CHAPTER – I

HISTORICAL SURVEY

1.1 Special functions include functions as well as orthogonal and non-orthogonal polynomials, such as, Hypergeometric functions, Meijir’s G-functions, Lame functions, Cylindrical functions, Matheiu functions, Wave functions, Legendre polynomials, Hermite polynomials etc. These functions and polynomials have been studied for long, but a systematic and proper study of the development of the subject has been done during the last decade only. This is one of the most important branch of the Mathematics for Scientists and Engineers. For example, orthogonal system of functions play an important role in analysis, mainly because functions pertaining to very general classes can be expanded in a series of orthogonal functions e.g. Fourier series, Fourier Bessel series etc.

Also orthogonal polynomials are of great importance in approximation theory, mathematical physics and theory of mechanical quadratures etc., For instance, Laguerre polynomials are encountered in problems involving the integration of Helm-Holtz equations in parabolic co-ordinates, the theory of Hydrogen atoms, the theory of propagation of electromagnetic waves along transmission lines etc.

During the last decade, the development of larger computers has made it possible to study functions with multiple series representations from numerical point of view. By far, the most important functions of this type are hypergeometric in character. In particular, since they occur in connection with such matters as statistical
distributions, functional equations and their characterizations, quantum theory, vibration of beams, conduction of heat, elasticity, telecommunications etc., as well as in agricultural and biological sciences. It is thus evident that a detailed study of the analytical behavior of such functions and polynomials will be of great importance in unifying and generalizing the methods of dealing a wide variety of problems in Applied Mathematics. Intensive study on unification and generalization of these multiple hypergeometric functions was made and is being studied by the Mathematicians in international fields.

Eminent mathematicians and physicists have written numerous treatises and text books in this field. Some of them are listed below:


6. Exton H. : Multiple Hypergeometric Functions and Applications, John Wiley and


17. Srivastava H. M. and Buschman R. G : Convolution Integral Equations with


20. Srivastava H. M. and Manocha H. L. : A Treatise on Generating Functions,


Univ. 1945.

Remark: A complete bibliography is given at the end of the thesis.

1.2 In this section, the definitions of the different functions and polynomials
defined by different researchers are given.

1. HUMBERT’S FUNCTIONS

Humbert (1920) has defined and discussed many functions as given below:
\[
\phi_1(\alpha, \beta; \gamma; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha + n)(\beta + m)}{(\gamma + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < 1, |y| < \infty
\] --- (1.2.1)

\[
\phi_2(\beta, \beta' \gamma; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\beta + m)(\beta' + n)}{(\gamma + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < \infty, |y| < \infty
\] --- (1.2.2)

\[
\phi_3(\beta; \gamma; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\beta + m)}{(\gamma + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < \infty, |y| < \infty
\] --- (1.2.3)

\[
\psi_1(\alpha, \beta; \gamma', x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha + m + n)(\beta' + m)}{(\gamma' + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < \infty, |y| < \infty
\] --- (1.2.4)

\[
\psi_2(\alpha; \gamma', x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha + m)}{(\gamma' + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < \infty, |y| < \infty
\] --- (1.2.5)

\[
E_1(\alpha, \alpha', \beta; \gamma; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha + m)(\alpha' + n)(\beta + m)}{(\gamma + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < 1, |y| < \infty
\] --- (1.2.6)

\[
E_2(\alpha, \beta; \gamma'; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha)(\beta + m)}{(\gamma' + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < 1, |y| < \infty
\] --- (1.2.7)

II. APPELL’S FUNCTIONS OF TWO VARIABLES

The Appell defined the four functions namely \(F_1, F_2, F_3\) and \(F_4\) as:

\[
F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha + m + n)(\beta + m)(\beta' + n)}{(\gamma + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < 1, |y| < 1
\] --- (1.2.8)

\[
F_2(\alpha; \beta, \beta'; \gamma'; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha + m + n)(\beta + m)(\beta' + n)}{(\gamma' + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| + |y| < 1
\] --- (1.2.9)

\[
F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha + m)(\alpha' + n)(\beta + m)(\beta' + n)}{(\gamma + m + n)} \frac{x^m y^n}{m! n!} \cdot |x| < 1, |y| < 1
\] --- (1.2.10)
\[ F_4(\alpha; \beta; \gamma; \gamma'; x, y) = \sum_{m, n = 0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_{m}(\gamma')_{n}} \frac{x^m y^n}{m! n!}, |x| + |y| < 1 \]  

--- (1.2.11)

### III. KAMPE’ DE FERIET FUNCTION

Kampe de Feriet [88] has generalized Appell’s functions and defined a function of two variables of order \( n \) as:

\[
F_{p,q}^{r,s}
\left[
\begin{array}{c}
(\alpha)_p : (\beta)_q : (\beta_1^1) ; \\
(\gamma)_p : (\delta)_q : (\delta_1^1) ;
\end{array}
\right]_{x, y} = \sum_{m, n = 0}^{\infty} \frac{(\alpha)_p m+n (\beta)_q m (\beta_1^1)_n}{(\gamma)_p m+n (\delta)_q m (\delta_1^1)_n} \frac{x^m y^n}{m! n!}.
\]  

--- (1.2.12)

where \((a_p)\) abbreviates the sequence of parameters \( a_1, a_2, \ldots, a_p \).

The series is absolutely convergent under the following set of conditions:

(i) \( p + q < r + s + 1, \) for all \( x \) and \( y \)

(ii) \( p + q = r + s + 1, \) \( p > r, |x|^{1/p-r} + |y|^{1/p-r} = 1 \)

(iii) \( p + q = r + s + 1, \) \( p \leq r, \) for \( |x| < 1, |y| < 1 \)

### IV. SARAN’S HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

In the year 1954, Saran [165] has defined the hypergeometric functions of three variables, which are given below:

\[
F_A(\alpha; \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \sum_{m, n, p = 0}^{\infty} \frac{(\alpha)_{m+n+p} (\beta_1)_m (\beta_2)_n (\beta_3)_p}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p} \frac{x^m y^n z^p}{m! n! p!}.
\]  

--- (1.2.13)
\[ F_B(a_1, a_2, a_3; \beta_1, \beta_2, \beta_3; x, y, z) \]
\[ = \sum_{m, n, p=0}^{\infty} \frac{a_1}{m} \frac{a_2}{n} \frac{a_3}{p} \frac{\beta_1}{m+n+p} \frac{\beta_2}{m+n+p} \frac{\beta_3}{m+n+p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \]  
\[ \text{--- (1.2.14)} \]

\[ F_C(\alpha; \beta; \gamma_1, \gamma_2, \gamma_3; x, y, z) \]
\[ = \sum_{m, n, p=0}^{\infty} \frac{\alpha}{m+n+p} \frac{\beta}{m+n+p} \frac{\gamma_1}{m} \frac{\gamma_2}{n} \frac{\gamma_3}{p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \]  
\[ \text{--- (1.2.15)} \]

V. THE GENERAL TRIPLE HYPERGEOMETRIC SERIES

In the year 1967, Srivastava [185] has introduced a general triple

hypergeometric series \( F^3(x,y,z) \) as

\[ F^3(x,y,z) = F^3 \left[ \begin{array}{c}
(a): (b); (b^1); (b^1); (c); (c^1); (c^1); \\
(e): (g); (g^1); (g^1); (h); (h^1); (h^1); \\
\end{array} \right] \]
\[ = \sum_{m, n, p=0}^{\infty} \frac{\wedge(m,n,p)}{m!} \frac{x^m}{n!} \frac{y^n}{p!} \]  
\[ \text{--- (1.2.16)} \]

where \( \wedge(m,n,p) = \)

\[ \begin{array}{cccccccc}
A & \prod_{j=1}^{m+n+p} (a_j) & B & \prod_{j=1}^{m+n+p} (b_j) & B^1 & \prod_{j=1}^{m+n+p} (b^1_j) & B^{11} & \prod_{j=1}^{m+n+p} (b^{11}_j) \\
B & \prod_{j=1}^{m+n+p} (c_j) & C & \prod_{j=1}^{m+n+p} (c^1_j) & C^1 & \prod_{j=1}^{m+n+p} (c^1_j) & C^{11} & \prod_{j=1}^{m+n+p} (c^{11}_j) \\
E & \prod_{j=1}^{m+n+p} (e_j) & G & \prod_{j=1}^{m+n+p} (g_j) & G^1 & \prod_{j=1}^{m+n+p} (g^1_j) & G^{11} & \prod_{j=1}^{m+n+p} (g^{11}_j) \\
\end{array} \]
VI. LAURICELLA’S HYPERGEOMETRIC FUNCTIONS OF n VARIABLES

While several authors, for example, Hermite (1865) and Didon (1870) have discussed what amounts to certain specialized multiple hypergeometric functions, it was left to Lauricella (1893) to approach this topic systematically. Beginning with Appell function, Lauricella proceeded to define and study the four important functions, which bear his name. These functions have the following multiple series representations:

\[
F_A^{(n)}(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n) \equiv \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \ldots (b_n)_{m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c_1)_{m_1 + \ldots + m_n} m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n} \frac{a}{m_1! \ldots m_n!}
\]

where \(|x_1| + \ldots + |x_n| < 1\)

\[
F_B^{(n)}(a_1, \ldots, a_n; b_1, \ldots, b_n; c; x_1, \ldots, x_n) \equiv \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a_1)_{m_1} \ldots (a_n)_{m_n} (b_1)_{m_1} \ldots (b_n)_{m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c)_{m_1 + \ldots + m_n} m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n} \frac{a_1}{m_1! \ldots m_n!}
\]

where \(\max\{|x_1|, \ldots, |x_n|\} < 1\)

\[
F_C^{(n)}(a, b; c_1, \ldots, c_n; x_1, \ldots, x_n) \equiv \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b)_{m_1 + \ldots + m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n} m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n} \frac{a}{m_1! \ldots m_n!}
\]

where \(\sqrt{|x_1| + \ldots + \sqrt{|x_n|}} < 1\).
\begin{align*}
F_D^{(n)}(a,b_1,\ldots,b_n; c; x_1,\ldots,x_n) &= \sum_{m_1,\ldots,m_n = 0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c)_{m_1+\ldots+m_n} m_1! \ldots m_n !} \\
\text{where } \max \{|x_1|, \ldots, |x_n| < 1 \}
\end{align*}

--- (1.2.20)

**VII. CONFLUENT FORMS OF LAURICELLA FUNCTIONS**

Two important confluent hypergeometric functions of \( n \) variables are

\begin{align*}
\phi_2^{(n)}(b_1,\ldots,b_n; c; x_1,\ldots,x_n) &= \sum_{m_1,\ldots,m_n = 0}^{\infty} \frac{(b_1)_{m_1} \cdots (b_n)_{m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c)_{m_1+\ldots+m_n} m_1! \ldots m_n !} \\
\psi_2^{(n)}(a; c_1,\ldots,c_n; x_1,\ldots,x_n) &= \sum_{m_1,\ldots,m_n = 0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n} m_1! \ldots m_n !} \\
\phi_2^{(n)}(b_1,\ldots,b_n; c; x_1,\ldots,x_n) &= \sum_{m_1,\ldots,m_n = 0}^{\infty} \frac{(b_1)_{m_1} \cdots (b_n)_{m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c)_{m_1+\ldots+m_n} m_1! \ldots m_n !} \\
\psi_2^{(n)}(a; c_1,\ldots,c_n; x_1,\ldots,x_n) &= \sum_{m_1,\ldots,m_n = 0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n} m_1! \ldots m_n !} \\
\text{--- (1.2.21)}
\end{align*}

**VIII. GENERALIZED LAURICELLA FUNCTION**

In 1969, Srivastava, H. M. and Doust, M. C. [192] introduced the generalized Lauricella function as follows:

\begin{align*}
F_{A,B}^{\phi}: & \sum_{r_1 = 0}^{\infty} \sum_{r_2 = 0}^{\infty} \cdots \sum_{r_m = 0}^{\infty} \frac{(a_A)_{r_1 r_2 \ldots r_m} [b_1]_{r_1} [b_2]_{r_2} \cdots [b_m]_{r_m} x_1^{r_1} \ldots x_m^{r_m}}{(\phi_1)_{r_1} [\psi_1]_{r_1} [\psi_2]_{r_2} \cdots [\psi_m]_{r_m} r_1! \ldots r_m !}
\text{--- (1.2.23)}
\end{align*}
which for particular values of $A, B^{(m)}, C, D^{(m)}, \theta, \phi, \psi$ and $\delta$ reduces to $F_A, F_B, F_C$ and $F_D$ respectively. For example if we take $C = 0, \theta_1 = \theta_2 = \ldots = \theta_m = 1 = \phi_1 = \phi_2 = \ldots = \phi_m = \psi_1 = \psi_2 = \ldots = \psi_m = \delta_1 = \delta_2 = \ldots = \delta_m$, then above function reduces to Lauricella function of first kind i.e. $F_A$.

**Deduction**: In particular, by taking $m = 2$, Srivastava and Panda [206] have given a slightly modified notion as

$$
E_{p; q; k}^{l; m; n} \left[ \begin{array}{c}
(a_p): (b_q); (c_k); \\
(\alpha_l): (\beta_m); (\gamma_n)
\end{array} \right]_{x, y} = \sum_{r, l = 0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+1} \prod_{j=1}^{q} (b_j)_{r+1} \prod_{j=1}^{k} (c_j)_{s} x^r y^s}{r! s!} \quad \text{--- (1.2.24)}
$$

### 1.3 LATEST RESEARCH WORK DONE BY DIFFERENT RESEARCHERS

A lot of work has been done by the different researchers on the topic of special functions by considering the different aspects of their definitions. Due to lack of space, it is not possible to list all of them here. But we shall try to give some of the important generalizations below:

1. Srivastava H. M., Lavoie J. L. and Tremblay R. [204] (1979) have defined the sequence

$$
f_n(x) = \frac{1}{n!} D^n_x \left\{ (ax+b)^n F(x) \right\},
$$

such that, for every sequence, the following generating relation holds:

$$
\sum_{n=0}^{\infty} \binom{m+n}{n} f_{n+m} (x)^n = (1-at)^{-m-1} f_{m} \left( \frac{x+bt}{1-at} \right). \quad \text{--- (1.3.1)}
$$

The above result is applicable not only to the Bessel polynomials, the classical
orthogonal polynomials and to their various generalizations studied in recent years,  
buts also to such other generalizations, as the Bessel functions and a certain class of  
generalized hypergeometric function.

2. In 1979, Srivastava H. M. [187] generalized the Carlitz theorem [31] and gave its  
applications. He defined the sequence by means of the generating relation

\[ A(z)B(z) = \sum_{n=0}^{\infty} f_n(x) \frac{z^n}{n!}, \]

--- (1.3.2)

3. Parihar, C. L. and Patel [154] (1979) have given modified Jacobi polynomial  
defined by using a difference operator as

\[ f_n(x,w) = \frac{-x}{n!(x-w)} \Delta^n x, w \left[ (x-w)[(\alpha+n)w](1-w)^w \right], \]

where \( \Delta_{x,w}f(x) = \frac{f(x+w) - f(x)}{w} \)

The hypergeometric form of the above polynomial is

\[ f_n(x,w) = \frac{(1+\alpha)_n}{n!} \left\{ \begin{array}{c} x \\ -n, \frac{x}{w}; 1+\alpha, w \end{array} \right\} \]

--- (1.3.3)

When \( w \to 0 \), the form reduces to \( L_n^{(\alpha)}(x) \).

4. Srivastava A. N. and Singh S. N. [181] (1979) have given an extension of several  
bilateral generating relations established earlier by W. A. Alsalam [7] and H. M.  
Srivastava [187], S. K. Chatterjea [40] and others, in the form of mixed trilateral  
generating relations:
\[ \sum_{n=0}^{\infty} S_{n+m}(x)\sigma_{n}^{q}(y,z) t^{n} = \frac{f(x,t)}{[g(x,t)]^{m}} F_{[q]} \left[ h(x,t), y, z \left[ \frac{t}{g} \right]^{q} \right] \]  \hspace{1cm} (1.3.4)

where \[ \sigma_{n}^{q}(y,z) = \sum_{n=0}^{q} a_{n} A_{n-qk} g_{k}(y) z^{k} \]  \hspace{1cm} (1.3.5)

and \[ F_{[q]}[x,y,z] = \sum_{n=0}^{\infty} a_{n} S_{nq+m}(x) g_{n}(y) t^{n} \]  \hspace{1cm} (1.3.6)

5. Furthermore, H. M. Srivastava and M. A. pathan [208], [209], (1979), (1980) made efforts to define a new class of bilateral generating functions, involving multiple series with arbitrary terms for certain general polynomial systems, which include, as their special, such hypergeometric polynomials as the classical Jacobi, Laguerre, Hermite and their various known generalizations. These are given as follows:

\[ \sum_{n=0}^{\infty} \frac{(-x)^{n}}{n!(\lambda+n)} \sum_{k=0}^{n} (-n)_{NK} (\lambda+n)_{NK} C_{k}^{w} \sum_{l,m=0}^{\infty} \frac{A_{l+m+n} B_{n+l} D_{m} x^{l} y^{m}}{(\lambda+2n+1)_{l} l! m!} \]  \hspace{1cm} (1.3.7)

for any integer \( N \geq 1 \) and sequence \( \{A_{n}\}, \{B_{n}\}, \{C_{n}\} \) and \( \{D_{n}\} \) are arbitrary complex numbers.

6. Agrawal, B. D. and Prasad, J. [5] in (1980), have defined and studied the general sequence of functions \( \left\{ R_{n; \lambda, k}^{(\alpha, \beta)}(x) \right\}, n = 0, 1, 2, \ldots \) defined by means of the generalized Rodrigue’s formula.

\[ R_{n; \lambda, k}^{(\alpha, \beta)}[x; a, b, c, d; p, q, r, s; w(x)] \]
\[ (ax^p + b)^{-\alpha} (cx^q + d)^{-\beta} k_n \frac{w(x)}{w(x)} T^n_{k; \lambda} \left[ (ax^p + b)^{\alpha + \gamma} (cx^q + d)^{\beta + \xi} w(x) \right], \quad \text{(1.3.8)} \]

where \( T^n_{k; \lambda} = x^k (\lambda + xD); D = \frac{d}{dx} \), \( n = 0, 1, 2, \ldots \) under suitable conditions.

7. Srivastava A. N. and Singh S. N. [181] (1979) and Singh S. N. and Rai P. N. [177] unified the polynomials given by Crombez [50], Karande and Thakare [92] etc., by means of generating relations

\[ e^{3 \left( \frac{t^3}{c^2} \right)} \frac{\exp(xt + yt)^2}{(\exp(t) - a)(\exp(t^2) - b)} = \sum_{n=0}^{\infty} A_n(x, y; a, b, c, k) \frac{t^n}{n!} \quad \text{(1.3.9)} \]

and

\[ \frac{4(e^{3 / 4})^k \exp(xt + yt)^2}{(\exp(t) - a)(\exp(t^2) - b)} = \sum_{n=0}^{\infty} P_n(x, y; a, b, k) \frac{t^n}{k!} \quad \text{(1.3.10)} \]

8. Prabhakar T. R. and Rekha [158] studied the sequence \( \{R_n^{\alpha}(x; a, r, b)\} \) by means of generating relation

\[ \sum_{n=0}^{\infty} R_n^{(\alpha)}(x; a, r, b) \frac{t^n}{n!} \left[ \frac{at^n}{(\exp(t) - b - 1)} \right]^\alpha \exp(xt) \quad \text{(1.3.11)} \]

9. Alpana Bhatnagar [22] (1982) has defined the polynomial set as

\[ f^n_N \left[ \alpha, (\alpha_\lambda), (\beta_{\mu}), (\beta_{\mu}^1); (\gamma_{\rho}), (\delta_{\nu}}, x, y \right] \]

\[ \frac{a_n}{n!} f_{\lambda}^{\mu} : N + \mu ; \mu \frac{1}{\rho} : \nu^1 \left[ \left( \alpha_\lambda \right); (\Delta(N; -n)), (\beta_{\mu}); (\beta_{\mu}^1); \right] x, y \right] \]

\[ \left( \gamma_{\rho} \right); (\delta_{\nu}); \left( \delta_{\nu}^1 \right), \quad \text{(1.3.12)} \]

where \( a \neq 0 \).
10. In 1981, further generalization of the polynomials defined by Crombez [50], Singh, S. N. and Rai P. N. [177] etc. has been done by Agrawal, B. D. and Prasad, J. [5] as

\[
\frac{m^l+m^l+m^l}{(1+wt)^{m^l}w} = \sum_{n=0}^{\infty} B_{n;h,w}^{l,m,m}(x,y) t^n/n!
\]  

(1.3.13)

11. Khanna I. K. and Preeti Pandey [99], (1982), have defined a polynomial set by means of a generating relation

\[
\sum_{n=0}^{\infty} P_n(x,y) t^n/n! = \frac{2(t/2)^k (1-xt)^{-\alpha}}{(1-yt)^{-\beta-a}}
\]  

(1.3.14)

where \(\alpha, \beta\) are real numbers and \(k\) is a non-negative integer.

12. Recently in the year 1982, Agarwal A.K. and Manocha H.L. [2] established the following extended linear generating relations for Gottlieb Polynomials, Meixner Polynomials, Cesaro Polynomials and Generalized Sylvester Polynomials as follows:

i) \(\sum_{n=0}^{\infty} \frac{m_{n+k}(x;\lambda)}{k!} t^n = (1-t)^{-k} \left(1-t e^{-\lambda}\right)^{-x} \phi(x;\log_e \frac{e^\lambda-t}{1-t})
\]  

(1.3.15)

where \(\phi_n(x;\lambda) = e^{-n\lambda} \frac{\Gamma(n)}{\Gamma(1)} \left(-n,-x;1-e^\lambda\right)
\]  

(1.3.16)

is the Gottlieb Polynomial set introduced by Gottlieb, M.J. in the year 1938 [p.303,161].

ii) \(\sum_{n=0}^{\infty} \frac{m_{n+k}(x;\beta,c)}{n!} t^n = \left(1-t/c\right)^x \left(1-t(1-c)\right)^{-x-\beta-k} m(x;\beta;(c-t)(1-t)^{-1})
\]  

(1.3.17)

where \(m_n(x;\lambda,\mu) = \frac{\Gamma(n)}{\Gamma(1)} \left(-n,-x;\lambda;1-\frac{1}{\mu}\right)
\]  

(1.3.18)
is the Meixner Polynomials[p.75,205].

\[ \sum_{n=0}^{\infty} \binom{m+n}{n} g_{m+n}^{(s)}(x) t^n = (1-t)^{-s-m-1} (1-xt)^{-t} g_{m}^{(s)} \left( \frac{x-xt}{1-xt} \right) \quad \text{--- (1.3.19)} \]

where ‘m’ being a non-negative integer and \( g_{n}^{(m)}(x) = \binom{m+n}{n} 2F_1 \left( \frac{-n,1}{-m-n};x \right) \)

is the Cesaro Polynomials[p.449,205]

\[ \sum_{n=0}^{\infty} \binom{n+k}{k} f_{n+k}(x;a) t^n = (1-t)^{-x-k} e^{ax} f_k(x;a(1-t)) \quad \text{--- (1.3.21)} \]

where \( f_n(x;a) = \frac{(ax)^n}{n!} 2F_0 \left( -n, x; -\frac{1}{ax} \right) \)

is the Generalized Sylvester Polynomial[2].

13. Recently in 1983, Agrawal, H. C. [6] defined the polynomial set \( \{f_n^{(\alpha)}(x)\} \) satisfying the function relation

\[ T(\Delta_{\alpha}) \{f_n^{(\alpha)}(x)\} = f_n^{(\alpha+1)}(x), \quad (n = 1, 2, \ldots) \quad \text{--- (1.3.23)} \]

\( f_n^{(\alpha)}(x) \) is the polynomial of order \( n \) in \( x \) and \( T \) is the operator of infinite order defined by

\[ T(\Delta_{\alpha}) = \sum_{k=0}^{\infty} n_k^{(\alpha)} \Delta_{\alpha}^k + n_{0}^{(\alpha)} \neq 0 \quad \text{--- (1.3.24)} \]

in which \( \Delta_{\alpha} \{ f(\alpha) \} = f(\alpha+1) - f(\alpha) \)

14. Khanna I. K. and Yadav, K. [100] (1986) have defined the generalized polynomial set \( \{R_n^A(x_m,y)\}, \quad n = 0, 1, 2, \ldots \) by means of the generating relation
\[
\sum_{n=0}^{\infty} \binom{n}{\gamma} \binom{\beta}{\mu} \binom{\alpha}{b} \binom{\delta}{c} \binom{\epsilon}{d} \binom{\sigma}{e} \binom{\theta}{f} \binom{\rho}{g} \binom{\omega}{h} \binom{\lambda}{i} 
\]

where

i) \((x_m), y\) and \(t\) are any finite numbers.

ii) \(\alpha, n\) and \(r\) are non-negative integers.

iii) \(\nu, (\mu_m), (\beta_m), (b_m)\) and \(a\) are any finite real numbers.

iv) \((c_m)\) are any real numbers which are neither zeros nor negative integers. and

v) \(m\) is a natural number.

15. Bavanari Satyanarayana [166], (1993), has defined two sets of generalized hypergeometric functions

\[\{I_{\alpha,\mu;\lambda;\beta}^{\mu;\lambda;\beta}(x, w)\}, n = 0, 1, 2, \ldots\] and

\[\{H_{\alpha,\mu;\lambda;\beta}^{\mu;\lambda;\beta}(x, w)\}, n = 0, 1, 2, \ldots\] by a difference operator formula as:

\[I_{\alpha,\mu;\lambda;\beta}^{\mu;\lambda;\beta}(x, w) = \frac{1}{n!(x - \mu w)^{[\alpha w]}} \Delta_{x,w}^n [(x - \mu w)^{[\alpha w]}]_{\mu+1} F_q ((a_{\mu}), -\frac{x}{w} + \lambda; (b_{\mu}); w)\]

--- (1.3.26)

and \(H_{\alpha,\mu;\lambda;\beta}^{\mu;\lambda;\beta}(x, w)\)

\[-\frac{1}{n!(x - \mu w)^{[\alpha w]}} \Delta_{x,w}^n [(x - \mu w)^{[\alpha w]}]_{\mu+1} F_q ((a_{\mu}), -\frac{x}{w} + \lambda; (b_{\mu}); w)\]

--- (1.3.27)

The hypergeometric forms of (1.3.18) and (1.3.19) are as follows:
\[ I_{n; \lambda; (b_q)}^{(a_p)}(x, w) = \frac{(1+\alpha)_n}{n!} F_{q;1;0}^{p;2;1} \left[ \begin{array}{c} (a_p) : -n, \frac{x}{w} - \mu + 1; -\frac{x}{w} + \lambda; w, w \\ (b_q) : 1 + \alpha \end{array} \right] \]  
--- (1.3.28)

and
\[ H_{n; \lambda; (b_q)}^{(a_p)}(x, w) = \frac{(-\alpha)_n w^{-2n}}{n! \left( \frac{x}{w} - \mu - \alpha + 1 \right)_n \left( -\frac{x}{w} + \mu + \alpha \right)_n} \times F_{q;1;0}^{p;2;1} \left[ \begin{array}{c} (a_p) : -n, \frac{x}{w} - \mu + 1; -\frac{x}{w} + \lambda; w, w \\ (b_q) : 1 + \alpha - n \end{array} \right] \]  
--- (1.3.29)

By means of the generating relations
\[ \sum_{n=0}^{\infty} \frac{[e_r]_n}{[(d_s)]_n} I_n^{(a_p)}(x, w) t^n = F^{(3)} \left[ \begin{array}{c} -:: (e_r), 1 + \alpha; (a_p); -:: -; -; \frac{x}{w} - \mu + 1; -\frac{x}{w} + \lambda; t, -wt, w \\ -:: (d_s); -; -; 1 + \alpha \end{array} \right] \]  
--- (1.3.30)

a class of extended linear generating relation for \( I_{n; \lambda; (b_q)}^{(a_p)}(x, w) \) has been derived as
\[ \sum_{n=0}^{\infty} \binom{n+m}{m} \frac{(\eta + m)_n}{(1+\alpha + m)_n} I_n^{(a_p)}(x, w) t^n \]
\[ = (1-t)^{-\eta-m} \binom{\alpha + m}{m} \left( \begin{array}{c} (a_p) : \frac{x}{w} - \mu + 1; -; -; \eta + m; -m; -\frac{x}{w} + \lambda; -wt \end{array} \right) F^{(3)} \left[ \begin{array}{c} (b_q) : 1 + \alpha \end{array} \right] \]  
--- (1.3.31)

For more generalizations, please see the following references:

1.4 KEYWORDS USED IN THE PRESENT THESIS

(i) Difference Quotients

“NORLUNDS” operator $\Delta_{x,w}$ which is defined by the relation

$$\Delta_{x,w} u(x) = \frac{u(x+w) - u(x)}{w}$$

It is called $\Delta_{x,w} u(x)$, which is evidently a divided difference, the first difference quotient of $u(x)$. The above result can be generalized as

$$\Delta^n_{x,w} u(x) = \Delta_{x,w}[\Delta^{n-1}_{x,w} u(x)]$$

From this we can prove the useful relations

1. $\Delta^n_{x,w} u(x+w) = w \Delta^n_{x,w} u(x) + \Delta^n_{x,w} u(x)$

2. $\lim_{w \to 0} \Delta^n_{x,w} u(x) = D^n u(x)$

(ii) Discrete Set

A set that has no accumulation points i.e. each point has a neighbourhood that contains no other points of the set.

(iii) Discrete Variable
A variable, whose possible values form a discrete set, is called a discrete variable.

**(iv) Generating Function**

The name “generating function” was introduced by Laplace, of which a brief account is available in Doetsch’s paper [56] published in 1937.

Let the function $F(x, t)$ have a formal power series expansion (which may not converge)

$$f(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n \quad \text{--- (1.4.1)}$$

The coefficient of $t^n$ in (1.4.1) is, in general, a function of $x$. It can be said that the expansion (1.4.1) of $F(x,t)$ has generated the set \{f_n(x)\} and that $F(x,t)$ is a generating function for \{f_n(x)\}. Also (1.4.1) is called as generating relation. If for some set of values of $x$, usually a region in the complex plane, the function $F(x,t)$ is analytic at $t = 0$, the series in (1.4.1) converge in same region around $t = 0$. However, convergence is not necessary for the relation (1.4.1) to define $f_n(x)$.

**(v) Bilinear and Bilateral Generating Relations**

Let $H(x,y,t)$ be expanded in power of $t$ in the form

$$H(x,y,t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n \quad \text{--- (1.4.2)}$$

where $h_n$ are independent of $x$ and $y$, $f_n(x)$ and $g_n(y)$ are different functions. Following the terminology used by Rainville [161, p.170(3)], we call $H(x,y,t)$ is a bilateral generating function and the equation (1.4.2) is a bilateral generating relation.
However, if a function \( G(x,y,t) \) can be expanded in the form

\[
G(x,y,t) = \sum_{n=0}^{\infty} g_n f_n(x) f_n(y) t^n \quad \text{--- (1.4.3)}
\]

where \( g_n \) are independent of \( x \) and \( y \), then \( G(x,y,t) \) is called a bilinear generating relation. Relation (1.4.2) and (1.4.3) are accordingly called the bilateral and bilinear generating relations respectively. A generating function may be used to define a set of functions to determine pure or differential recurrence relations, to evaluate certain integrals.

(vi) Linear Difference Equation

An expressed relation between an independent variable \( x \) and one or more dependent variables or functions \( f, g, \ldots \), and any successive differences of \( f, g, \) etc., as

\[ \Delta f(x) = f(x+h) - f(x), \quad \Delta^2 f(x) = \Delta^2 f(x+2h) - 2f(x+h)+f(x), \] etc. or equivalently, the results of any successive applications of the operator \( E \), where \( E f(x) = f(x+h) \).

(vii) Contiguous Function Relations

Gauss defined the contiguous function by increasing or decreasing the parameter in any function by unity. For example, contiguous functions for \( F(a,b;c;z) \) are:

\[ F(a+1,b;c;z); \ F(a-1,b;c;z); \ F(a,b-1;c;z); \ F(a,b;c+1;z); \ F(a,b;c-1;z) \] and it is denoted by \( F(a+); \ F(a-); \ F(b+); \ F(b-); \ F(c+); \ F(c-) \). Any relation involving \( F \) and any two or more contiguous functions is called a contiguous function relation.
(viii) Series Re-arrangement

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n - mk) \]  \hspace{1cm} --- (1.4.5)

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n + mk) \]  \hspace{1cm} --- (1.4.6)

1.5 NOTATIONS

i) \[ (a)_{n} = \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)\ldots(a+n-1) & \text{if } n = 1, 2, \ldots, a \neq 0 \end{cases} \]  \hspace{1cm} --- (1.5.1)

ii) \[ \binom{m}{n} = \frac{m!}{n!(m-n)!} \]  \hspace{1cm} --- (1.5.2)

\[ = \frac{(-1)^n \binom{-m}{n}}{n!}; \quad 0 \leq n \leq m \]  \hspace{1cm} --- (1.5.3)

iii) \[ (a_p) = a_1, a_2, \ldots, a_p \]  \hspace{1cm} --- (1.5.4)

iv) \[ [(a_p)] = a_1, a_2, \ldots, a_p \]  \hspace{1cm} --- (1.5.5)

v) \[ [(a_p)]_n = (a_1)_n \cdot (a_2)_n \cdot \ldots \cdot (a_p)_n \]  \hspace{1cm} --- (1.5.6)

vi) \[ \Delta(n; m) = \frac{m}{n}, \frac{m+1}{n}, \ldots, \frac{m+n-1}{n} \]  \hspace{1cm} --- (1.5.7)

vii) \[ \Delta[(n; m)] = \prod_{i=1}^{n} \left( \frac{m+i-1}{n} \right) \]  \hspace{1cm} --- (1.5.8)

viii) \[ \Delta_k [(n; m)] = \prod_{i=1}^{n} \left( \frac{m+i-1}{n} \right)_k \]  \hspace{1cm} --- (1.5.9)
\[ U_j = \frac{\prod_{s=1}^{p} (a_s - b_j)}{\prod_{s=1,(j)}^{q} (b_s - b_j)} \] --- (1.5.10)

\[ W_{j,k} = \frac{\prod_{s=1,(k)}^{p} (a_s - b_j)}{\prod_{s=1,(j)}^{q} (b_s - b_j)} \] --- (1.5.10A)

\[ S_n = \frac{(a_1 + n)(a_2 + n) \ldots (a_p + n)}{(b_1 + n)(b_2 + n) \ldots (b_q + n)} \] --- (1.5.11)

\[ T_{n,k} = \frac{S_n}{a_n + n} \] --- (1.5.12)

\[ A = \sum_{s=1}^{p} a_s, \] --- (1.5.13)

\[ B = \sum_{s=1}^{q} b_s \] --- (1.5.14)

The following notations have been adopted throughout this work:

\[ \{jw\}_x = x(x+w)(x+2w)(x+3w) - \ldots - (x+kw-w) \] --- (1.5.15)

so that \[ \{jw\}_x = \left( \frac{x}{w} \right)^j w^j \] --- (1.5.16)

\[ \{jw\} \{x^{kw}\}_j = \{jw\}_x \{jw\}_x (x-w) \{jw\}_x (x-2w) - \ldots - \{jw\}_x (x-kw+w) \] --- (1.5.17)

By a well-known lemma Rainville, E.D[161,22(2)], we have
\[
(\alpha)_{kn} = k^n \left( \frac{\alpha}{k} \right)_n \left( \frac{\alpha+1}{k} \right)_n \left( \frac{\alpha+2}{k} \right)_n \cdots \left( \frac{\alpha+k-1}{k} \right)_n
\]  --- (1.5.18)

for a positive integer ‘\(k\)’ and a non-negative integer ‘\(n\)’.

xvii) \( (a)_{n-t} = \frac{(-1)^t (a)}{1-a-n} \), \( (1.5.19) \)

xviii) \( (-n)_t = \frac{(-1)^t n!}{(n-t)!} \), \( (1.5.20) \)

1.6 Important Formulae

Following formulae (Milne-Thomson [221] and Jordan [85] have been frequently used throughout the present thesis.

\[
\Delta_{x,w} f(x) = \frac{f(x+w) - f(x)}{w}
\]  --- (1.6.1)

\[
\chi[\alpha w] = x(x-w)(x-2w) \ldots (x-\alpha w + w)
\]  --- (1.6.2)

so that \( \lim_{w \to 0} \Delta_{x,w} f(x) = \frac{d}{dx} f(x) \) \( (1.6.3) \)

\[
\Delta_{x,w} \chi[\alpha w] = \alpha \chi[\alpha w - w]
\]  --- (1.6.4)

\[
\Delta_{x,w}^n (u_x v_x) = \sum_{k=0}^{n} \binom{n}{k} \Delta_{x,w}^{n-k} u_{x+kw} \Delta_{x,w}^k v_x
\]  --- (1.6.5)

\[
\Delta_{x,w}^{-1} (u_x v_x) = \sum_{p=0}^{\infty} (-1)^p \Delta_{x,w}^p v_x \Delta_{x,w}^{-p-1} u_{x+pw}
\]  --- (1.6.6)

\[
\Delta_{x,w} \left( \frac{x}{w} \right)_R = \frac{\left( \frac{x+1}{w} \right)_R - \left( \frac{x}{w} \right)_R}{w} = \frac{R}{w} \left( \frac{x}{w} + 1 \right)_R - 1
\]  --- (1.6.7)
\[ \Delta_{x,w} \left( \frac{-x}{w} \right) = \frac{\left( \frac{-x}{w} + 1 \right)}{R} \frac{\left( \frac{-x}{w} \right)}{R} \frac{\left( \frac{-R}{w} \right)}{R-1} \quad \text{--- (1.6.8)} \]

\[ \Delta_{x,w} (1-w)^w = \frac{x+1}{w} - \frac{x}{w} = -(1-w)^w \quad \text{--- (1.6.9)} \]

By a special case of Newton’s generalized binomial theorem

\[ \frac{1}{(1-x)^s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} x^k \quad \text{--- (1.6.10)} \]

By Gauss’s theorem \( \_2 F_1(a,b;c;1) = \frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)} \quad \text{--- (1.6.11)} \)