Chapter 1

Introduction

Riemann's revolutionary ideas generalized the geometry of surfaces which had been studied earlier by Gauss, Bolyai and Lobachevsky. Later this lead to an exact definition of the modern concept of an abstract Riemannian manifold. The development of the 20th century has turned Riemannian geometry into one of the most important parts of modern Mathematics. Levi-Civitae and Ricci developed the concept of parallel translation in the classical language of tensors. This approach received a tremendous impetus from Einstein's work on relativity. E. Cartan initiated research and methods that were independent of a particular coordinate system (invariant methods).

In 1958, Boothby and Wang initiated the study of odd dimensional manifolds with contact and almost contact structures. Sasaki and Hatakeyama reinvestigated them using tensor calculus in 1961. Almost contact metric structures and Sasakian structures viz., almost Sasakian, nearly Sasakian etc., were proposed by Sasaki [86] in 1960 and 1965 respectively. Later Kenmotsu [58] defined a class of almost contact Riemannian manifold, called Kenmotsus manifold, similar to Sasakian manifolds in 1972. A class of almost contact metric manifolds known as trans-Sasakian manifolds was introduced by Oubina [77] in 1985. This
class contains $\alpha$ Sasakian, $\beta$-Kenmotsu and cosymplectic manifolds. An almost contact metric structure on a manifold $M$ is called a trans-Sasakian structure if the product manifold $M \times R$ belongs to the class $W_4$ [51], a class of Hermitian manifolds which are closely related to a locally conformal Kahler manifolds. Trans-Sasakan manifolds were studied extensively by Marrero [66], Tripathi [108], De [34, 42, 43] and others.

*Manifolds* are important objects in Mathematics and physics because they allow more complicated structure to be expressed and understood in terms of the relatively well understood properties of simpler spaces.

A Manifold is an abstract mathematical space in which every point has a neighborhood which resembles Euclidian space. In discussing manifolds, the idea of dimension is important. In an one-dimensional manifold, every point has a neighborhood that looks like a segment of a line. Examples of one-manifolds include lines, circles, and two separate circles. In a two-manifold, every point has a neighborhood that looks like a disc. Examples include a plane, the surface of a sphere and the surface of a torus.

Any manifold can be described by a collection (or atlas) of charts. Each chart specifies a coordinate system on a piece of the manifold, which is a function from that piece of the manifold into a Euclidian space.

The notion of a differentiable manifold refines the notion of a manifold by requiring the transition from one chart to another to be differentiable.

A *differentiable manifold* is a topological manifold with a globally defined differentiable structure. Any topological manifold can be given a differentiable structure locally by using the homeomorphisms in its atlas, combined with the standard differentiable structure on the Euclidean space. In other words, the
homeomorphism can be used to give a local coordinate system.

A (smooth) Riemannian metric on a manifold $M$ is an association to every $p \in M$ a symmetric positive definite bilinear form $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ (hence an inner product) such that in every local coordinates $(x^1, \ldots, x^n)$, $g_p$ is given by

$$g_p \left( a^i \frac{\partial}{\partial x^i}, b^j \frac{\partial}{\partial x^j} \right) = g_{ij}(p)a^ib^j$$

with smooth coefficients $g_{ij}(p)$. The pair $(M, g)$ is called Riemannian manifold.

The study of Riemannian manifolds comprises the subject called Riemannian geometry. A Riemannian metric makes it possible to define various geometric notions on a Riemannian manifold, such as angles, lengths of curves, areas (or volumes), curvature, gradients of functions and divergence of vector fields.

An $n$-dimensional differentiable manifold $M$ is called a contact manifold if it carries a global differentiable 1-form $\eta$-such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$. This 1-form $\eta$ is called the contact form of $M$. Where the exponent denotes the $n^{th}$ exterior power. we call $\eta$-a contact form.

Let $M$ be an almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a (1,1) tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ then

$\phi^2 = -I + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0$,

$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$,

$g(X, \phi Y) = -g(\phi X, Y)$,

$g(X, \xi) = \eta(X)$, for all $X, Y \in \chi(M)$, where $\chi(M)$ is the algebra of all $C^\infty$ vector fields on $M$.  

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By providing additional conditions on contact manifolds one can obtain the structures like Sasakian, K-contact, Kenmotsu and trans-Sasakian manifolds.

An almost contact manifold $M(\phi, \xi, \eta, g)$ is said to be Sasakian manifold if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$$

and $\nabla_X \xi = -\phi X$, for any $X, Y \in \chi(M)$,

where $\nabla$ denotes the Riemannian connection of $g$.

Example: An odd-dimensional sphere $S^{2n-1}$ is Sasakian.

An almost contact metric manifold $M$ is called a Kenmotsu manifold if it satisfies

$$(\nabla_X \phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X, \quad X, Y \in \chi(M).$$

From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \quad (\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y).$$

Let $M$ be an $n$-dimensional contact metric manifold with contact metric structure $(\phi, \xi, \eta, g)$. If the characteristic vector field $\xi$ is a killing vector field, then the contact manifold $M$ with this structure is called $K$-contact Riemannian manifold or simply a $K$-contact Manifold.

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be trans-Sasakian manifold [77] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$ [51] for the Hermitian manifolds, where $J$ is the almost complex structure [9] on $M \times \mathbb{R}$ defined by $J(Z, f \frac{d}{dt}) = (\phi Z - f \xi, \eta(Z) \frac{d}{dt})$. For the vector field $Z$ on $M$, smooth function $f$ on $M \times \mathbb{R}$ and $G$ an Hermitian metric on $M \times \mathbb{R}$. This may be expressed by the condition:

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X),$$

for some smooth functions $\alpha$ and $\beta$ on $M$ and we say that trans-Sasakian structure is of type $(\alpha, \beta)$. From the above equation it follows that
\[ \nabla_X \xi = -\alpha \phi X + \beta (X - \eta(X))\xi, \]
\[ (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \]

Example: Let \((x, y, z)\) be a cartesian coordinate in \(\mathbb{R}^3\). Then the metric structure \((\phi, \xi, \eta, g)\) given by
\[
\xi = \frac{\partial}{\partial z}, \eta = dz - ydx, \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}
\]
is a trans-Sasakian structure of type \((-\frac{1}{2e^z}, \frac{1}{2})\) in \(\mathbb{R}^3\). In general in a 3-dimensional \(K\)-contact manifold with structure tensor \((\phi, \xi, \eta, g)\) and for a non-constant function \(f\), if we define \(g^* = fg + (1 - f)\eta \otimes \eta\), then \((\phi, \xi, \eta, g^*)\) is a trans-Sasakian structure of type \((-\frac{1}{f}, \frac{1}{2} \xi \log f)\) [28, 56].

Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds.

A contact metric manifold is said to be an Einstein manifold if its Ricci tensor \(S\) is of the form \(S = ag\), where \(a\) is a constant.

An almost contact metric manifold is said to be \(\eta\)-Einstein if its Ricci tensor is of the form \(S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)\), where \(a\) and \(b\) are some smooth functions on \(M\).

The notion of local symmetry of a Riemannian manifold has been weaken by many authors in several ways to a different extent. In the context of contact geometry the notion of \(\phi\)-symmetry is introduced and studied by Bocckx, Buccken and Vanhecke [24] with several examples. As a weaker version of local symmetry, Takahashi [100] introduced the notion of locally \(\phi\)-symmetry on a Sasakian manifold. Generalizing the notion of \(\phi\)-symmetry, De et al. [45] introduced the
notion of $\phi$-recurrent Sasakian manifold.

The entire work presented in the thesis has been divided into six chapters including the introductory chapter 1.

In Chapter 2, we study $(\epsilon, \delta)$ trans-Sasakian structure and its curvature conditions. The study of manifolds with indefinite metrics is of interest from the stand point of physics and relativity. The investigation of manifolds with indefinite metrics was taken up by several authors. In 1993, Bejancu and Duggal [23] introduced the concept of $(\epsilon)$-Sasakian manifolds and X Xufeng and C Xicili [115] established that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds. Rakesh Kumar et al.[63] studied the curvature conditions of these manifolds. Tripathi et al. [103] introduced and studied $(\epsilon)$-almost para contact manifolds. De and Sarkar [40] have introduced $(\epsilon)$-Kenmotsu manifolds and studied conformally flat, Weyl semi- symmetric, $\phi$- recurrent $(\epsilon)$-Kenmotsu manifolds. The existence of a new structure on indefinite metrics influences the curvature. Nagaraja et al. [70] introduced $(\epsilon, \delta)$ trans-Sasakian manifolds which generalize the notion of both $(\epsilon)$-Sasakian and $(\epsilon)$-Kenmotsu manifolds [32].

An $(\epsilon)$-almost contact metric manifold is called a $(\epsilon, \delta)$-trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - c\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X)$$

holds for some smooth functions $\alpha$ and $\beta$ on $M$ and $\epsilon = \pm 1, \delta = \pm 1$. For $\beta = 0, \alpha = 1$, $(\epsilon, \delta)$-trans-Sasakian manifold reduces to $(\epsilon)$-Sasakian and for $\alpha = 0, \beta = 1$ it reduces to $(\epsilon)$-Kenmotsu manifold. Chapter 2 is structured in 8 sections. First section is kept for preliminaries. In Section 2.2 we give objectives. In Section 2.3, we give an example of $(\epsilon, \delta)$-trans-Sasakian structure and we present some basic results. Also an explicit formula for the curvature
tensor and Ricci tensors are obtained. Further we prove in a three dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold the characteristic vector field \(\xi\) belongs to the \(k\)-nullity distribution. In Section 2.4, we study \(\phi\)-recurrent \((\varepsilon, \delta)\)-trans-Sasakian manifold and we show that such a manifold reduces to a manifold of constant curvature for constants \(\alpha\) and \(\beta\).

Prasad [12] defined and studied a tensor field \(\overline{P}\) on a Riemannian manifold of dimension \(n > 2\) which includes the projective curvature tensor \(P\). This tensor field \(\overline{P}\) is known as pseudo-projective curvature tensor.

The pseudo-projective curvature tensor is defined as follows.

\[
\overline{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{r}{n} \left( \frac{a}{n - 1} + b \right) [g(Y, Z)X - g(X, Z)Y],
\]

where \(a\) and \(b\) are constants such that \(a, b \neq 0\), \(R\) is the curvature tensor, \(S\) is the Ricci tensor and \(r\) is the scalar curvature.

In [15, 16, 112], the authors Bagewadi, Venkatesha, Prakasha and others have extended this notion to Kenmotsu and LP-Sasakian manifolds and obtained the conditions for these manifolds to be of Einstein, \(\eta\)-Einstein and pseudo-projectively flat. A Sasakian manifold is called pseudo-projective semi-symmetric if \(R(X, Y) \cdot \overline{P} = 0\), where \(R(X, Y)\) is the derivation on \(\overline{P}\).

In Section 2.5, we prove pseudo-projectively flat \((\varepsilon, \delta)\)-trans-Sasakian manifold is \(\eta\)-Einstein. In Section 2.6, we prove \((\varepsilon, \delta)\)-trans-Sasakian manifold satisfying \(R(X, Y) \cdot \overline{P} = 0\) is projectively flat and trans-Sasakian manifold satisfying \(R(X, Y) \cdot Rsc\overline{P} = 0\) is an Einstein manifold. Further we obtain an expression for scalar curvature. In Section 2.7, we compute \(\xi\)-sectional
curvature and the $\phi$-sectional curvature of an $(\varepsilon, \delta)$-trans-Sasakian manifold. Further we establish relations among sectional curvature, $\phi$-sectional curvature and totally real bisectional curvature. In the last section we show that a 3-dimensional $(\varepsilon, \delta)$-trans-Sasakian manifold is of negative curvature if $\beta \leq \alpha$.

Chapter 3 deals with the study of pseudo-projective curvature tensor in Sasakian manifolds and $(\varepsilon, \delta)$ trans-Sasakian structure. Jaiswal and Ojha [55] studied weakly pseudo-projective symmetric and Ricci-symmetric manifolds and investigated the nature of scalar curvature in these manifolds.

This chapter consists of 5 sections of which first and second are devoted to preliminaries and objectives. In Section 3.3, we study pseudo-projective curvature tensor in Sasakian manifolds and we show that pseudo-projectively flat Sasakian manifold is locally isometric to unit sphere and Sasakian manifold is locally isometric to unit sphere if and only if $R(X, Y) \cdot \overline{\mathcal{P}} = 0$. In this section, we also show that in a Sasakian space-time manifold with conservative pseudo-projective curvature tensor and also in a divergence free Sasakian-space-form the integral curves of the characteristic vector field are geodesics. In Section 3.4, we show that a Sasakian manifold is pseudo-projective Ricci-semi-symmetric if and only if it is Ricci-semi-symmetric. Also a Sasakian manifold satisfying the condition $\overline{\mathcal{P}} \cdot Ric \overline{\mathcal{P}} = 0$ is an $\eta$-Einstein manifold.

In Section 3.5, we study $(\varepsilon, \delta)$-trans-Sasakian manifolds and establish relation between 1-forms in a weakly pseudo-projective symmetric $(\varepsilon, \delta)$-trans-Sasakian manifolds. Also we deduce that weakly pseudo-projective $\phi$-symmetric, weakly pseudo-projective $\phi$-Ricci-symmetric and $\phi$-pseudo-projectively flat $(\varepsilon, \delta)$-trans-Sasakian manifolds are $\eta$-Einstein manifolds.
In Chapter 4, we study generalized Sasakian-space-forms. Alegre et al. [1] initiated the study of generalized Sasakian-space-forms and presented some examples. In [2], the authors studied the structure of generalized Sasakian-space-forms and proved that a $K$-contact generalized Sasakian-space-form is Sasakian and if its dimension is greater than 5 then it is a Sasakian-space-form. Further the authors proved that any three dimensional trans-Sasakian manifold with $\alpha$ and $\beta$ depending only on the direction of $\xi$ is a generalized Sasakian-space-form. The authors De and Sarkar [41, 33] studied curvature properties of generalized Sasakian-space-forms and obtained important results.

In [1], the authors introduced the notion of generalized Sasakian-space-form $M(f_1, f_2, f_3)$ as an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ whose Riemannian curvature tensor $R$ satisfies

$$R(X, Y)Z = f_1[g(Y, Z)X - g(X, Z)Y] + f_2[\phi(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z] + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi],$$

where $f_1, f_2$ and $f_3$ are differentiable functions on $M$.

We also study $K$-torse-forming vector fields in trans-Sasakian generalized Sasakian-space-forms $M(f_1, f_2, f_3)$. Torse forming vector fields were introduced by Yano [117] in 1944 and complex analogue of a torse-forming vector field was introduced by Yamaguchi [116] in 1979. This vector field is called $K$-torse forming vector field.

We discuss preliminaries and objectives of this chapter in the first two sections. In the next section 4.3, we study generalized Sasakian-space-form
satisfying generalized recurrent and generalized Ricci recurrent conditions. These conditions are defined as follows:

\[(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + (n - 1)B(X)g(Y, Z),\]

where \(A\) and \(B\) are two 1-forms and \(B\) is non-zero.

Here we prove that in a generalized recurrent \(M(f_1, f_2, f_3)\), \(f_1 \neq f_3\) holds. Further we prove that the 1-forms \(A(X)\) and \(B(X)\) are related by \(B(X) = c A(X)\), where \(c = f_1 - f_3\) and a generalized recurrent \(M(f_1, f_2, f_3)\) is always a co-symplectic manifold.

In Section 4.4, we study generalized Sasakian-space-forms satisfying \(\phi\)-Ricci recurrent [21],[29] conditions. A generalized recurrent \(M(f_1, f_2, f_3)\) is called generalized \(\phi\)-Ricci recurrent if \(\phi^2((\nabla_X Q)(Y)) = A(X)QY + (n - 1)B(X)Y\), where \(Q\) is the Ricci operator, \(A(X)\) and \(B(X)\) are non-zero 1-forms.

Here we prove that in an \(\alpha\)-Sasakian (or an \(\beta\)-Kenmotsu) generalized Sasakian-space-form which is \(\phi\)-Ricci recurrent the relation \((n - 2)f_3 - 3f_2 = 0\) holds. Also we show that an \(\alpha\)-Sasakian (or an \(\beta\)-Kenmotsu) generalized Sasakian-space-form with constant \(\zeta\)-sectional curvature is \(\phi\)-Ricci-symmetric if and only if \((n - 2)f_3 + 3f_2 = 0\).

In Section 4.5, we study \(\phi\)-concircular recurrent \(M(f_1, f_2, f_3)\). The concircular curvature tensor \(\bar{C}\) [119, 118] in \(M(f_1, f_2, f_3)\) is given by:

\[
\bar{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n - 1)}[g(Y, Z)X - g(X, Z)Y].
\]
Definition: $M(f_1, f_2, f_3)$ is said to be $\phi$-concircular recurrent if

$$\phi^2((\nabla_W \tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z,$$

for arbitrary vector fields $X, Y, Z, W$ on $M$. If a nonzero 1-form $A(W)$ vanishes identically, then the manifold will be called a $\phi$-symmetric manifold.

In this section, we prove that in a $\phi$-concircular recurrent $M(f_1, f_2, f_3)$, $f_1$ is a constant if and only if $(n-2)f_3 + 3f_2 = 0$. Further the scalar curvature $r = n(n-1)(f_1 - f_3)$. Also we prove that $f_1$ is constant in a $\phi$-concircular symmetric $M(f_1, f_2, f_3)$. In Section 4.6, we study $\phi$-conharmonically recurrent manifolds. The conharmonic curvature tensor is given by [95]

$$N(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$

Here we show that in a $\phi$-conharmonically recurrent $M(f_1, f_2, f_3)$, $f_1$ and $3f_2 - f_3$ are constants if and only if $nf_1 + 6f_2 - 2f_3 = 0$.

In Section 4.7, we study pseudo-projectively flat generalized Sasakian-space-form $M(f_1, f_2, f_3)$ and we show that an $n$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is pseudo-projectively flat if and only if $a + (n-1)b \neq 0$, where $a \neq 0, b \neq 0$ and $3f_2 + (n-2)f_3 = 0$.

In Section 4.8, we study pseudo-projectively semi-symmetric generalized Sasakian-space-form and we prove that an $n$-dimensional generalized Sasakian-space-form is pseudo-projectively semi-symmetric if and only if the space form is either pseudo-projectively flat or $f_1 = f_3$. 

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In section 4.9, we study the $K$-torse-forming vector fields in a generalized Sasakian-space-form.

A vector field $\rho$ defined by $g(X, \rho) = \omega(X)$ for any vector field $X$, is said to be a $K$-torse-forming vector field if

$$(\nabla_X \omega) Z = a g(X, Z) + b g(\phi X, Z) + B(X) \omega(Z) - D(X) \omega(\phi Z),$$

where $g(X, \rho) = \omega(X)$, $a$ and $b$ are functions and $B(X)$ and $D(X)$ are 1-forms. In this section we prove that if in a generalized Sasakian-space-form a torse-forming vector field $\rho$ is orthogonal to $\xi$ then $f_1 = f_3$. Also in a non co-symplectic trans-Sasakian manifold of dimension $n \geq 5$ the $K$-torse-forming vector field $\xi$ is proper. Further in a trans-Sasakian generalized Sasakian-space-form of dimension 5 or more with $f_1 \neq f_3$, the Ricci tensor $S$ is semi-conjugated with the $K$-torseforming vector field $\xi$ if and only if $3f_2 + (n - 2)f_3 = 0$. In this section we also show that in a $(0, \beta)$-trans-Sasakian generalized Sasakian-space-form of dimension $\geq 5$, the Ricci tensor $S$ is semi-conjugated with the $K$-torse-forming vector field $\xi$ if and only if $3f_2 + (n - 2)f_3 = 0$.

In Section 4.10, we study infinitesimal contact transformation. A vector field $V$ on a contact manifold with contact form $\eta$ is said to be an infinitesimal contact transformation [93] if $V$ satisfies $(L_V \eta) X = \sigma \eta(X)$ for a scalar function $\sigma$ where $L_V$ denotes the Lie differentiation with respect to $V$. Especially if $\sigma$ vanishes identically, then it is called an infinitesimal strict contact transformation. In this section, we show that in a trans-Sasakian generalized Sasakian-space-form, if $\xi$ is a $K$-torse-forming vector field with $a$ and $b$ as constants, then the infinitesimal contact transformation which leaves the Ricci tensor invariant is an infinitesimal strict contact transformation.
In Chapter 5, we study $\tau$-curvature tensor in $(k, \mu)$-contact manifolds and in generalized Sasakian-space-forms. Tripathi et al. [102] introduced the $\tau$-curvature tensor which in particular cases reduces to known curvatures like conformal, concircular and projective curvature tensors and some recently introduced curvature tensors like $M$-projective curvature tensor, $W_i$-curvature tensor($i = 0, \ldots, 9$) and $W_j^r$-curvature tensors($j = 0, 1$). Tripathi et al. [104, 105] studied $\tau$-curvature tensor in $K$-contact, Sasakian and Semi-Riemannian manifolds.

In an $n$-dimensional Riemannian manifold $M$, the $\tau$-curvature tensor is given by [102]

$$\tau(X, Y)Z = a_0 R(X, Y)Z + a_1 S(Y, Z)X + a_2 S(X, Z)Y + a_3 S(X, Y)Z + a_4 g(Y, Z)QX + a_5 g(X, Z)QY + a_6 g(X, Y)QZ + a_7 \tau(g(Y, Z)X - g(X, Z)Y),$$

where $a_0, \ldots, a_7$ are some smooth functions on $M$.

Blair et al. [8] studied the $(k, \mu)$-nullity conditions on a contact metric manifold and gave several examples. The study of $(k, \mu)$-contact manifolds is interesting as it contains both Sasakian and non-Sasakian manifolds. Boeckx[13] gave a complete classification of $(k, \mu)$-contact manifolds. The authors De et al.[30, 57, 52] studied $(k, \mu)$-contact manifolds with $\phi$-recurrency and $\phi$-symmetry conditions.

After briefly discussing the preliminaries and objectives of this chapter we show, in section 5.3, that a $\tau$-flat $(k, \mu)$-contact manifold is $\phi$-symmetric either for a manifold of constant curvature or $a_7 = 0$. In Section 5.4, we consider $\phi$-$\tau$-symmetric $(k, \mu)$-contact manifolds and prove the conditions for a $(k, \mu)$-manifold to be $\phi$-$\tau$-symmetric or $\phi$-symmetric. Further we also study $\phi$-$\tau$-Ricci recurrent $(k, \mu)$-manifold and obtain a condition for these manifolds to be
Einstein and \( \eta \)-Einstein. In Section 5.5, we study \( \xi - \tau \)-flat \((k, \mu)\) manifold and obtain conditions for these manifolds to be \( \eta \)-Einstein. Further we obtain expression for scalar curvature for \(a_4 = 0\) and \(a_7 \neq 0\). In section 5.6 we study \((k, \mu)\)-contact manifold with semi-symmetric condition \(\tau S = 0\) and it is shown that under such condition manifold is reduced to \(\eta\)-Einstein manifold provided \(a_3 \neq 0\). In the last section, we study \(\tau\)-flat generalized Sasakian-space-form. Further we study \(\tau - \phi\)-semi-symmetric generalized Sasakian-space-form and show that a \(\tau - \phi\)-semi-symmetric generalized Sasakian-space-form is \(\eta\)-Einstein provided \((2a_1 + na_2) \neq 0\).

**Chapter 6** deals about \((N(k), \xi)\)-semi-Riemannian 3-manifolds. The notion of \((N(k), \xi)\)-semi-Riemannian structure was introduced and studied by Tripathi and Gupta [105] to unify \(N(k)\)-contact metric, Sasakian, \((\epsilon)\)-Sasakian, Kenmotsu, para-Sasakian, \((\epsilon)\)-para-Sasakian structures.

The first two sections of this chapter brief out preliminaries and objectives of the chapter. In subsequent section 6.3, we show that \((N(k), \xi)\)-semi-Riemannian 3-manifold is a space form if and only if the scalar curvature \(r\) of the manifold is equal to \(6k\). In Section 6.4, we establish that a Ricci -semi-symmetric \((N(k), \xi)\)-semi-Riemannian 3-manifold is a space form. In Section 6.5, a necessary and sufficient condition for an \((N(k), \xi)\) semi-Riemannian 3-manifold to be locally \(\phi\)-symmetric is obtained. In Section 6.6, we study \((N(k), \xi)\)-semi-Riemannian 3-manifold satisfying \(\eta\)-parallel condition. In this section, we show that the scalar curvature \(r\) is constant in an \((N(k), \xi)\)-semi-Riemannian 3-manifold with \(\eta\)-parallel Ricci tensor. Further we show that an \((N(k), \xi)\)-semi-Riemannian 3-manifold with \(\eta\)-parallel Ricci tensor is locally \(\phi\)-symmetric.

The thesis ends with Bibliography.