Chapter 1

INTRODUCTION

“That was the wonderful thing about Ramanujan. He discovered so much and yet he left so much more in his garden for other people to discover. In the forty-four years since that happy day, I have intermittently been coming back to Ramanujan’s garden. Every time when I come back, I find fresh flowers blooming”

- Freeman Dyson

Srinivasa Ramanujan was born on 22 December 1887 in his maternal grandmother’s house in Erode about 250 miles southwest of Madras. At the age of 12, Ramanujan was able to solve all the problems from the book “Plane Trigonometry” by S. L. Loney [55]. A book entitled “Elementary Results in Pure Mathematics” written by G. S. Carr [35] inspired Ramanujan very much in his life. At the age of 16, Ramanujan joined Government Arts college in Kumbakonam on the strength of his good school work, he also got scholarship. Ramanujan devoted all his time to mathematics and neglected other subjects. He lost the scholarship, as he failed in F. A. examination. Ramanujan later tried to obtain F. A. degree from Pachaiyappa’s College in Madras, but failed as he consecrated his time on
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mathematics. This marked the end of his formal education.

Later, in 1912 Ramanujan became a clerk in Madras port trust office. In a historic letter dated Jan. 16, 1913, Ramanujan communicated 11 page letter to Professor G. H. Hardy which contained works on divergent series. Hardy after going through the letter realized that the work is original and most of the results were path breaking. Hardy then made up his mind that Ramanujan should be brought to Cambridge and provide him with the necessary education. Ramanujan was initially reluctant to go abroad because of his caste prejudices. Due to some continuous efforts by Mr. E. H. Neville, a young mathematician who visited his home and obtained Ramanujan's confidence, after which Ramanujan left to England in March 1914.

Soon after his arrival in Cambridge he started his work with Hardy. Hardy did not attempt to convert Ramanujan into classroom mathematician but enabled him to produce his original ideas. During his stay in Cambridge Ramanujan published around twenty one research papers containing theorems on definite integrals, modular equations, Riemann's zeta function, infinite series, summation of series, analytic number theory, asymptotic formulae, modular functions, partitions and combinatorial analysis. Ramanujan was awarded the B. A. degree by research in March 1916 for his work on Highly composite numbers which appeared in Journal of London Mathematical Society.

Unfortunately, early in 1917, Ramanujan fell ill and during his remaining time in England he spent most of the time at nursing homes. In 1918, Ramanujan was elected as a Fellow of the Royal Society and Fellow of Trinity College. In 1919, Ramanujan returned to India but as a sick man. Despite all the tender attention and the best attention from the doctors, he died on 26 April 1920. Ramanujan was
working on mathematics four days before the end. Ramanujan was 32 years, 4 months and 4 days old when he died.

Ramanujan left behind more than 3000 results compiled in three notebooks and “lost notebook”. After the death of Ramanujan, many mathematicians including Hardy strongly urged that his notebooks should be edited and published. Ramanujan’s work first came to light in Ramanujan’s *Collected Papers* [98], however due to lack of funds prevented the notebooks being published. G. N. Watson and B. M. Wilson in 1929 were assigned to edit the notebooks of Ramanujan by Hardy. Alas, Wilson passed away prematurely in 1935. Watson wrote around 30 papers inspired by the works of Ramanujan and his interest declined in late 1930’s. Thus the project of editing Ramanujan’s notebooks was incomplete. Finally, in 1957 a photostat unedited version of the notebooks in two volumes were published by Tata Institute of Fundamental Research in Bombay.

B. C. Berndt undertook the task of editing the notebooks in 1977. Berndt with the help of many mathematicians was able to publish five volumes [16]–[20] which consists of proofs to almost all claims made by Ramanujan in his notebooks. In the spring of 1976, while searching through papers of the late Watson at Trinity College, Cambridge, G. E. Andrews found a sheaf of 138 pages in the handwriting of Srinivasa Ramanujan. In view of the fame of Ramanujan’s earlier notebooks, Andrews called these papers Ramanujan’s “lost notebook”. This work, comprising about 650 results with no proofs, arises from the last year of Ramanujan’s life and represents some of his deepest work. Some of the topics found in the “lost notebook” are q-series, theta–functions, mock theta functions, continued fractions, partitions and infinite series. For a neat history on Ramanujan’s “lost notebook” one can refer [9].
In 1988, Narosa [99] published these scattered sheets, unpublished manuscripts and letters written to Hardy. In last two decades many mathematicians across the globe have got inspired by the works of Ramanujan and substantial research have been carried out. Andrews and Berndt have published three volumes [9]–[11] out of planned five volumes which consists proofs to the claims made by Ramanujan in his “lost notebook” and other unpublished papers.

In this thesis an attempt has been made for the contributions in the theory of modular equations in the spirit of Ramanujan. We have established several new modular equations and various applications therefrom.

1.1 Ramanujan’s theta–function

Theta–function plays central role in the Ramanujan’s theory. Ramanujan introduced his general theta–function \( f(a, b) \) in Chapter 16 of his second notebook [97] as:

\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.
\] (1.1.1)

By using the Jacobi's triple product identity, Ramanujan's general theta–function takes the form

\[
f(a, b) := (-a, ab)_{\infty} (-b, ab)_{\infty} (ab, ab)_{\infty},
\] (1.1.2)

where \((a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n)\), it is tacitly assumed in the sequel that \(|q| < 1\) always. The above equation (1.1.2) gives the product representation for Ramanujan's theta–function. Following Ramanujan, we define the three special cases of
\[ f(a, b) : \]
\[ \varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad (1.1.3) \]
\[ \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \quad (1.1.4) \]
\[ f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}. \quad (1.1.5) \]

We also define
\[ \chi(q) := (-q; q^2)_{\infty}. \]

1.2 Modular equations

The theory of modular equation actually began with the work of A. M. Legendre [54] and C. G. J. Jacobi [49], [50]. Legendre derived modular equation of degree 3 and Jacobi derived modular equations of degrees 3 and 5 in his Fundamenta Nova in 1829. Later many mathematicians predominantly C. Guetzlaff, L. A. Sohncke, H. Schrörter, L. Schláfli, F. Klein, A. Hurwitz, E. Fiedler, A. Cayley, R. Fricke, R. Russell and H. Weber have contributed to the theory of modular equations. More information about modular equations can be found in the books of A. Enneper [43], Weber [111], [112], Klein [52], [53], Fricke [44] and Berndt [18]–[20]. M. Hanna [46] has conducted a brief survey on modular equations until 1929 and also established some new modular equations of higher degrees.

In early 20th century, Ramanujan began to discover several new modular equations. However in his notebooks, Ramanujan perhaps recorded more modular equations than those of the predecessors combined all together. Ramanujan con-
structured over 200 modular equations. Chapter 15-21 in his second notebook [97]
is entirely devoted to modular equations.

There are many definitions of a modular equations in the literature. We give
the definition of a modular equation as understood by Ramanujan. Firstly, the
complete elliptic integral of the first kind $K(\kappa)$ for $0 < \kappa < 1$ is defined by

$$K(\kappa) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2)^2}{(n!)^2} \kappa^{2n}$$

(1.2.1)

where $\,_{2}F_{1}(a, b; c; p) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} p^n, \quad 0 \leq |p| < 1, \quad (a)_0 = 1, \quad (a)_n = a (a+1) \cdots (a+n-1)$ for $n$ a positive integer and $a, b$ and $c$ are complex numbers such that $c \neq 0, -1, -2, \ldots$. The number $\kappa$ is called the modulus of $K$.

One of the most fundamental results recorded by Ramanujan as Entry 6 of his
second notebook [97] in the theory of elliptic functions is the inversion formula
given by:

$$\varphi^2(q) = \,_{2}F_{1} \left( \frac{1}{2}, \frac{1}{2}; 1; \kappa^2 \right) := \varphi$$

(1.2.2)

and

$$\gamma := \pi \frac{\,_{2}F_{1} \left( \frac{1}{2}, \frac{1}{2}; 1; \kappa^2 \right)}{\,_{2}F_{1} \left( \frac{1}{2}, \frac{1}{2}; 1; \kappa^2 \right)}$$

(1.2.3)

where $K' = K(\kappa'), \kappa' := \sqrt{1 - \kappa^2}$ is called the complementary modulus. The base
$q$ is defined by $q := e^{-\pi \kappa'/K}$ then

$$\varphi(e^{-\gamma}) = \varphi$$

(1.2.4)
Following Ramanujan we set $\alpha = \kappa^2$. In Chapter 17 of his second notebook [97], Ramanujan has recorded a list of formulas at different arguments for $\varphi$, $\psi$, $f$ and $\chi$ in terms of $\alpha$, $q$ and $\varepsilon$. These formulas serves as foundation for Ramanujan’s modular equations.

Now we define a modular equation in the classical theory. Let $K$, $K'$, $L$ and $L'$ denote the complete elliptic integrals of the first kind associated with the moduli $\kappa$, $\kappa'$, $l$ and $l' := \sqrt{1-t^2}$ respectively, where $0 < \kappa, l < 1$. For a fixed positive integer $n$, suppose that

$$n \frac{K'}{K} = \frac{L'}{L}. \quad (1.2.5)$$

Then a modular equation of degree $n$ is the relation between $\kappa$ and $l$ induced by the equation (1.2.5). Following Ramanujan, we set $\beta = i^2$. Then we say $\beta$ is of degree $n$ over $\alpha$. The multiplier $m$ is defined by

$$m := \frac{z_1}{z_n} = \frac{\frac{1}{2} F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \alpha \right)}{\frac{1}{2} F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \beta \right)} = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \quad (1.2.6)$$

By the definition of $q$, if we set $q' = e^{-\pi L'/L}$ then the equation (1.2.5) is equivalent relation connecting $q$ with $q^n$. By the inversion formula $\kappa$ and $l$ can be expressed in the terms of theta–functions. Thus, a modular equation can also be identified as an identity involving theta–functions at the arguments $q$ and $q^n$.

Many methods have been developed to prove modular equations, but a method that may be applicable to prove one class of equations or for a degree may not be applicable for another class of equations or degree. Generally, modular equation can be expressed as a theta–function identity and it is sufficient that we prove this theta–function identity.
As we are unaware of the method that Ramanujan employed to establish these equations. There is no single method that one can use to discover or construct modular equations, one has to use a variety of tools. Generally, as the degree of the modular equation increases, the difficulty of establishing modular equations rises sharply.

1.3 Continued fractions

A continued fraction is an expression of the form

\[ r = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}} \]

(1.3.1)

where \(a_i\) and \(b_i\) are either real numbers, or complex numbers. We employ the following notation for general continued fraction

\[ r = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \frac{b_4}{a_4 + \cdots}}} \cdots} \]

Continued fractions offer a useful means of expressing numbers and functions. In the early ages, 300 B.C.–200 A.D., mathematicians used other algorithms and methods to express numbers and to express solutions of Diophantine equations. In mathematics, a Diophantine equation is an indeterminate polynomial equation that allows the variables to be integers only. Indian mathematician Aryabhata (475-550 A.D.) used a continued fraction to solve a linear equation. In the six-
teenth century, two Italian mathematicians, Rafael Bombelli (1526-72) and Pietro Cataldi (1548-1626), expressed $\sqrt{13}$ and $\sqrt{18}$, respectively, as periodic continued fractions. John Wallis (1616-1703) did go further and through his work, continued fractions became a subject of study in its own right. Later, the theory grew when Leonard Euler (1707-83), Johan Lambert (1728-77) and Joseph Louis Lagrange (1736-1813) worked on the topic. Much of the modern theory was developed in Euler’s 1737 work, De Fractionibus Continuis.

Ramanujan’s contribution to the theory of continued fraction expansions is immense. Apart from marvelous results communicated by Ramanujan in his letters to Hardy, the important contributions in the area of continued fractions can be found in Chapter 12 and Chapter 16 of his second notebook [97] and in also ‘lost’ notebook [99].

Ramanujan never published a paper devoted to continued fractions; indeed the only mention of a continued fraction in his published papers is the Rogers-Ramanujan continued fraction

$$\frac{1}{1 + \frac{aq}{1 + \frac{aq^2}{1 + \frac{aq^3}{1 + \cdots}}}} = \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)}}{(q; q)_n}$$

which is the particular case of more general continued fractions mentioned by Ramanujan in Entries 15 and 16 of Chapter 16 of his second notebook [97], viz.,

$$\frac{1}{1 + \frac{aq}{1 + \frac{aq^2}{1 + \frac{aq^3}{1 + \cdots}}}} = \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)}}{(q; q)_n}$$

(1.3.2)
and

\[
\frac{1}{1 + \frac{aq}{1 + \frac{aq^2}{1 + \frac{aq^3}{\ddots}}}} + \frac{aq^n}{1 + \frac{aq^{n+1}}{\ddots}} = \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)}}{(q;q)_n (-bq;q)_n} / \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (-bq;q)_n}.
\] (1.3.3)

The identity (1.3.2) was first established by L. J. Rogers [100]. The proof of the equation (1.3.2) was also given by Watson [108], V. Ramamani [92] and Andrews [8].

In his ‘lost’ notebook [99], Ramanujan has stated several continued fraction identities equivalent to and more general than (1.3.3). Some of these are

\[
\frac{\sum_{n=0}^{\infty} \lambda^n q^{n(n+1)}}{(q;q)_n (-bq;q)_n} / \sum_{n=0}^{\infty} \lambda^n q^{n^2} / (q;q)_n (-bq;q)_n = \frac{1}{1 + \frac{\lambda q}{1 + \frac{bq + \lambda q^2}{\ddots}} + \frac{\lambda q^{2n+1}}{1 + \frac{bq^{n+1}}{1 + \frac{\lambda q^{2n+2}}{\ddots}}}}.
\]

One of the monumental works of Ramanujan on continued fractions can be found in page 44 of his ‘lost’ notebook is given below:

\[
\frac{1}{1 + \frac{q^2 + aq}{1 + \frac{q^4 + bq^2}{1 + \frac{q^6 + aq^3}{1 + \ddots}}}}.
\] (1.3.4)

where Ramanujan states 9 cases for \(a = 0, \pm 1; b = 0, \pm 1\). Ramanujan without stating the definition of order, classifies these 9 cases of continued fractions into
order 4 (two cases), order 5 (one case), order 6 (four cases) and order 8 (2 cases).
M. S. Mahadeva Naika, K. Sushan Bairy and M. Manjunatha [66] have defined the order of continued fraction in the sense of Ramanujan.

Many mathematicians have contributed to the theory of continued fractions. Some of them are C. Adiga and D. D. Somashekara [4], W. A. Al-Salam and M. E. H. Ismail [6], Andrews [7], N. A. Bhagirathi [26], [27], S. Bhargava and Adiga [28], Bhargava, Adiga and Somashekara [31], [32], L. Carlitz [33], Carlitz and R. Scoville [34], R. Y. Denis [39], [40], [41], [42], B. Gordon [45], M. D. Hirschhorn [48], K. G. Ramanathan [93], [94], S. N. Singh [103], [104], A. Verma, Denis and K. Srinivasa Rao [107].

In Chapter 2, we established several new $P$–$Q$ modular equations of degrees two, three and nine involving Ramanujan’s theta–functions $\varphi$ and $\psi$. These modular equations are analogous to those equations recorded by Ramanujan [19, pp. 204–237]

In Chapter 3, we present several new modular identities for Ramanujan’s cubic continued fraction, a continued fraction of order 12 and a parameter involving Ramanujan’s cubic continued fraction.

In Chapter 4, we offer some applications of $P$–$Q$ modular equations established in Chapter 2 by establishing several new general formulas for explicit evaluations of ratios of Ramanujan’s theta–function $\varphi$ and $\psi$. Several particular values of ratios of theta–functions are evaluated. These particular values are employed to establish several new explicit evaluations of Ramanujan–Göllnitz–Gordon continued fraction, Ramanujan–Selberg continued fraction, Ramanujan’s cubic continued fraction, a continued fraction of Eisenstein, a continued fraction of order 12 and also Ramanujan’s remarkable product of theta–functions.
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In Chapter 5, we established several new Schl"afli type modular equations in the cubic theory and as well as the quartic theory. Following Ramanujan, we introduced cubic and quartic class invariants. Some evaluations of these invariants are established. We also explicitly evaluate cubic and quartic singular modulus.

1.4 Preliminary results

In this section, we collect some relevant identities which are useful in establishing our results concerned with the classical theory.

Lemma 1.4.1. [18, Ch. 16, Entry 24(ii), (iv), p. 39] We have

\[ f^3(-q) = \varphi^2(-q) \psi(q), \]  
(1.4.1)

\[ f^3(-q^2) = \varphi(-q) \psi^2(q). \]  
(1.4.2)

Lemma 1.4.2. [18, Ch. 16, Entry 27(i), p. 43] If \( \alpha \beta = \pi \), then

\[ \sqrt{\alpha} \varphi(e^{-\alpha^2}) = \sqrt{\beta} \varphi(e^{-\beta^2}). \]  
(1.4.3)

Lemma 1.4.3. [18, Ch. 16, Entry 27(iii), p. 43] If \( \alpha \beta = \pi^2 \), then

\[ \sqrt{\alpha} e^{-\alpha/12} f(-e^{-2\alpha}) = \sqrt{\beta} e^{-\beta/12} f(-e^{-2\beta}). \]  
(1.4.4)

Lemma 1.4.4. [18, Ch. 17, Entry 10(i), (iv), (vi) p. 122] For \( 0 < x < 1 \), we have

\[ \varphi(e^{-x}) = \sqrt{x}. \]  
(1.4.5)
\[ \varphi (e^{-2x}) = \sqrt{z} \left( \frac{1}{2} \left( 1 + \sqrt{1-x} \right) \right)^{1/2}, \quad (1.4.6) \]

\[ \varphi \left( e^{-x/2} \right) = \sqrt{z} (1 + \sqrt{x})^{1/2}. \quad (1.4.7) \]

**Lemma 1.4.5.** [18, Ch. 17, Entry 11(ii), p. 123] For \(0 < x < 1\), we have

\[ \psi \left( e^{-x} \right) = \sqrt{\frac{1}{2} z} \{ x (1-x) e^x \}^{1/8}. \quad (1.4.8) \]

**Lemma 1.4.6.** [18, Ch. 17, Entry 12(ii), (iv), (v), (vi), p. 124] For \(0 < x < 1\), we have

\[ f \left( e^{-x} \right) = \sqrt{z} 2^{-1/6} \{ x (1-x) e^x \}^{1/24}, \quad (1.4.9) \]

\[ f \left( e^{-4x} \right) = \sqrt{z} 2^{-2/3} (1-x)^{1/24} \{ xe^x \}^{1/6}, \quad (1.4.10) \]

\[ \chi \left( e^{-x} \right) = 2^{1/6} (1-x)^{1/12} (xe^x)^{-1/24}, \quad (1.4.11) \]

\[ \chi \left( e^{-4x} \right) = 2^{1/6} (1-x)^{1/12} (xe^x)^{-1/24}. \quad (1.4.12) \]

**Lemma 1.4.7.** [18, Ch. 18, Eq. (24.22), p. 215] If \( \beta \) is of degree 4 over \( \alpha \), then

\[ \beta = \left( \frac{1 - \sqrt{1 - \alpha}}{1 + \sqrt{1 - \alpha}} \right)^4. \quad (1.4.13) \]

**Lemma 1.4.8.** [18, Ch. 19, Entry 13(i), p. 280] If \( \beta \) is of degree 5 over \( \alpha \), then

\[ (\alpha \beta)^{1/2} + ((1 - \alpha)(1 - \beta))^{1/2} + 2 \{ 16 \alpha \beta (1 - \alpha)(1 - \beta) \}^{1/6} = 1. \quad (1.4.14) \]

**Lemma 1.4.9.** [18, Ch. 19, Entry 19(i), p. 314] If \( \beta \) is of degree 7 over \( \alpha \), then

\[ (\alpha \beta)^{1/8} + ((1 - \alpha)(1 - \beta))^{1/8} = 1. \quad (1.4.15) \]
Lemma 1.4.10. [18, Ch. 19, Entry 24(v), p. 217] If $\beta$ is of degree 8 over $\alpha$, then

$$\left(1 - (1 - \alpha)^{1/4}\right) \left(1 - \beta^{1/4}\right) = 2\sqrt{2} (\beta (1 - \alpha))^{1/8}. \quad (1.4.16)$$

Lemma 1.4.11. [18, Ch. 20, Entry 3(x), (xi), p. 352] If $\beta$ is of degree 9 over $\alpha$ and $m$ is the multiplier connecting $\alpha$ and $\beta$, then

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1 - \beta}{1 - \alpha}\right)^{1/8} - \left(\frac{\beta (1 - \beta)}{\alpha (1 - \alpha)}\right)^{1/8} = \sqrt{m}, \quad (1.4.17)$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1 - \alpha}{1 - \beta}\right)^{1/8} - \left(\frac{\alpha (1 - \alpha)}{\beta (1 - \beta)}\right)^{1/8} = \frac{3}{\sqrt{m}}. \quad (1.4.18)$$

Lemma 1.4.12. [18, Ch. 20, Entry 7(i), p. 363] If $\beta$ is of degree 11 over $\alpha$, then

$$\{\alpha \beta\}^{1/4} + \{(1 - \alpha) (1 - \beta)\}^{1/4} + 2 \{16 \alpha \beta (1 - \alpha)(1 - \beta)\}^{1/12} = 1. \quad (1.4.19)$$

Lemma 1.4.13. [18, Ch. 20, Entry 21(i), p. 435] If $\beta$ is of degree 15 over $\alpha$, then

$$(\alpha \beta)^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8} + \{\alpha \beta (1 - \alpha)(1 - \beta)\}^{1/8}$$

$$= \left\{ \frac{1}{2} \left(1 + \sqrt{\alpha \beta} + \sqrt{(1 - \alpha)(1 - \beta)}\right) \right\}^{1/2}. \quad (1.4.20)$$

Lemma 1.4.14. [18, Ch. 20, Entry 12(iii), (iv), pp. 397-398] If $\beta$ has degree 17 over $\alpha$ and $m$ is the multiplier connecting $\alpha$ and $\beta$, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1 - \beta}{1 - \alpha}\right)^{1/4} + \left(\frac{\beta (1 - \beta)}{\alpha (1 - \alpha)}\right)^{1/4}$$

$$- 2 \left(\frac{\beta (1 - \beta)}{\alpha (1 - \alpha)}\right)^{1/8} \left\{1 + \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1 - \beta}{1 - \alpha}\right)^{1/8}\right\}, \quad (1.4.21)$$
\[
\frac{17}{m} = \left( \frac{\alpha}{\beta} \right)^{1/4} + \left( \frac{1 - \alpha}{1 - \beta} \right)^{1/4} + \left( \frac{\alpha (1 - \alpha)}{\beta (1 - \beta)} \right)^{1/4} \\
- 2 \left( \frac{\alpha (1 - \alpha)}{\beta (1 - \beta)} \right)^{1/8} \left\{ 1 + \left( \frac{\alpha}{\beta} \right)^{1/8} + \left( \frac{1 - \alpha}{1 - \beta} \right)^{1/8} \right\}.
\]

(1.4.22)

**Lemma 1.4.15.** [18, Ch. 20, Entry 3(xii), pp. 352–353] Let \(\alpha, \beta\) and \(\gamma\) be of the first, third and ninth degrees respectively. Let \(m\) denote the multiplier connecting \(\alpha, \beta\) and \(m'\) be the multiplier relating \(\gamma, \delta\), then

\[
\left( \frac{\beta^2}{\alpha \gamma} \right)^{1/4} + \left( \frac{(1 - \beta)^2}{(1 - \alpha) (1 - \gamma)} \right)^{1/4} - \left( \frac{\beta^2 (1 - \beta)^2}{\alpha \gamma (1 - \alpha) (1 - \gamma)} \right)^{1/4} = -3 \frac{m'}{m}, \tag{1.4.23}
\]

\[
\left( \frac{\alpha \gamma}{\beta^2} \right)^{1/4} + \left( \frac{(1 - \alpha) (1 - \gamma)}{(1 - \beta)^2} \right)^{1/4} - \left( \frac{\alpha \gamma (1 - \alpha) (1 - \gamma)}{\beta^2 (1 - \beta)^2} \right)^{1/4} = \frac{m'}{m}. \tag{1.4.24}
\]

**Lemma 1.4.16.** [18, Ch. 20, Entry 11(viii), (ix), p. 384] Let \(\alpha, \beta, \gamma\) and \(\delta\) be of the first, third, fifth and fifteenth degrees respectively. Let \(m\) denote the multiplier connecting \(\alpha\) and \(\beta\) and \(m'\) be the multiplier relating \(\gamma\) and \(\delta\). then

\[
\left( \frac{\alpha \delta}{\beta \gamma} \right)^{1/8} + \left( \frac{(1 - \alpha) (1 - \delta)}{(1 - \beta) (1 - \gamma)} \right)^{1/8} - \left( \frac{\alpha \delta (1 - \alpha) (1 - \delta)}{\beta \gamma (1 - \beta) (1 - \gamma)} \right)^{1/8} = \sqrt{\frac{m'}{m}}, \tag{1.4.25}
\]

\[
\left( \frac{\beta \gamma}{\alpha \delta} \right)^{1/8} + \left( \frac{(1 - \beta) (1 - \gamma)}{(1 - \alpha) (1 - \delta)} \right)^{1/8} - \left( \frac{\beta \gamma (1 - \beta) (1 - \gamma)}{\alpha \delta (1 - \alpha) (1 - \delta)} \right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{1.4.26}
\]

**Lemma 1.4.17.** [18, Ch. 20, Entry 13(i), (ii), p. 401] Let \(\alpha, \beta, \gamma\) and \(\delta\) be of the first, third, seventh and twenty first degrees respectively. Let \(m\) denote the
multiplier connecting $\alpha$ and $\beta$ and $m'$ be the multiplier relating $\gamma$ and $\delta$. then

\[
\left( \frac{\beta \gamma}{\alpha \delta} \right)^{1/4} + \left( \frac{(1 - \beta)(1 - \gamma)}{(1 - \alpha)(1 - \delta)} \right)^{1/4} - \left( \frac{\beta \gamma (1 - \beta)}{\alpha \delta (1 - \alpha)(1 - \delta)} \right)^{1/4} + 4 \left( \frac{\beta \gamma (1 - \beta)(1 - \gamma)}{\alpha \delta (1 - \alpha)(1 - \delta)} \right)^{1/6} = \frac{m}{m'},
\]

(1.4.27)

\[
\left( \frac{\alpha \delta}{\beta \gamma} \right)^{1/4} + \left( \frac{(1 - \alpha)(1 - \delta)}{(1 - \beta)(1 - \gamma)} \right)^{1/4} - \left( \frac{\alpha \delta (1 - \alpha)(1 - \delta)}{\beta \gamma (1 - \beta)(1 - \gamma)} \right)^{1/4} + 4 \left( \frac{\alpha \delta (1 - \alpha)(1 - \delta)}{\beta \gamma (1 - \beta)(1 - \gamma)} \right)^{1/6} = \frac{m'}{m}.
\]

(1.4.28)

**Lemma 1.4.18.** [18, Ch. 20, Entry 14(i), (ii), p. 408] Let $\alpha$, $\beta$, $\gamma$ and $\delta$ be of the first, third, eleventh and thirty third degrees respectively. Let $m$ denote the multiplier connecting $\alpha$ and $\beta$ and $m'$ be the multiplier relating $\gamma$ and $\delta$. then

\[
\left( \frac{\beta \delta}{\alpha \gamma} \right)^{1/8} + \left( \frac{(1 - \beta)(1 - \delta)}{(1 - \alpha)(1 - \gamma)} \right)^{1/8} - \left( \frac{\beta \delta (1 - \beta)(1 - \delta)}{\alpha \gamma (1 - \alpha)(1 - \gamma)} \right)^{1/8} - 2 \left( \frac{\beta \delta (1 - \beta)(1 - \delta)}{\alpha \gamma (1 - \alpha)(1 - \gamma)} \right)^{1/12} = \sqrt{mm'},
\]

(1.4.29)

\[
\left( \frac{\alpha \gamma}{\beta \delta} \right)^{1/8} + \left( \frac{(1 - \alpha)(1 - \delta)}{(1 - \beta)(1 - \gamma)} \right)^{1/8} - \left( \frac{\alpha \gamma (1 - \alpha)(1 - \delta)}{\beta \delta (1 - \beta)(1 - \gamma)} \right)^{1/8} - 4 \left( \frac{\alpha \gamma (1 - \alpha)(1 - \delta)}{\beta \delta (1 - \beta)(1 - \gamma)} \right)^{1/12} = \frac{3}{\sqrt{mm'}}.
\]

(1.4.30)

**Lemma 1.4.19.** [18, Ch. 20, Entry 19(iv), p. 426] If $\beta$, $\gamma$ and $\delta$ are of degrees 3, 13 and 39 respectively over $\alpha$, then

\[
\left( \frac{\alpha \delta}{\beta \gamma} \right)^{1/8} + \left( \frac{(1 - \alpha)(1 - \delta)}{(1 - \beta)(1 - \gamma)} \right)^{1/8} - \left( \frac{\alpha \delta (1 - \alpha)(1 - \delta)}{\beta \gamma (1 - \beta)(1 - \gamma)} \right)^{1/8} + 2 \left( \frac{\alpha \delta (1 - \alpha)(1 - \delta)}{\beta \gamma (1 - \beta)(1 - \gamma)} \right)^{1/12} = \sqrt{\frac{m'}{m}}.
\]

(1.4.31)
\[ \left( \frac{\beta \gamma}{\alpha \delta} \right)^{1/8} + \left( \frac{(1 - \beta)(1 - \gamma)}{(1 - \alpha)(1 - \delta)} \right)^{1/8} = \left( \frac{\beta \gamma(1 - \beta)(1 - \gamma)}{\alpha \delta(1 - \alpha)(1 - \delta)} \right)^{1/8} + 2 \left( \frac{\beta \gamma(1 - \beta)(1 - \gamma)}{\alpha \delta(1 - \alpha)(1 - \delta)} \right)^{1/12} = \sqrt{\frac{m}{m'}}. \] (1.4.32)

**Lemma 1.4.20.** [18, Ch. 20, Entry I(i), (ii), (iii), p. 345] We have

\[ 1 + \frac{1}{V^3(q)} = \frac{\psi^4(q)}{q \psi^4(q^3)}. \] (1.4.33)

\[ 1 + \frac{\psi(-q^{1/3})}{q^{1/3} \psi(-q^3)} = \left( 1 + \frac{\psi^4(-q)}{q \psi^4(-q^3)} \right)^{1/3}, \] (1.4.34)

\[ 1 + 3q \frac{\psi(-q^9)}{\psi(-q)} = \left( 1 + 9q \frac{\psi^4(-q^3)}{\psi^4(-q)} \right)^{1/3}, \] (1.4.35)

\[ \frac{\varphi(q^{1/3})}{\varphi(q)} = 1 + \left( \frac{\varphi^4(q)}{\varphi^4(q^3)} - 1 \right)^{1/3}, \] (1.4.36)

\[ \frac{3 \varphi(q^9)}{\varphi(q)} = 1 + \left( \frac{9 \varphi^4(q^3)}{\varphi^4(q)} - 1 \right)^{1/3}. \] (1.4.37)

where \( V(q) \) is Ramanujan’s cubic continued fraction.

**Lemma 1.4.21.** [19, Ch. 25, Entry 51, p. 204] Let \( X := \frac{f^2(-q)}{q^{1/6} f^2(-q^3)} \) and \( Y := \frac{f^2(-q^2)}{q^{1/3} f^2(-q^6)} \), then

\[ XY + \frac{9}{XY} = \left( \frac{Y}{X} \right)^3 + \left( \frac{X}{Y} \right)^3. \] (1.4.38)

**Lemma 1.4.22.** [19, Ch. 25, Entry 56, p. 210] If \( P := \frac{f(-q)}{q^{1/3} f(-q^9)} \) and \( Q := \frac{f(-q^2)}{q^{2/3} f(-q^{18})} \), then

\[ P^3 + Q^3 = P^2 Q^2 + 3PQ. \] (1.4.39)
Lemma 1.4.23. [59, Theorem 2.2] We have

\[
\frac{f^3(-q)}{f^3(-q^9)} = \frac{\psi(q)}{\psi(q^9)} \left( \frac{\psi(q) - 3q\psi(q^9)}{\psi(q) - q\psi(q^9)} \right)^2, \tag{1.4.40}
\]

\[
\frac{f^3(-q)}{qf^3(-q^9)} = \frac{\varphi^2(-q)}{\varphi(-q^9)} \frac{\varphi(-q) - 3\varphi(-q^9)}{\varphi(-q) - \varphi(-q^9)} \tag{1.4.41}
\]

\[
\frac{f^3(-q^9)}{q^2f^3(-q^{18})} = \frac{\varphi(-q)}{\varphi^2(-q^9)} \left( \frac{\varphi(-q) - 3\varphi(-q^9)}{\varphi(-q) - \varphi(-q^9)} \right)^2 \tag{1.4.42}
\]

\[
\frac{f^3(-q^9)}{f^3(-q^{18})} = \frac{\psi^2(q)}{\psi^2(q^9)} \frac{\psi(q) - 3q\psi(q^9)}{\psi(q) - q\psi(q^9)} \tag{1.4.43}
\]

Lemma 1.4.24. [2, Theorem 5.1] If

\[P := \frac{\psi(-q)}{q^{1/4}\psi(-q^3)}\]  \quad and \quad \[Q := \frac{\varphi(q)}{\varphi(q^3)}\]

then

\[Q^4 + P^4Q^4 = 9 + P^4. \tag{1.4.44}\]

Lemma 1.4.25. [115, Theorem 2.1] Let

\[P := \frac{f(-q)}{q^{1/12}f(-q^3)}\]  \quad and \quad \[Q := \frac{f(-q^{17})}{q^{17/12}f(-q^{51})}\]

then

\[
\left( \frac{Q}{P} \right)^9 - \left( \frac{P}{Q} \right)^9 - 238 \left[ \left( \frac{Q}{P} \right)^6 + \left( \frac{P}{Q} \right)^6 \right] + 1853 \left[ \left( \frac{Q}{P} \right)^3 - \left( \frac{P}{Q} \right)^3 \right] \\
- 17 \left( \frac{Q^8}{P^8} + \frac{3^2P^4}{Q^8} \right) - 17 \left( \frac{3^2Q^4}{P^8} + \frac{P^8}{Q^4} \right) + 34 \left( Q^7P - \frac{3^4}{Q^7P} \right) \\
- 34 \left( QP^7 - \frac{3^3}{Q^7P} \right) - 442 \left( \frac{Q^5}{P} - \frac{3^2P}{Q^5} \right) - 442 \left( \frac{3^2Q}{P^5} - \frac{P^5}{Q} \right) \tag{1.4.45}
\]

\[= (PQ)^8 + \left( \frac{3}{PQ} \right)^8 - 34 \left[ (PQ)^6 + \left( \frac{3}{PQ} \right)^6 \right] + 425 \left[ (PQ)^4 + \left( \frac{3}{PQ} \right)^4 \right] - 2380 \left[ (PQ)^2 + \left( \frac{3}{PQ} \right)^2 \right] + 8568.\]
Lemma 1.4.26. [115, Theorem 2.3] Let $P := \frac{f(-q)}{q^{1/12}f(-q^3)}$ and $Q := \frac{f(-q^{19})}{q^{19/12}f(-q^{57})}$, then

$$
\left(\frac{Q}{P}\right)^{10} - \left(\frac{P}{Q}\right)^{10} - 76 \left[\left(\frac{Q}{P}\right)^{8} - \left(\frac{P}{Q}\right)^{8}\right] + 912 \left[\left(\frac{Q}{P}\right)^{6} - \left(\frac{P}{Q}\right)^{6}\right]
+ 6650 \left[\left(\frac{Q}{P}\right)^{4} - \left(\frac{P}{Q}\right)^{4}\right] + 9481 \left[\left(\frac{Q}{P}\right)^{2} - \left(\frac{P}{Q}\right)^{2}\right]
- 570 \left(\frac{3Q}{P^7} + \frac{P^7}{Q}\right)
- 19 \left(\frac{Q^9}{P^3} + \frac{33P^3}{Q^9}\right)
- 19 \left(\frac{33Q^3}{P^9} + \frac{P^9}{Q^3}\right)
- 570 \left(\frac{Q^7}{P} + \frac{33P}{Q^7}\right)
+ 19 \left(Q^8P^4 - \frac{36}{Q^8P^4}\right)
- 19 \left(Q^4P^8 - \frac{36}{Q^4P^8}\right)
- 2166 \left(Q^5P + \frac{33}{Q^5P}\right)
- 2166 \left(QP^5 + \frac{33}{QP^5}\right) = (PQ)^9 + \left(\frac{3}{PQ}\right)^9 + 3211 \left[(PQ)^3 + \left(\frac{3}{PQ}\right)^3\right].
$$

(1.4.46)

Lemma 1.4.27 (Identity Theorem). Suppose $f(z)$ is analytic in a domain $D$ and that $\{z_n\}$ is a sequence of distinct points converging to a point $z_0$ in $D$. If $f(z_n) = 0$ for each $n$, then $f(z) \equiv 0$ throughout $D$.

Lemma 1.4.28. [20, Ch. 34, p. 247] If $\beta$ has degree $r$ over $\alpha$, then $\beta$ has degree $p$ over $1 - \alpha$, where $p$ and $r$ are co-prime positive integers and $q = \exp\left(-\pi\sqrt{p/r}\right)$. 