \lambda_4 S_4(\tau_i) + \lambda_5 [\tau_i + S_1(\tau_i) + S_2(\tau_i) + S_3(\tau_i) + S_4(\tau_i)] = k Q_i \]

or

\[ [r(k-1) + \lambda_5] \tau_i + (\lambda_5 - \lambda_1) S_1(\tau_i) + (\lambda_5 - \lambda_2) S_2(\tau_i) + (\lambda_5 - \lambda_3) S_3(\tau_i) + (\lambda_5 - \lambda_4) S_4(\tau_i) = k Q_i. \] 

---(7.3.1)

Substitute ‘\( S_5(\tau_i) \)’ from equation (7.2.10) in equation (7.2.5) and it becomes

\[ r(k-1) S_1(\tau_i) - S_1(\tau_i) [ \lambda_1 p_{11}^4 + \lambda_2 p_{12}^3 + \lambda_3 p_{13}^2 + \lambda_4 p_{14} + \lambda_5 p_{15} ] - S_2(\tau_i) [ \lambda_1 p_{21}^4 + \lambda_2 p_{22}^3 + \lambda_3 p_{23}^2 + \lambda_4 p_{24} + \lambda_5 p_{25} ] - S_3(\tau_i) [ \lambda_1 p_{31}^4 + \lambda_2 p_{32}^3 + \lambda_3 p_{33}^2 + \lambda_4 p_{34} + \lambda_5 p_{35} ] - S_4(\tau_i) [ \lambda_1 p_{41}^4 + \lambda_2 p_{42}^3 + \lambda_3 p_{43}^2 + \lambda_4 p_{44} + \lambda_5 p_{45} ] + [\tau_i + S_1(\tau_i) + S_2(\tau_i) + S_3(\tau_i) + S_4(\tau_i)] - n_1 \lambda_1 \tau_i = k S_1(Q_i) \]

or

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As, we know that

\[ p_{11}^1 + p_{12}^1 + p_{13}^1 + p_{14}^1 + p_{15}^1 = n_1 - 1. \]

\[ p_{11}^2 + p_{12}^2 + p_{13}^2 + p_{14}^2 + p_{15}^2 = n_1 \]

\[ p_{11}^3 + p_{12}^3 + p_{13}^3 + p_{14}^3 + p_{15}^3 = n_1 \]

\[ p_{11}^4 + p_{12}^4 + p_{13}^4 + p_{14}^4 + p_{15}^4 = n_1 \]

\[ p_{11}^5 + p_{12}^5 + p_{13}^5 + p_{14}^5 + p_{15}^5 = n_1 \]

using

\[ n_1 = p_{11}^5 + p_{12}^5 + p_{13}^5 + p_{14}^5 + p_{15}^5 \]

\[ p_{15}^1 = n_1 - 1 - p_{11}^1 - p_{12}^1 - p_{13}^1 - p_{14}^1 \]

\[ p_{15}^2 = n_1 - p_{11}^2 - p_{12}^2 - p_{13}^2 - p_{14}^2 \]

\[ p_{15}^3 = n_1 - p_{11}^3 - p_{12}^3 - p_{13}^3 - p_{14}^3 \]

\[ p_{15}^4 = n_1 - p_{11}^4 - p_{12}^4 - p_{13}^4 - p_{14}^4 \]

\[ p_{15}^5 = n_1 - p_{11}^5 - p_{12}^5 - p_{13}^5 - p_{14}^5 \]

in above equation (7.2.5), it becomes
\[ \lambda_1 p^5_{11} + \lambda_2 p^5_{12} + \lambda_3 p^5_{13} + \lambda_4 p^5_{14} + \lambda_5 p^5_{15} - \lambda_4 p^5_{11} - \lambda_4 p^5_{12} - \lambda_4 p^5_{13} - \lambda_4 p^5_{14} - \lambda_4 p^5_{15} \] 
\[ \tau_i + S_1(\tau_i) \quad r(k-1) - \lambda_1 p^5_{11} - \lambda_2 p^5_{12} - \lambda_3 p^5_{13} - \lambda_4 p^5_{14} - \lambda_5 (n_1 - 1 - p^5_{11} - p^5_{12} - p^5_{13} - p^5_{14}) + \lambda_2 p^5_{11} + \lambda_2 p^5_{12} + \lambda_3 p^5_{13} + \lambda_4 p^5_{14} + \lambda_5 (n_1 - p^5_{11} - p^5_{12} - p^5_{13} - p^5_{14}) - \lambda_1 p^5_{11} - \lambda_2 p^5_{12} - \lambda_3 p^5_{13} - \lambda_4 p^5_{14} - \lambda_5 (n_1 - p^5_{11} - p^5_{12} - p^5_{13} - p^5_{14}) \]

and after solving it, we get

\[ [(\lambda_2 - \lambda_1) p^5_{12} + (\lambda_3 - \lambda_1) p^5_{13} + (\lambda_4 - \lambda_1) p^5_{14} + (\lambda_5 - \lambda_1) p^5_{15}] \tau_i + [r(k-1) + \lambda_5 + (\lambda_5 - \lambda_1) (p^5_{11} - p^5_{12}) + (\lambda_5 - \lambda_3) (p^5_{13} - p^5_{14}) + (\lambda_5 - \lambda_4) (p^5_{14} - p^5_{15})] S_1(\tau_i) + [(\lambda_5 - \lambda_1) (p^5_{11} - p^5_{12}) + (\lambda_5 - \lambda_2) (p^5_{12} - p^5_{13}) + (\lambda_5 - \lambda_3) (p^5_{13} - p^5_{14}) + (\lambda_5 - \lambda_4) (p^5_{14} - p^5_{15})] S_2(\tau_i) + [(\lambda_5 - \lambda_1) (p^5_{11} - p^5_{12}) + (\lambda_5 - \lambda_2) (p^5_{12} - p^5_{13}) + (\lambda_5 - \lambda_3) (p^5_{13} - p^5_{14}) + (\lambda_5 - \lambda_4) (p^5_{14} - p^5_{15})] S_3(\tau_i) + [(\lambda_5 - \lambda_1) (p^5_{11} - p^5_{12}) + (\lambda_5 - \lambda_2) (p^5_{12} - p^5_{13}) + (\lambda_5 - \lambda_3) (p^5_{13} - p^5_{14}) + (\lambda_5 - \lambda_4) (p^5_{14} - p^5_{15})] S_4(\tau_i) = k S_1(\tau_i) \quad \text{--- (7.3.2)} \]

Substitute ‘S_2(\tau_i)’ from equation (7.2.10) in equation (7.2.6), it becomes

\[ r(k-1) S_2(\tau_i) - S_1(\tau_i) \quad [\lambda_1 p^5_{21} + \lambda_2 p^5_{22} + \lambda_3 p^5_{23} + \lambda_4 p^5_{24} + \lambda_5 p^5_{25} - S_2(\tau_i) \quad [\lambda_1 p^5_{21} + \lambda_2 p^5_{22} + \lambda_3 p^5_{23} + \lambda_4 p^5_{24} + \lambda_5 p^5_{25}] - S_3(\tau_i) \quad [\lambda_1 p^5_{21} + \lambda_2 p^5_{22} + \lambda_3 p^5_{23} + \lambda_4 p^5_{24} + \lambda_5 p^5_{25}] - S_4(\tau_i) \quad [\lambda_1 p^5_{21} + \lambda_2 p^5_{22} + \lambda_3 p^5_{23} + \lambda_4 p^5_{24} + \lambda_5 p^5_{25}] + [\tau_i + S_1(\tau_i) + S_2(\tau_i) + S_3(\tau_i) + S_4(\tau_i)] = -n_2 \lambda_2 \tau_i = k S_2(\tau_i) \]

or

\[ [\lambda_1 p^5_{21} + \lambda_2 p^5_{22} + \lambda_3 p^5_{23} + \lambda_4 p^5_{24} + \lambda_5 p^5_{25} - n_2 \lambda_2] \tau_i + [\lambda_1 p^5_{21} + \lambda_2 p^5_{22} + \lambda_3 p^5_{23} + \lambda_4 p^5_{24} + \lambda_5 p^5_{25}] S_1(\tau_i) + [r(k-1) - \lambda_1 p^2_{21} -
\[
\lambda_2 p_{22}^2 - \lambda_3 p_{23}^2 - \lambda_4 p_{24}^2 - \lambda_5 p_{25}^2 + \lambda_1 p_{21}^5 + \lambda_2 p_{22}^5 + \lambda_3 p_{23}^5 + \lambda_4 p_{24}^5 + \lambda_5 p_{25}^5 \] S_2(\tau_i) + \\
[ \lambda_1 p_{21}^5 + \lambda_2 p_{22}^5 + \lambda_3 p_{23}^5 + \lambda_4 p_{24}^5 + \lambda_5 p_{25}^5 - \lambda_1 p_{21}^3 - \lambda_2 p_{22}^3 - \lambda_3 p_{23}^3 - \lambda_4 p_{24}^3 - \lambda_5 p_{25}^3 ] S_3(\tau_i) + \\
[ \lambda_1 p_{21}^5 + \lambda_2 p_{22}^5 + \lambda_3 p_{23}^5 + \lambda_4 p_{24}^5 + \lambda_5 p_{25}^5 - \lambda_1 p_{21}^4 - \lambda_2 p_{22}^4 - \lambda_3 p_{23}^4 - \lambda_4 p_{24}^4 - \lambda_5 p_{25}^4 ] S_4(\tau_i) = k S_2(Q_i)
\]

As, we know that

\[
p_{121}^1 + p_{122}^1 + p_{123}^1 + p_{124}^1 + p_{125}^1 = n_2
\]

\[
p_{121}^2 + p_{122}^2 + p_{123}^2 + p_{124}^2 + p_{125}^2 = n_2 - 1
\]

\[
p_{121}^3 + p_{122}^3 + p_{123}^3 + p_{124}^3 + p_{125}^3 = n_2
\]

\[
p_{121}^4 + p_{122}^4 + p_{123}^4 + p_{124}^4 + p_{125}^4 = n_2
\]

\[
p_{121}^5 + p_{122}^5 + p_{123}^5 + p_{124}^5 + p_{125}^5 = n_2
\]

using

\[
n_2 = p_{121}^5 + p_{122}^5 + p_{123}^5 + p_{124}^5 + p_{125}^5
\]

\[
p_{121}^2 = n_2 - p_{121}^1 - p_{122}^1 - p_{123}^1 - p_{124}^1
\]

\[
p_{121}^3 = n_2 - 1 - p_{121}^2 - p_{122}^2 - p_{123}^2 - p_{124}^2 - p_{125}^2
\]

\[
p_{121}^4 = n_2 - p_{121}^3 - p_{122}^3 - p_{123}^3 - p_{124}^3 - p_{125}^3
\]

\[
p_{121}^5 = n_2 - p_{121}^4 - p_{122}^4 - p_{123}^4 - p_{124}^4 - p_{125}^4
\]

\[
p_{121}^5 = n_2 - p_{121}^5 - p_{122}^5 - p_{123}^5 - p_{124}^5
\]

in above equation (7.2.6), it becomes

\[
[ \lambda_1 p_{21}^5 + \lambda_2 p_{22}^5 + \lambda_3 p_{23}^5 + \lambda_4 p_{24}^5 + \lambda_5 p_{25}^5 - (p_{21}^5 + p_{22}^5 + p_{23}^5 + p_{24}^5 + p_{25}^5)\lambda_2 ] \tau_i + [ \]
\[ \lambda_1 p_{21}^5 + \lambda_2 p_{22}^5 + \lambda_3 p_{23}^5 + \lambda_4 p_{24}^5 + \lambda_5 (n_2 - p_{21}^5 - p_{22}^5 - p_{23}^5 - p_{24}^5) - \lambda_1 p_{21}^4 - \lambda_2 p_{22}^4 - \lambda_3 p_{23}^4 - \lambda_4 p_{24}^4 - \lambda_5 (n_2 - p_{21}^4 - p_{22}^4 - p_{23}^4 - p_{24}^4)] S_1(t_i) + [r(k-1) - \lambda_1 p_{21}^2 - \lambda_2 p_{22}^2 - \lambda_3 p_{23}^2 - \lambda_4 p_{24}^2 - \lambda_5 (n_2 - p_{21}^2 - p_{22}^2 - p_{23}^2 - p_{24}^2)] S_2(t_i) + [\lambda_1 p_{21}^5 + \lambda_2 p_{22}^5 + \lambda_3 p_{23}^5 + \lambda_4 p_{24}^5 + \lambda_5 (n_2 - p_{21}^5 - p_{22}^5 - p_{23}^5 - p_{24}^5)] S_3(t_i) + [\lambda_1 p_{21}^4 - \lambda_2 p_{22}^4 - \lambda_3 p_{23}^4 - \lambda_4 p_{24}^4 - \lambda_5 (n_2 - p_{21}^4 - p_{22}^4 - p_{23}^4 - p_{24}^4)] S_4(t_i) = k S_2 (Q_i) \]

and after solving it, we get

\[
[ (\lambda_1 - \lambda_2) p_{21}^5 + (\lambda_3 - \lambda_2) p_{23}^5 + (\lambda_4 - \lambda_2) p_{24}^5 + (\lambda_5 - \lambda_2) p_{25}^5 ] t_i + [ (\lambda_5 - \lambda_1) (p_{21}^5 - p_{21}^5) + (\lambda_5 - \lambda_3) (p_{23}^5 - p_{23}^5) + (\lambda_5 - \lambda_4) (p_{24}^5 - p_{24}^5) ] S_1(t_i) + [r(k-1)] + (\lambda_5 - \lambda_1) (p_{22}^5 - p_{22}^5) + (\lambda_5 - \lambda_3) (p_{24}^5 - p_{24}^5) ] S_2(t_i) + [(\lambda_5 - \lambda_1) (p_{22}^5 - p_{22}^5) + (\lambda_5 - \lambda_3) (p_{24}^5 - p_{24}^5) ] S_3(t_i) + [(\lambda_5 - \lambda_1) (p_{22}^5 - p_{22}^5) + (\lambda_5 - \lambda_3) (p_{24}^5 - p_{24}^5) ] S_4(t_i) = k S_2 (Q_i) \]

--- (7.3.3)

Equation (7.2.7) becomes

\[
r(k-1) S_3(t_i) - S_1(t_i) [ \lambda_1 p_{31}^4 + \lambda_2 p_{32}^4 + \lambda_3 p_{33}^4 + \lambda_4 p_{34}^4 + \lambda_5 p_{35}^4 ] - S_2(t_i) [ \lambda_1 p_{31}^3 + \lambda_2 p_{32}^3 + \lambda_3 p_{33}^3 + \lambda_4 p_{34}^3 + \lambda_5 p_{35}^3 ] - S_3(t_i) [ \lambda_1 p_{31}^2 + \lambda_2 p_{32}^2 + \lambda_3 p_{33}^2 + \lambda_4 p_{34}^2 + \lambda_5 p_{35}^2 ] - S_4(t_i) [ \lambda_1 p_{31} + \lambda_2 p_{32} + \lambda_3 p_{33} + \lambda_4 p_{34}^2 + \lambda_5 p_{35}^2 ] + [t_i + S_1(t_i) + S_2(t_i) + S_3(t_i) + S_4(t_i)] \]

or

\[
[ \lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 p_{35}^5 - n_3 \lambda_3 t_i = k S_3 (Q_i) \]

or

\[
[ \lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 p_{35}^5 - n_3 \lambda_3 ] t_i + [ \lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 p_{35}^5 - (\lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 p_{35}^5) ] S_1(t_i) + [\lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 p_{35}^5 ] S_2(t_i) + [\lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 p_{35}^5 ] S_3(t_i) + [ \lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 p_{35}^5 ] S_4(t_i) = k S_2 (Q_i) \]

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r(k – 1) - \lambda_1 p^3_{31} - \lambda_2 p^3_{32} - \lambda_3 p^3_{33} - \lambda_4 p^3_{34} - \lambda_5 p^3_{35} + \lambda_1 p^5_{31} + \lambda_2 p^5_{32} + \lambda_3 p^5_{33} + \lambda_4 p^5_{34} + \lambda_5 p^5_{35} ] S_3(t_i) + [ \lambda_1 p^5_{31} + \lambda_2 p^5_{32} + \lambda_3 p^5_{33} + \lambda_4 p^5_{34} + \lambda_5 p^5_{35} - \lambda_1 p^4_{31} - \lambda_2 p^4_{32} - \lambda_3 p^4_{33} - \lambda_4 p^4_{34} - \lambda_5 p^4_{35} ] S_4(t_i) = k S_3(Q_i)

As, we know that

p^1_{31} + p^1_{32} + p^1_{33} + p^1_{34} + p^1_{35} = n_3

p^2_{31} + p^2_{32} + p^2_{33} + p^2_{34} + p^2_{35} = n_3

p^3_{31} + p^3_{32} + p^3_{33} + p^3_{34} + p^3_{35} = n_3 - 1

p^4_{31} + p^4_{32} + p^4_{33} + p^4_{34} + p^4_{35} = n_3

p^5_{31} + p^5_{32} + p^5_{33} + p^5_{34} + p^5_{35} = n_3

using

n_3 = p^5_{31} + p^5_{32} + p^5_{33} + p^5_{34} + p^5_{35}

p^1_{35} = n_3 - p^1_{31} - p^1_{32} - p^1_{33} - p^1_{34}

p^2_{35} = n_3 - p^2_{31} - p^2_{32} - p^2_{33} - p^2_{34}

p^3_{35} = n_3 - 1 - p^3_{31} - p^3_{32} - p^3_{33} - p^3_{34}

p^4_{35} = n_3 - p^4_{31} - p^4_{32} - p^4_{33} - p^4_{34}

p^5_{35} = n_3 - p^5_{31} - p^5_{32} - p^5_{33} - p^5_{34}

in above equation (7.2.7), it becomes

[ \lambda_1 p^5_{31} + \lambda_2 p^5_{32} + \lambda_3 p^5_{33} + \lambda_4 p^5_{34} + \lambda_5 p^5_{35} - (p^5_{31} + p^5_{32} + p^5_{33} + p^5_{34} + p^5_{35}) \lambda_3 ] t_i +

[ \lambda_1 p^5_{31} + \lambda_2 p^5_{32} + \lambda_3 p^5_{33} + \lambda_4 p^5_{34} + \lambda_5 (n_3 - p^5_{31} - p^5_{32} - p^5_{33} - p^5_{34}) - \lambda_1 p^1_{31} - \lambda_2 p^1_{32} -

\( \lambda_3 p_{33}^1 - \lambda_4 p_{34}^1 - \lambda_5 (n_3 - p_{31}^1 + p_{32}^1 + p_{33}^1 + p_{34}^1) \) \( S_1(\tau_i) + [\lambda_1 p_{31}^5 + \lambda_2 p_{32}^5 + \lambda_3 p_{33}^5 + \lambda_4 p_{34}^5 + \lambda_5 (n_3 - p_{31}^5 - p_{32}^5 - p_{33}^5 - p_{34}^5) - \lambda_1 p_{31}^2 + \lambda_2 p_{32}^2 + \lambda_3 p_{33}^2 + \lambda_4 p_{34}^2 - \lambda_5 (n_3 - p_{31}^2 - p_{32}^2 - p_{33}^2 - p_{34}^2)] \) \( S_2(\tau_i) + [r(k - 1) - \lambda_1 p_{31}^3 - \lambda_2 p_{32}^3 - \lambda_3 p_{33}^3 - \lambda_4 p_{34}^3 - \lambda_5 (n_3 - p_{31}^3 - p_{32}^3 - p_{33}^3 - p_{34}^3) - \lambda_1 p_{31}^4 - \lambda_2 p_{32}^4 - \lambda_3 p_{33}^4 - \lambda_4 p_{34}^4 - \lambda_5 (n_3 - p_{31}^4 - p_{32}^4 - p_{33}^4 - p_{34}^4)] \) \( S_3(\tau_i) = k S_3 (Q_i) \)

and after solving it, we get

\[
\begin{align*}
&[ (\lambda_1 - \lambda_3) p_{31}^5 + (\lambda_2 - \lambda_3) p_{32}^5 + (\lambda_4 - \lambda_3) p_{33}^5 + (\lambda_5 - \lambda_3) p_{35}^5 ] \tau_i + [ (\lambda_5 - \lambda_1)(p_{31}^1 - p_{31}^5) + (\lambda_5 - \lambda_2)(p_{32}^1 - p_{32}^5) + (\lambda_5 - \lambda_3)(p_{33}^1 - p_{33}^5) + (\lambda_5 - \lambda_4)(p_{34}^1 - p_{34}^5) ] \) \( S_1(\tau_i) + [ (\lambda_5 - \lambda_1)(p_{31}^3 - p_{31}^5) + (\lambda_5 - \lambda_2)(p_{32}^3 - p_{32}^5) + (\lambda_5 - \lambda_3)(p_{33}^3 - p_{33}^5) + (\lambda_5 - \lambda_4)(p_{34}^3 - p_{34}^5)] \) \( S_2(\tau_i) + [r(k - 1) + \lambda_5 + (\lambda_5 - \lambda_1)(p_{31}^1 - p_{31}^5) + (\lambda_5 - \lambda_2)(p_{32}^1 - p_{32}^5) + (\lambda_5 - \lambda_3)(p_{33}^1 - p_{33}^5) + (\lambda_5 - \lambda_4)(p_{34}^1 - p_{34}^5)] \) \( S_3(\tau_i) + [ (\lambda_5 - \lambda_1)(p_{31}^3 - p_{31}^5) + (\lambda_5 - \lambda_2)(p_{32}^3 - p_{32}^5) + (\lambda_5 - \lambda_3)(p_{33}^3 - p_{33}^5) + (\lambda_5 - \lambda_4)(p_{34}^3 - p_{34}^5)] \) \( S_4(\tau_i) = k S_3 (Q_i) \)
\]

---(7.3.4)

Equation (7.2.8) becomes

\[
r(k-1) \) \( S_4(\tau_i) - S_1(\tau_i) [ \lambda_1 p_{41}^1 + \lambda_2 p_{42}^1 + \lambda_3 p_{43}^1 + \lambda_4 p_{44}^1 + \lambda_5 p_{45}^1 ] - S_2(\tau_i) [ \lambda_1 p_{41}^2 + \lambda_2 p_{42}^2 + \lambda_3 p_{43}^2 + \lambda_4 p_{44}^2 + \lambda_5 p_{45}^2 ] - S_3(\tau_i) [ \lambda_1 p_{41}^3 + \lambda_2 p_{42}^3 + \lambda_3 p_{43}^3 + \lambda_4 p_{44}^3 + \lambda_5 p_{45}^3 ] - S_4(\tau_i) [ \lambda_1 p_{41}^4 + \lambda_2 p_{42}^4 + \lambda_3 p_{43}^4 + \lambda_4 p_{44}^4 + \lambda_5 p_{45}^4 ] + [ \tau_i + S_1(\tau_i) + S_2(\tau_i) + S_3(\tau_i)] [ \lambda_1 p_{41}^5 + \lambda_2 p_{42}^5 + \lambda_3 p_{43}^5 + \lambda_4 p_{44}^5 + \lambda_5 p_{45}^5 ] - n_4 \) \( \lambda_4 \) \( \tau_i = k S_4 (Q_i) \)
\]

or

\[
[ \lambda_1 p_{41}^5 + \lambda_2 p_{42}^5 + \lambda_3 p_{43}^5 + \lambda_4 p_{44}^5 + \lambda_5 p_{45}^5 - n_4 \) \( \lambda_4 \) \( \tau_i + [ \lambda_1 p_{41}^5 + \lambda_2 p_{42}^5 + \lambda_3 p_{43}^5 + \lambda_4 p_{44}^5 + \lambda_5 p_{45}^5 ] \) \( S_1(\tau_i) + [ \lambda_1 p_{41}^5 + \lambda_2 p_{42}^5 + \lambda_3 p_{43}^5 + \lambda_4 p_{44}^5 + \lambda_5 p_{45}^5 ] \) \( S_2(\tau_i) + [ \lambda_1 p_{41}^5 + \lambda_2 p_{42}^5 + \lambda_3 p_{43}^5 + \lambda_4 p_{44}^5 + \lambda_5 p_{45}^5 ] \) \( S_3(\tau_i) + [ \lambda_1 p_{41}^5 + \lambda_2 p_{42}^5 + \lambda_3 p_{43}^5 + \lambda_4 p_{44}^5 + \lambda_5 p_{45}^5 ] \) \( S_4(\tau_i) = k S_4 (Q_i) \)
\]

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\[ + \lambda_3 p^5_{43} + \lambda_4 p^5_{44} + \lambda_5 p^5_{45} - (\lambda_1 p^2_{41} + \lambda_2 p^2_{42} + \lambda_3 p^2_{43} + \lambda_4 p^2_{44} + \lambda_5 p^2_{45}) S_2(t_i) + [\right. \]

\[ \left. \lambda_1 p^5_{41} + \lambda_2 p^5_{42} + \lambda_3 p^5_{43} + \lambda_4 p^5_{44} + \lambda_5 p^5_{45} - (\lambda_1 p^3_{41} + \lambda_2 p^3_{42} + \lambda_3 p^3_{43} + \lambda_4 p^3_{44} + \lambda_5 p^3_{45}) S_3(t_i) + [r(k-1) - (\lambda_1 p^4_{41} + \lambda_2 p^4_{42} + \lambda_3 p^4_{43} + \lambda_4 p^4_{44} + \lambda_5 p^4_{45}) + \lambda_1 p^5_{41} + \lambda_2 p^5_{42} + \lambda_3 p^5_{43} + \lambda_4 p^5_{44} + \lambda_5 p^5_{45}] S_4(t_i) = k S_4(Q_i) \]

As, we know that

\[ p^1_{41} + p^1_{42} + p^1_{43} + p^1_{44} + p^1_{45} = n_4 \]

\[ p^2_{41} + p^2_{42} + p^2_{43} + p^2_{44} + p^2_{45} = n_4 \]

\[ p^3_{41} + p^3_{42} + p^3_{43} + p^3_{44} + p^3_{45} = n_4 \]

\[ p^4_{41} + p^4_{42} + p^4_{43} + p^4_{44} + p^4_{45} = n_4 - 1 \]

\[ p^5_{41} + p^5_{42} + p^5_{43} + p^5_{44} + p^5_{45} = n_4 \]

using

\[ n_4 = p^5_{41} + p^5_{42} + p^5_{43} + p^5_{44} + p^5_{45} \]

\[ p^1_{45} = n_4 - p^1_{41} - p^1_{42} - p^1_{43} - p^1_{44} \]

\[ p^2_{45} = n_4 - p^2_{41} - p^2_{42} - p^2_{43} - p^2_{44} \]

\[ p^3_{45} = n_4 - p^3_{41} - p^3_{42} - p^3_{43} - p^3_{44} \]

\[ p^4_{45} = n_4 - 1 - p^4_{41} - p^4_{42} - p^4_{43} - p^4_{44} \]

\[ p^5_{45} = n_4 - p^5_{41} - p^5_{42} - p^5_{43} - p^5_{44} \]

in above equation \((7.2.8)\), it becomes

\[ [\lambda_1 p^5_{41} + \lambda_2 p^5_{42} + \lambda_3 p^5_{43} + \lambda_4 p^5_{44} + \lambda_5 p^5_{45} - (p^5_{41} + p^5_{42} + p^5_{43} + p^5_{44} + p^5_{45}) \lambda_4] \tau_i + [\]
\[
\lambda_1 \beta_1^4 + \lambda_2 \beta_2^4 + \lambda_3 \beta_3^4 + \lambda_4 \beta_4^4 + \lambda_5 (n_4 - \beta_4^4 - \beta_2^4 - \beta_3^4) - \lambda_4 \beta_4^4 - \lambda_2 \beta_2^4 - \\
\lambda_3 \beta_3^4 - \lambda_4 \beta_4^4 - \lambda_5 (n_4 - \beta_4^4 - \beta_2^4 - \beta_3^4 - \beta_1^4 + \beta_1^4 - \beta_1^4) S_1(\tau_i) + \left[ \lambda_1 \beta_1^4 + \lambda_2 \beta_2^4 + \lambda_3 \beta_3^4 + \lambda_4 \beta_4^4 + \lambda_5 (n_4 - \beta_4^4 - \beta_2^4 - \beta_3^4) - \lambda_4 \beta_4^4 - \lambda_2 \beta_2^4 - \lambda_3 \beta_3^4 - \lambda_4 \beta_4^4 - \lambda_5 (n_4 - \beta_4^4 - \beta_2^4 - \beta_3^4 - \beta_1^4) \right] S_2(\tau_i) + \left[ \lambda_1 \beta_1^4 + \lambda_2 \beta_2^4 + \lambda_3 \beta_3^4 + \lambda_4 \beta_4^4 + \lambda_5 (n_4 - \beta_4^4 - \beta_2^4 - \beta_3^4 - \beta_1^4) \right] S_3(\tau_i) + \left[ \lambda_1 \beta_1^4 + \lambda_2 \beta_2^4 + \lambda_3 \beta_3^4 + \lambda_4 \beta_4^4 + \lambda_5 (n_4 - \beta_4^4 - \beta_2^4 - \beta_3^4 - \beta_1^4) \right] S_4(\tau_i) = k S_4 (Q_i)
\]

After solving, we get

\[
[ (\lambda - \lambda_4) \beta_1^4 + (\lambda - \lambda_4) \beta_2^4 + (\lambda - \lambda_4) \beta_3^4 + (\lambda - \lambda_4) \beta_4^4 ] \tau_i + \left[ (\lambda - \lambda_4) \beta_1^4 + (\lambda - \lambda_4) \beta_2^4 + (\lambda - \lambda_4) \beta_3^4 + (\lambda - \lambda_4) \beta_4^4 \right] S_1(\tau_i) + \left[ (\lambda - \lambda_4) \beta_2^4 + (\lambda - \lambda_4) \beta_3^4 + (\lambda - \lambda_4) \beta_4^4 \right] S_2(\tau_i) + \left[ (\lambda - \lambda_4) \beta_3^4 + (\lambda - \lambda_4) \beta_4^4 \right] S_3(\tau_i) + \left[ (\lambda - \lambda_4) \beta_4^4 \right] S_4(\tau_i) = k S_4 (Q_i)
\]

--- (7.3.5)

Therefore, we have

\[
A_1 \tau_i + A_2 S_1(\tau_i) + A_3 S_2(\tau_i) + A_4 S_3(\tau_i) + A_5 S_4(\tau_i) = K Q_i
\]

\[
B_1 \tau_i + B_2 S_1(\tau_i) + B_3 S_2(\tau_i) + B_4 S_3(\tau_i) + B_5 S_4(\tau_i) = K S_1(\tau_i)
\]

\[
C_1 \tau_i + C_2 S_1(\tau_i) + C_3 S_2(\tau_i) + C_4 S_3(\tau_i) + C_5 S_4(\tau_i) = K S_2(\tau_i)
\]

\[
D_1 \tau_i + D_2 S_1(\tau_i) + D_3 S_2(\tau_i) + D_4 S_3(\tau_i) + D_5 S_4(\tau_i) = K S_3(\tau_i)
\]

\[
E_1 \tau_i + E_2 S_1(\tau_i) + E_3 S_2(\tau_i) + E_4 S_3(\tau_i) + E_5 S_4(\tau_i) = K S_4(\tau_i)
\]

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where

\begin{align*}
A_1 &= r(k-1) + \lambda_5 \\
A_2 &= (\lambda_5 - \lambda_1) \\
A_3 &= (\lambda_5 - \lambda_2) \\
A_4 &= (\lambda_5 - \lambda_3) \\
A_5 &= (\lambda_5 - \lambda_4)
\end{align*}

\begin{align*}
B_1 &= (\lambda_5 - \lambda_1) p^5_{11} + (\lambda_2 - \lambda_1) p^5_{12} + (\lambda_3 - \lambda_1) p^5_{13} + (\lambda_4 - \lambda_1) p^5_{14} + (\lambda_5 - \lambda_1) p^5_{15} \\
B_2 &= r(k-1) + \lambda_5 + (\lambda_5 - \lambda_1)(p^1_{11} - p^5_{11}) + (\lambda_5 - \lambda_2)(p^1_{12} - p^5_{12}) \\
&\quad + (\lambda_5 - \lambda_3)(p^1_{13} - p^5_{13}) + (\lambda_5 - \lambda_4)(p^1_{14} - p^5_{14}) \\
B_3 &= (\lambda_5 - \lambda_1)(p^2_{11} - p^5_{11}) + (\lambda_5 - \lambda_2)(p^2_{12} - p^5_{12}) + (\lambda_5 - \lambda_3)(p^2_{13} - p^5_{13}) \\
&\quad + (\lambda_5 - \lambda_4)(p^2_{14} - p^5_{14}) \\
B_4 &= (\lambda_5 - \lambda_1)(p^3_{11} - p^5_{11}) + (\lambda_5 - \lambda_2)(p^3_{12} - p^5_{12}) + (\lambda_5 - \lambda_3)(p^3_{13} - p^5_{13}) \\
&\quad + (\lambda_5 - \lambda_4)(p^3_{14} - p^5_{14}) \\
B_5 &= (\lambda_5 - \lambda_1)(p^4_{11} - p^5_{11}) + (\lambda_5 - \lambda_2)(p^4_{12} - p^5_{12}) + (\lambda_5 - \lambda_3)(p^4_{13} - p^5_{13}) \\
&\quad + (\lambda_5 - \lambda_4)(p^4_{14} - p^5_{14}) \\
C_1 &= (\lambda_4 - \lambda_2) p^5_{21} + (\lambda_2 - \lambda_2) p^5_{22} + (\lambda_3 - \lambda_2) p^5_{23} + (\lambda_4 - \lambda_2) p^5_{24} + (\lambda_5 - \lambda_2) p^5_{25} \\
C_2 &= (\lambda_5 - \lambda_1)(p^1_{21} - p^5_{21}) + (\lambda_5 - \lambda_2)(p^1_{22} - p^5_{22}) + (\lambda_5 - \lambda_3)(p^1_{23} - p^5_{23}) \\
&\quad + (\lambda_5 - \lambda_4)(p^1_{24} - p^5_{24}) \\
C_3 &= r(k-1) + \lambda_5 + (\lambda_5 - \lambda_1)(p^2_{21} - p^5_{21}) + (\lambda_5 - \lambda_2)(p^2_{22} - p^5_{22}) \\
&\quad + (\lambda_5 - \lambda_3)(p^2_{23} - p^5_{23}) + (\lambda_5 - \lambda_4)(p^2_{24} - p^5_{24}) \\
C_4 &= (\lambda_5 - \lambda_1)(p^3_{21} - p^5_{21}) + (\lambda_5 - \lambda_2)(p^3_{22} - p^5_{22}) + (\lambda_5 - \lambda_3)(p^3_{23} - p^5_{23})
\end{align*}
\[ C_5 = (\lambda_5 - \lambda_1)(p_{21}^4 - p_{21}^5) + (\lambda_5 - \lambda_2)(p_{22}^4 - p_{22}^5) + (\lambda_5 - \lambda_3)(p_{23}^4 - p_{23}^5) + (\lambda_5 - \lambda_4)(p_{24}^4 - p_{24}^5) \]

\[ D_1 = (\lambda_1 - \lambda_3)p_{31}^5 + (\lambda_2 - \lambda_3)p_{32}^5 + (\lambda_3 - \lambda_3)p_{33}^5 + (\lambda_4 - \lambda_3)p_{34}^5 + (\lambda_5 - \lambda_3)p_{35}^5 \]

\[ D_2 = (\lambda_5 - \lambda_1)(p_{31}^1 - p_{31}^5) + (\lambda_5 - \lambda_2)(p_{32}^1 - p_{32}^5) + (\lambda_5 - \lambda_3)(p_{33}^1 - p_{33}^5) + (\lambda_5 - \lambda_4)(p_{34}^1 - p_{34}^5) \]

\[ D_3 = (\lambda_5 - \lambda_1)(p_{31}^2 - p_{31}^5) + (\lambda_5 - \lambda_2)(p_{32}^2 - p_{32}^5) + (\lambda_5 - \lambda_3)(p_{33}^2 - p_{33}^5) + (\lambda_5 - \lambda_4)(p_{34}^2 - p_{34}^5) \]

\[ D_4 = r(k-1) + \lambda_5 + (\lambda_5 - \lambda_1)(p_{31}^3 - p_{31}^5) + (\lambda_5 - \lambda_2)(p_{32}^3 - p_{32}^5) + (\lambda_5 - \lambda_3)(p_{33}^3 - p_{33}^5) + (\lambda_5 - \lambda_4)(p_{34}^3 - p_{34}^5) \]

\[ D_5 = (\lambda_5 - \lambda_1)(p_{31}^4 - p_{31}^5) + (\lambda_5 - \lambda_2)(p_{32}^4 - p_{32}^5) + (\lambda_5 - \lambda_3)(p_{33}^4 - p_{33}^5) + (\lambda_5 - \lambda_4)(p_{34}^4 - p_{34}^5) \]

\[ E_1 = (\lambda_1 - \lambda_4)p_{41}^5 + (\lambda_2 - \lambda_4)p_{42}^5 + (\lambda_3 - \lambda_4)p_{43}^5 + (\lambda_4 - \lambda_4)p_{44}^5 + (\lambda_5 - \lambda_4)p_{45}^5 \]

\[ E_2 = (\lambda_5 - \lambda_1)(p_{41}^1 - p_{41}^5) + (\lambda_5 - \lambda_2)(p_{42}^1 - p_{42}^5) + (\lambda_5 - \lambda_3)(p_{43}^1 - p_{43}^5) + (\lambda_5 - \lambda_4)(p_{44}^1 - p_{44}^5) \]

\[ E_3 = (\lambda_5 - \lambda_1)(p_{41}^2 - p_{41}^5) + (\lambda_5 - \lambda_2)(p_{42}^2 - p_{42}^5) + (\lambda_5 - \lambda_3)(p_{43}^2 - p_{43}^5) + (\lambda_5 - \lambda_4)(p_{44}^2 - p_{44}^5) \]

\[ E_4 = (\lambda_5 - \lambda_1)(p_{41}^3 - p_{41}^5) + (\lambda_5 - \lambda_2)(p_{42}^3 - p_{42}^5) + (\lambda_5 - \lambda_3)(p_{43}^3 - p_{43}^5) + (\lambda_5 - \lambda_4)(p_{44}^3 - p_{44}^5) \]
\[ + (\lambda_5 - \lambda_4)(p_{44}^3 - p_{44}^5) \]

\[ E_5 = r(k-1) + \lambda_5 + (\lambda_5 - \lambda_1)(p_{41}^4 - p_{41}^5) + (\lambda_5 - \lambda_2)(p_{42}^4 - p_{42}^5) \]

\[ + (\lambda_5 - \lambda_3)(p_{43}^4 - p_{43}^5) + (\lambda_5 - \lambda_4)(p_{44}^4 - p_{44}^5) \]

Using these constants, we get a solution

\[ \hat{\tau}_i = k[Q_i F - GS_1 + HS_2 - IS_3 + JS_4] / \Delta \]

where

\[ \Delta = [A_1 F - A_2 G + A_3 H - A_4 I + A_5 J] \]

\[ F = \begin{bmatrix} B_2 B_3 B_4 B_5 \\ C_2 C_3 C_4 C_5 \\ D_2 D_3 D_4 D_5 \\ E_2 E_3 E_4 E_5 \end{bmatrix} \quad G = \begin{bmatrix} A_2 A_3 A_4 A_5 \\ C_2 C_3 C_4 C_5 \\ D_2 D_3 D_4 D_5 \\ E_2 E_3 E_4 E_5 \end{bmatrix} \quad H = \begin{bmatrix} A_2 A_3 A_4 A_5 \\ B_2 B_3 B_4 B_5 \\ D_2 D_3 D_4 D_5 \\ E_2 E_3 E_4 E_5 \end{bmatrix} \]

\[ I = \begin{bmatrix} A_2 A_3 A_4 A_5 \\ B_2 B_3 B_4 B_5 \\ C_2 C_3 C_4 C_5 \\ E_2 E_3 E_4 E_5 \end{bmatrix} \quad J = \begin{bmatrix} A_2 A_3 A_4 A_5 \\ B_2 B_3 B_4 B_5 \\ C_2 C_3 C_4 C_5 \\ D_2 D_3 D_4 D_5 \end{bmatrix} \]

and the variance of an estimated elementary contrast among treatment effects are

\[ V(\tau_i - \tau_j) = 2k \sigma^2 [F + G] / \Delta \quad \text{if i and j are 1st associate} \]

\[ V(\tau_i - \tau_j) = 2k \sigma^2 [F - H] / \Delta \quad \text{if i and j are 2nd associate} \]

\[ V(\tau_i - \tau_j) = 2k \sigma^2 [F + I] / \Delta \quad \text{if i and j are 3rd associate} \]

\[ V(\tau_i - \tau_j) = 2k \sigma^2 [F - J] / \Delta \quad \text{if i and j are 4th associate} \]

\[ V(\tau_i - \tau_j) = 2k \sigma^2 [F] / \Delta \quad \text{if i and j are 5th associate} \]

and the efficiency factors of five kinds are given by
The overall efficiency factor is

\[ E = (v - 1) \Delta / rk [(v - 1) F + n_1 G - n_2 H + n_3 I - n_4 J] \]

### 7.4 Generalization of Analysis of Higher Associate Class PBIB Design

After solving the normal equations of a PBIB design with five associate classes, we get an idea about the general representation of intra-block analysis of higher associate class PBIB design. We represent the reduced normal equations of a higher associate class PBIB design in matrix form and get a solution from these equations without solving complex mathematical work.

Let us consider a \( m \geq 2 \) associate class PBIB design with parameters

\( v, b, r, k, \lambda_i, n_i, p_{ijk} \) where \( i, j, k = 1,2,\ldots,m \)

The Reduced Normal equations under the restriction (7.1.5) for estimating linear combination of treatment effects of intra-block model of \( m \)-associate class PBIB design in matrix form as

\[
M \begin{bmatrix}
\tau_i \\
S_j(\tau_i)
\end{bmatrix} = k \begin{bmatrix}
Q_i \\
S_j(\tau_i)
\end{bmatrix} \text{ where } j = 1, 2,\ldots,m-1.
\]

\( E_1 = \Delta / rk [F + G] \)

\( E_2 = \Delta / rk [F - H] \)

\( E_3 = \Delta / rk [F + I] \)

\( E_4 = \Delta / rk [F - J] \)

\( E_5 = \Delta / rk [F] \)
M is a square coefficient matrix of order m, \[ \begin{bmatrix} \tau_i \\ S_j (\tau_i) \end{bmatrix} \] is column matrix of order m x1, column matrix \[ \begin{bmatrix} Q_i \\ S_j (\tau_i) \end{bmatrix} \] is order mx1 and k be a scalar.

For (m≥3), square matrix of order m in which each row/ column has m distinct elements. The elements of this coefficient matrix in notational form as.

Elements in 1st row: \[ E_{11} = r(k-1) + \lambda_m \] \[ E_{1C} = (\lambda_m - \lambda_{c-1}) \quad \forall \quad C = 2,3,…,m. \]

Elements in 1st column: \[ E_{R1} = \sum_{j=1}^{m} (\lambda_j - \lambda_r) p_{rj}^m \] where \( r = R - 1 \) \( \forall \quad R = 2,3,…,m. \)

Other Elements: \[ E_{RC} = \sum_{j=1}^{m-1} (\lambda_m - \lambda_j) (p_{rj}^c - p_{rj}^m) \] where \( r = R - 1, \quad c = C-1, \quad R,C = 2,3,…,m \)

under the condition \( R \neq C. \)

Diagonal elements: \[ E_{RC} = r(k-1) + \lambda_m + \sum_{j=1}^{m-1} (\lambda_m - \lambda_j) (p_{rj}^c - p_{rj}^m) \] \( \forall \quad R = C. \)

By applying least square method, solution coming from above system of m normal equations under the process mentioned in section (7.2) is

\[ \tau_i = k \left[ \sum_{j=1}^{m} S_j \right] \] / \( \Delta \)

\[ = k \left[ \sum_{j=2}^{m} (-1)^{j-1} M_j \right] \] / \( \Delta \)

where \( \Delta \) is determinant of square matrix of M and \( M_j \) is minor of 1st element of jth row of matrix M , where \( j = 1,2,…,m-1. \)

The efficiency factor of m- kind is given by

\[ E_j = \Delta / rk \left[ M_j + (-1)^j M_{j+1} \right] \]

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here \( j = 1, 2, \ldots, m \) under the condition that \( M_{j+1} \) does not exists for \( j+1 > m \).

The overall efficiency factor is

\[
E = \frac{(v-1) \Delta}{rk} \left[ (v-1) M_1 + \sum_{j=1}^{m-1} (-1)^{j-1} \eta_j M_{j+1} \right]
\]

### 7.5 Numerical Representation of General Analysis

Now, we verify the generalization of analysis for higher (\( m \geq 3 \)) associate class PBIB design practically by taking particular values of ‘\( m \)’.

As we know that the intra- block model for analysis of PBIB design in which there is no interaction between blocks and treatments is

\[
y_{ij} = \mu + \tau_i + b_j + \varepsilon_{ij}
\]

where \( E(\varepsilon_{ij}) = 0, \varepsilon_{ij} \) are identical independent random variables

and we are interested only for estimating linear combinations of treatment effects for \( m ( m \geq 3) \) associate class PBIB design

#### 7.5.1 Analysis of three associate class PBIB design

For \( m = 3 \):

Using least square method, the reduced normal equations in matrix form of a PBIB design with \( m = 3 \) for treatment effects under the restriction

\[
\tau_i + S_1(\tau_i) + S_2(\tau_i) + S_3(\tau_i) = 0
\]

is

\[
M \left[ \begin{array}{c}
\tau_i \\
S_j(\tau_i)
\end{array} \right] = k \left[ \begin{array}{c}
Q_i \\
S_j(\tau_i)
\end{array} \right] \quad \text{where } j = 1, 2.
\]

Clearly \( M \) is a square coefficient matrix of order 3. R,C = 2,3.
and \( \Delta \) be the determinant of matrix \( M \).

The reduced matrix form can be written as

\[
\begin{bmatrix}
E_{11} & E_{12} & E_{13} \\
E_{21} & E_{22} & E_{23} \\
E_{31} & E_{32} & E_{33}
\end{bmatrix}
\begin{bmatrix}
\tau_i \\
S_1(\tau_i) \\
S_2(\tau_i)
\end{bmatrix} = k
\begin{bmatrix}
Q_i \\
S_1(\tau_i) \\
S_2(\tau_i)
\end{bmatrix}
\]

or

\[
E_{11} \tau_i + E_{12} S_1(\tau_i) + E_{13} S_2(\tau_i) = k Q_i
\]

\[
E_{21} \tau_i + E_{22} S_1(\tau_i) + E_{23} S_2(\tau_i) = k S_1(\tau_i)
\]

\[
E_{31} \tau_i + E_{32} S_1(\tau_i) + E_{33} S_2(\tau_i) = k S_2(\tau_i)
\]

Here

Elements in 1st row:

\[
E_{11} = r(k-1) + \lambda_m = r(k-1) + \lambda_3
\]

\[
E_{1C} = (\lambda_m - \lambda_{C-1}) \forall C=2,3
\]

\[
E_{12} = (\lambda_3 - \lambda_1) \quad \text{and} \quad E_{13} = (\lambda_3 - \lambda_2)
\]

Elements in 1st Column: \( E_{R1} = \sum_{j=1}^{m}(\lambda_j - \lambda_r) p_{rj}^m \) where \( r = R - 1 \) \( \forall R = 2,3 \)

are

\[
E_{21} = \sum_{j=1}^{3}(\lambda_j - \lambda_1) p_{1j}^3 = (\lambda_1 - \lambda_1) p_{11}^3 + (\lambda_2 - \lambda_1) p_{12}^3 + (\lambda_3 - \lambda_1) p_{13}^3
\]

\[
E_{31} = \sum_{j=1}^{3}(\lambda_j - \lambda_2) p_{2j}^3 = (\lambda_1 - \lambda_2) p_{21}^3 + (\lambda_2 - \lambda_2) p_{22}^3 + (\lambda_3 - \lambda_2) p_{23}^3
\]

Diagonal elements: \( E_{RC} = r(k-1) + \lambda_m + \sum_{j=1}^{m-1}(\lambda_m - \lambda_j)(p_{rj}^r - p_{rj}^m) \forall R = C. \)
\[ E_{22} = r(k-1) + \lambda_3 + \sum_{j=1}^{2}(\lambda_3 - \lambda_j)(p_{ij}^1 - p_{ij}^3) \]
\[ = r(k-1) + \lambda_3 + (\lambda_3 - \lambda_1)(p_{11}^1 - p_{11}^3) + (\lambda_3 - \lambda_2)(p_{12}^1 - p_{12}^3). \]

\[ E_{33} = r(k-1) + \lambda_3 + \sum_{j=1}^{2}(\lambda_3 - \lambda_j)(p_{2j}^2 - p_{2j}^3) \]
\[ = r(k-1) + \lambda_3 + (\lambda_3 - \lambda_1)(p_{21}^2 - p_{21}^3) + (\lambda_3 - \lambda_2)(p_{22}^2 - p_{22}^3). \]

Other Elements: \( E_{RC} = \sum_{j=1}^{m-1}(\lambda_m - \lambda_j)(p_{ij}^c - p_{ij}^m) \) where \( r = R - 1, c = C - 1, \]
\( R, C = 2, 3 \) under the condition \( R \neq C. \)

\[ E_{23} = \sum_{j=1}^{2}(\lambda_3 - \lambda_j)(p_{2j}^1 - p_{1j}^3) = (\lambda_3 - \lambda_1)(p_{11}^1 - p_{11}^3) + (\lambda_3 - \lambda_2)(p_{12}^1 - p_{12}^3) \]
\[ E_{32} = \sum_{j=1}^{2}(\lambda_3 - \lambda_j)(p_{2j}^1 - p_{2j}^3) = (\lambda_3 - \lambda_1)(p_{21}^1 - p_{21}^3) + (\lambda_3 - \lambda_2)(p_{22}^1 - p_{22}^3) \]

Clearly, the elements of coefficient square matrix \( M \) is equivalent to the coefficients of normal equations originally introduced by Rao (1947b) as

\[ A_1 = r(k-1) + \lambda_3 = E_{11} \quad A_2 = (\lambda_3 - \lambda_1) = E_{12} \quad A_3 = (\lambda_3 - \lambda_2) = E_{13} \]

\[ B_1 = (\lambda_3 - \lambda_2)p_{12}^3 + (\lambda_3 - \lambda_1)p_{13}^3 = E_{21} \]
\[ B_2 = r(k-1) + \lambda_3 + (\lambda_3 - \lambda_1)(p_{11}^1 - p_{11}^3) + (\lambda_3 - \lambda_2)(p_{12}^1 - p_{12}^3) = E_{22} \]
\[ B_3 = (\lambda_3 - \lambda_1)(p_{21}^2 - p_{11}^3) + (\lambda_3 - \lambda_2)(p_{22}^2 - p_{12}^3) = E_{23} \]

\[ C_1 = (\lambda_4 - \lambda_2)p_{21}^3 + (\lambda_3 - \lambda_2)p_{23}^3 = E_{31} \]
\[ C_2 = (\lambda_3 - \lambda_1)(p_{21}^1 - p_{21}^3) + (\lambda_3 - \lambda_2)(p_{22}^1 - p_{22}^3) = E_{32} \]
\[ C_3 = r(k-1) + \lambda_3 + (\lambda_3 - \lambda_1)(p_{21}^2 - p_{21}^3) + (\lambda_3 - \lambda_2)(p_{22}^2 - p_{22}^3) = E_{33} \]

and solution coming from above system for \( m=3 \) is

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\[ \hat{\tau}_i = k \left[ Q_i M_1 - S_1 M_2 + S_2 M_3 \ldots + (-1)^{m-1} S_{m-1} \right] / \Delta \]

\[ = k \left[ Q_i M_1 + \sum_{j=2}^{m} (-1)^{j-1} M_j S_{j-1} \right] / \Delta \]

\[ = k \left[ Q_i M_1 + \sum_{j=2}^{3} (-1)^{j-1} M_j S_{j-1} \right] / \Delta = k \left[ Q_i M_1 - M_2 S_1 + M_3 S_2 \right] / \Delta \]

and the efficiency factors of three kinds are given by

\[ E_j = \Delta / rk \left[ M_1 + (-1)^{j-1} M_{j+1} \right] \; ; \; M_{j+1} \; \text{does not exist for} \; j+1 > 3, \; \text{here} \; j = 1,2,3. \]

Therefore

\[ E_1 = \Delta / rk \left[ M_1 + M_2 \right] \]

\[ E_2 = \Delta / rk \left[ M_1 - M_3 \right] \]

\[ E_3 = \Delta / rk \left[ M_1 + M_4 \right], \; M_4 \; \text{does not exist due to} \; j+1 > 3 \]

The overall efficiency factor is

\[ E = (v-1) \Delta / rk \left[ (v-1) M_1 + \sum_{j=1}^{m-1} (-1)^{j-1} n_j M_{j+1} \right] \]

\[ = (v-1) \Delta / rk \left[ (v-1) M_1 + \sum_{j=1}^{2} (-1)^{j-1} n_j M_{j+1} \right] \]

\[ = (v-1) \Delta / rk \left[ (v-1) M_1 + n_1 M_2 - n_2 M_3 \right] \; , \; \text{where} \]

\[ M_1 = B_2 C_3 - B_3 C_2, \; M_2 = A_2 C_3 - A_3 C_2, \; M_3 = A_2 B_3 - A_3 B_2 \]

7.5.2 Analysis of four associate class PBIB design

For m=4:

Using least square method, the reduced normal equations in matrix form of a PBIB design with m= 4 for treatment effects under the restriction
\[ \tau_i + S_1(\tau_i) + S_2(\tau_i) + S_3(\tau_i) + S_4(\tau_i) = 0 \]

is

\[ M \begin{bmatrix} \tau_i \\ S_j(\tau_i) \end{bmatrix} = k \begin{bmatrix} Q_i \\ S_j(\tau_i) \end{bmatrix} \text{ where } j = 1, 2, 3. \]

Clearly, \( M \) is a square coefficient matrix of order 4. \( R, C = 2, 3, 4 \).

and \( \Delta \) be the determinant of matrix \( M \).

The reduced matrix form can be written as

\[
\begin{bmatrix}
E_{11} & E_{12} & E_{13} & E_{14} \\
E_{21} & E_{22} & E_{23} & E_{24} \\
E_{31} & E_{32} & E_{33} & E_{34} \\
E_{41} & E_{42} & E_{43} & E_{44}
\end{bmatrix}
\begin{bmatrix}
\tau_i \\
S_1(\tau_i) \\
S_2(\tau_i) \\
S_3(\tau_i)
\end{bmatrix}
= k
\begin{bmatrix}
Q_i \\
S_1(\tau_i) \\
S_2(\tau_i) \\
S_3(\tau_i)
\end{bmatrix}
\]

or

\[
E_{11} \tau_i + E_{12} S_1(\tau_i) + E_{13} S_2(\tau_i) + E_{14} S_3(\tau_i) = k Q_i
\]

\[
E_{21} \tau_i + E_{22} S_1(\tau_i) + E_{23} S_2(\tau_i) + E_{24} S_3(\tau_i) = k S_1(\tau_i)
\]

\[
E_{31} \tau_i + E_{32} S_1(\tau_i) + E_{33} S_2(\tau_i) + E_{34} S_3(\tau_i) = k S_2(\tau_i)
\]

\[
E_{41} \tau_i + E_{42} S_1(\tau_i) + E_{43} S_2(\tau_i) + E_{44} S_3(\tau_i) = k S_3(\tau_i)
\]

Here

Elements in 1\textsuperscript{st} row: \( E_{11} = r(k-1) + \lambda_m = r(k-1) + \lambda_4 \)

\( E_{1C} = (\lambda_m - \lambda_{C-1}) \forall C = 2, 3, 4 \) are

\( E_{12} = (\lambda_4 - \lambda_1) \), \( E_{13} = (\lambda_4 - \lambda_2) \) and \( E_{14} = (\lambda_4 - \lambda_3) \)
Elements in 1st Column: $E_{R1} = \sum_{j=1}^{m}(\lambda_j - \lambda_{r})p_{rj}^m$ where $r = R - 1 \forall R = 2, 3, 4$

are

$E_{21} = \sum_{j=1}^{4}(\lambda_j - \lambda_1)p_{1j}^4$

$= (\lambda_4 - \lambda_1)p_{11}^4 + (\lambda_2 - \lambda_1)p_{12}^4 + (\lambda_3 - \lambda_1)p_{13}^4 + (\lambda_4 - \lambda_1)p_{14}^4$

$E_{31} = \sum_{j=1}^{4}(\lambda_j - \lambda_2)p_{2j}^4$

$= (\lambda_1 - \lambda_2)p_{21}^4 + (\lambda_2 - \lambda_2)p_{22}^4 + (\lambda_3 - \lambda_2)p_{23}^4 + (\lambda_4 - \lambda_2)p_{24}^4$

$E_{41} = \sum_{j=1}^{4}(\lambda_j - \lambda_3)p_{3j}^4$

$= (\lambda_1 - \lambda_3)p_{31}^4 + (\lambda_2 - \lambda_3)p_{32}^4 + (\lambda_3 - \lambda_3)p_{33}^4 + (\lambda_4 - \lambda_3)p_{34}^4$

Diagonal elements: $E_{RC} = r(k-1) + \lambda_m + \sum_{j=1}^{m-1}(\lambda_m - \lambda_j)(p_{rj}^c - p_{rj}^m) \forall r = R - 1, R, C = 2, 3, 4.$

$E_{22} = r(k-1) + \lambda_4 + \sum_{j=1}^{3}(\lambda_4 - \lambda_j)(p_{1j}^4 - p_{1j}^4)$

$= r(k-1) + \lambda_4 + (\lambda_4 - \lambda_1)(p_{11}^4 - p_{11}^4) + (\lambda_4 - \lambda_2)(p_{12}^4 - p_{12}^4) + (\lambda_4 - \lambda_3)(p_{13}^4 - p_{13}^4)$

$E_{33} = r(k-1) + \lambda_4 + \sum_{j=1}^{3}(\lambda_4 - \lambda_j)(p_{2j}^4 - p_{2j}^4)$

$= r(k-1) + \lambda_4 + (\lambda_4 - \lambda_1)(p_{21}^4 - p_{21}^4) + (\lambda_4 - \lambda_2)(p_{22}^4 - p_{22}^4) + (\lambda_4 - \lambda_3)(p_{23}^4 - p_{23}^4)$

$E_{44} = r(k-1) + \lambda_4 + \sum_{j=1}^{3}(\lambda_4 - \lambda_j)(p_{3j}^4 - p_{3j}^4)$

$= r(k-1) + \lambda_4 + (\lambda_4 - \lambda_1)(p_{31}^4 - p_{31}^4) + (\lambda_4 - \lambda_2)(p_{32}^4 - p_{32}^4) + (\lambda_4 - \lambda_3)(p_{33}^4 - p_{33}^4)$

Other Elements: $E_{RC} = \sum_{j=1}^{m-1}(\lambda_m - \lambda_j)(p_{rj}^c - p_{rj}^m)$ where $r = R - 1, c = C-1, R,C = 2,3,4$ under the condition $R \neq C.$
\[ E_{23} = \sum_{j=1}^{3} (\lambda_4 - \lambda_j) (p_{ij}^2 - p_{ij}^4) \]
\[ = (\lambda_4 - \lambda_1) (p_{11}^2 - p_{11}^4) + (\lambda_4 - \lambda_2) (p_{12}^2 - p_{12}^4) + (\lambda_4 - \lambda_3) (p_{13}^2 - p_{13}^4) \]

\[ E_{24} = \sum_{j=1}^{3} (\lambda_4 - \lambda_j) (p_{2j}^3 - p_{2j}^4) \]
\[ = (\lambda_4 - \lambda_1) (p_{21}^3 - p_{21}^4) + (\lambda_4 - \lambda_2) (p_{22}^3 - p_{22}^4) + (\lambda_4 - \lambda_3) (p_{23}^3 - p_{23}^4) \]

\[ E_{32} = \sum_{j=1}^{3} (\lambda_4 - \lambda_j) (p_{1j}^1 - p_{1j}^4) \]
\[ = (\lambda_4 - \lambda_1) (p_{11}^1 - p_{11}^4) + (\lambda_4 - \lambda_2) (p_{12}^1 - p_{12}^4) + (\lambda_4 - \lambda_3) (p_{13}^1 - p_{13}^4) \]

\[ E_{34} = \sum_{j=1}^{3} (\lambda_4 - \lambda_j) (p_{3j}^3 - p_{3j}^4) \]
\[ = (\lambda_4 - \lambda_1) (p_{31}^3 - p_{31}^4) + (\lambda_4 - \lambda_2) (p_{32}^3 - p_{32}^4) + (\lambda_4 - \lambda_3) (p_{33}^3 - p_{33}^4) \]

\[ E_{42} = \sum_{j=1}^{3} (\lambda_4 - \lambda_j) (p_{4j}^1 - p_{4j}^4) \]
\[ = (\lambda_4 - \lambda_1) (p_{41}^1 - p_{41}^4) + (\lambda_4 - \lambda_2) (p_{42}^1 - p_{42}^4) + (\lambda_4 - \lambda_3) (p_{43}^1 - p_{43}^4) \]

\[ E_{43} = \sum_{j=1}^{3} (\lambda_4 - \lambda_j) (p_{3j}^2 - p_{3j}^4) \]
\[ = (\lambda_4 - \lambda_1) (p_{31}^2 - p_{31}^4) + (\lambda_4 - \lambda_2) (p_{32}^2 - p_{32}^4) + (\lambda_4 - \lambda_3) (p_{33}^2 - p_{33}^4) \]

Clearly, the elements of coefficient square matrix \( M \) is equivalent to the coefficients of normal equations originally introduced by Garg and Mishra (2013) as

\[ A_1 = r \ (k-1) + \lambda_4 \quad A_2 = \lambda_4 - \lambda_1 \quad A_3 = \lambda_4 - \lambda_2 = E_{13} \quad A_4 = \lambda_4 - \lambda_3 \]

\[ B_1 = (\lambda_2 - \lambda_1) \ p_{12}^4 + (\lambda_3 - \lambda_1) \ p_{13}^4 + (\lambda_4 - \lambda_1) \ p_{14}^4 \]

\[ B_2 = r(k-1) + \lambda_4 + (\lambda_4 - \lambda_1)(p_{11}^1 - p_{11}^4) + (\lambda_4 - \lambda_2)(p_{12}^1 - p_{12}^4) + (\lambda_4 - \lambda_3)(p_{13}^1 - p_{13}^4) \]

\[ B_3 = (\lambda_4 - \lambda_1)(p_{11}^2 - p_{11}^4) + (\lambda_4 - \lambda_2)(p_{12}^2 - p_{12}^4) + (\lambda_4 - \lambda_3)(p_{13}^2 - p_{13}^4) \]
\[ B_4 = (\lambda_4 - \lambda_1) (p_{11}^3 - p_{11}^4) + (\lambda_4 - \lambda_2) (p_{12}^3 - p_{12}^4) + (\lambda_4 - \lambda_3) (p_{13}^3 - p_{13}^4) \]

\[ C_1 = (\lambda_4 - \lambda_2) p_{21}^4 + (\lambda_3 - \lambda_2) p_{23}^4 + (\lambda_4 - \lambda_2) p_{24}^4 \]

\[ C_2 = (\lambda_4 - \lambda_1) (p_{11}^1 - p_{11}^4) + (\lambda_4 - \lambda_2) (p_{12}^1 - p_{12}^4) + (\lambda_4 - \lambda_3) (p_{13}^1 - p_{13}^4) \]

\[ C_3 = r(k-1) + \lambda_4 + (\lambda_4 - \lambda_1) (p_{21}^2 - p_{21}^4) + (\lambda_4 - \lambda_2) (p_{22}^2 - p_{22}^4) + (\lambda_4 - \lambda_3) (p_{23}^2 - p_{23}^4) \]

\[ C_4 = (\lambda_4 - \lambda_1) (p_{31}^3 - p_{31}^4) + (\lambda_4 - \lambda_2) (p_{32}^3 - p_{32}^4) + (\lambda_4 - \lambda_3) (p_{33}^3 - p_{33}^4) \]

\[ D_1 = (\lambda_1 - \lambda_3) p_{31}^4 + (\lambda_2 - \lambda_3) p_{32}^4 + (\lambda_4 - \lambda_3) p_{34}^4 \]

\[ D_2 = (\lambda_4 - \lambda_1) (p_{31}^1 - p_{31}^4) + (\lambda_4 - \lambda_2) (p_{32}^1 - p_{32}^4) + (\lambda_4 - \lambda_3) (p_{33}^1 - p_{33}^4) \]

\[ D_3 = (\lambda_4 - \lambda_1) (p_{31}^2 - p_{31}^4) + (\lambda_4 - \lambda_2) (p_{32}^2 - p_{32}^4) + (\lambda_4 - \lambda_3) (p_{33}^2 - p_{33}^4) \]

\[ D_4 = r(k-1) + \lambda_4 + (\lambda_4 - \lambda_1) (p_{31}^3 - p_{31}^4) + (\lambda_4 - \lambda_2) (p_{32}^3 - p_{32}^4) + (\lambda_4 - \lambda_3) (p_{33}^3 - p_{33}^4) \]

and solution coming from above system for \( m=4 \) is

\[ \tau_i = k \left[ \frac{Q_i M_1 - S_1 M_2 + S_2 M_3 + \ldots + (-1)^{m-1} S_{m-1}}{\Delta} \right] / \Delta \]

\[ = k \left[ Q_i M_1 + \sum_{j=2}^{m} (-1)^{j-1} M_j S_{j-1} \right] / \Delta \]

\[ = k \left[ Q_i M_1 + \sum_{j=2}^{4} (-1)^{j-1} M_j S_{j-1} \right] / \Delta \]

\[ = k \left[ Q_i M_1 - M_2 S_1 + M_3 S_2 - M_4 S_3 \right] / \Delta \]

and the efficiency factors of four kinds are given by

\[ E_j = \Delta / rk \left[ M_1 + (-1)^{j-1} M_{j+1} \right] ; \quad M_{j+1} \text{ does not exists for } j+1 > 4, \quad \text{here } j = 1, 2, 3, 4. \]

Therefore

\[ E_1 = \Delta / rk \left[ M_1 + M_2 \right] \]
\( E_2 = \Delta / \text{rk} \left[ M_1 - M_3 \right] \)

\( E_3 = \Delta / \text{rk} \left[ M_1 + M_4 \right] \)

\( E_4 = \Delta / \text{rk} \left[ M_1 + M_5^* \right], \ M_5^* \text{ does not exist due to } j+1 > 4 \)

The overall efficiency factor is

\[
E = (v-1) \Delta / \text{rk} \left[ (v-1) M_1 + \sum_{j=1}^{m-1} (-1)^{j-1} n_j M_{j+1} \right]
\]

\[
= (v-1) \Delta / \text{rk} \left[ (v-1) M_1 + \sum_{j=1}^{3} (-1)^{j-1} n_j M_{j+1} \right]
\]

\[
= (v-1) \Delta / \text{rk} \left[ (v-1) M_1 + n_1 M_2 - n_2 M_3 + n_3 M_4 \right], \text{ where}
\]

\( M_1 = \left[ B_2 (C_3 D_4 - C_4 D_3 ) - B_3 ( C_2 D_4 - C_4 D_2 ) + B_4 ( C_2 D_3 - C_3 D_2 ) \right] \)

\( M_2 = \left[ A_2 (C_3 D_4 - C_4 D_3 ) - A_3 ( C_2 D_4 - C_4 D_2 ) + A_4 ( C_2 D_3 - C_3 D_2 ) \right] \)

\( M_3 = \left[ A_2 (B_3 D_4 - B_4 D_3 ) - A_3 ( B_2 D_4 - B_4 D_2 ) + A_4 ( B_2 D_3 - B_3 D_2 ) \right] \)

\( M_4 = \left[ A_2 (B_3 C_4 - B_4 C_3 ) - A_3 ( B_2 C_4 - B_4 C_2 ) + A_4 ( B_2 C_3 - B_3 C_2 ) \right] \)

7.6 Summary and Discussion

Using reduced normal equations for \( m=5 \) associate class PBIB design, we get a general solution for five associate class PBIB design and estimate five types of elementary treatment contrasts among treatment effects. Five kinds of efficiencies as well as average efficiency factor (A.E.F) also calculated from the variance of five types by doing complex mathematical calculations and then express these results in a general form for analysis of five associate class PBIB design.

Also, we have explored the general solution of reduced normal equations for \( m \) (\( m \geq 2 \)) associate class PBIB design with \( m \) kinds of efficiencies as well as average
efficiency factor (A.E.F) in a formulation form. The merit of this generalization is that, it gives a provision to get ‘m’ kinds of efficiencies along with A.E.F of a m associate class PBIB design with particular parametric combinations from this formulation frame without doing complex mathematical computational work. In other words, we can say it is an generalized analysis framework in which we substitute the parameters $v, b, r, k, \lambda_i, n_i, p_{jk}^i$ of m associate class PBIB design and able to approach the whole information of analysis directly.