Chapter-5

CERTAIN SUBCLASSES OF P-VALENT ANALYTIC FUNCTIONS
CHAPTER 5

Chapter 5. Certain Subclasses of P-Valent Analytic Functions

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CHAPTER 5
CERTAIN SUBCLASSES OF P-VALENT ANALYTIC FUNCTIONS

5.1 INTRODUCTION

Irmak et al. (23) and Prajapat (68) investigated certain subclasses of multivalent analytic functions and obtained some sufficient conditions for functions belonging to these classes. Motivated by the above works, in this chapter, a new subclass of $p$-valent analytic functions $B(\gamma, \beta, p, \alpha)$ is introduced.

Let $A(p)$ denote the class of functions represented by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}),$$

which are analytic and $p$-valent in $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

A function $f \in A(p)$ is said to be a member of the class $B(\gamma, \beta, p, \alpha)$ if and only if it satisfies

$$\text{Re}\left\{ \beta \gamma z^3 f'''(z) + (2\beta \gamma + \beta - \gamma)z^2 f''(z) + zf'(z) \right\} > \alpha,$$  \quad (0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),  \quad (5.1)

for some $\alpha$, for all $z \in U$. Note that the condition (5.1) implies that

$$\left| \left( \frac{\beta \gamma z^3 f'''(z) + (2\beta \gamma + \beta - \gamma)z^2 f''(z) + zf'(z)}{\beta \gamma z^2 f''(z) + (\beta - \gamma)zf'(z) + (1 - \beta + \gamma)f(z)} \right) - p \right| < p - \alpha$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}).$$

We note that $B(0, \beta, p, \alpha) \equiv T_\beta(p; \alpha)$ is the class studied by Irmak and Raina in (23). The important subclasses such as $B(0, 0, p, \alpha) = S^*(p, \alpha)$, $p$-valently starlike function of order $\alpha$ and
$B(0,1,p,\alpha) = C(p,\alpha)$, $p$-valently convex function of order $\alpha$ are reduced from the aforesaid class.

Sufficient conditions are obtained for functions to be a element of the class $B(\gamma,\beta,p,\alpha)$. Also, certain inequalities for $p$-valent functions which characterize the properties of starlikeness and convexity in $U$ are considered. Furthermore sufficient conditions are found for function to be univalent.

The result obtained in this chapter, generalizes the result obtained by H. Irmak and R. K. Raina (23). Also, our result unifies the result for a functions belonging to the class of $p$-valently starlike function of order $\alpha$ and $p$-valently convex function of order $\alpha$.

5.1.1 Preliminaries

In order to derive our main results, we need the following Lemmas.

**Lemma 5.1.1.** “Let $w(z)$ be the non-constant and analytic function in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0$, then

$$z_0w'(z_0) = kw(z_0) \quad (5.2)$$

where $k \geq 1$ is a real number” (24, Jack, 1971).

**Lemma 5.1.2.** “Let $\Omega$ be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi(z)$ is a mapping from $\mathbb{C}^2 \times U$ to $\mathbb{C}$ which satisfies $\Phi(ix,y;z) \notin \Omega$ for $z \in U$, and for all real $x, y$ such that $y \leq -n(1+x^2)/2$. If the function $q(z) = 1 + q_nz^n + q_{n+1}z^{n+1} + \cdots$ is analytic in $U$ such that $\Phi(q(z), zq'(z); z) \in \Omega$ for all $z \in U$, then $\text{Re } q(z) > 0$” (48, Miller and Mocanu, 1987).

**Lemma 5.1.3.** “Let $\delta$ be the complex number, $\text{Re } \delta > 0$, and $\lambda$ be a complex number, $|\lambda| \leq 1, \lambda \neq -1$ and let $h(z) = z + a_2z^2 + \cdots$ be a regular
function on $U$. If
\[
|\lambda| |z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zh''(z)}{h'(z)} \leq 1
\]
for all $z \in U$, then the function
\[
F_\delta(z) = \left(\delta \int_0^z t^{\delta-1}h'(t)dt\right)^{1/\delta}
\]
\[
= z + \frac{2a_2}{\delta+1}z^2 + \left(\frac{3a_3}{\delta+2} + \frac{2\delta(1-\delta)a_2^2}{(\delta+1)^2}\right)z^3 + \ldots
\]
is regular and univalent in $U$” (66, Pescar, 1996).

**Lemma 5.1.4.** “Let $\delta$ be a complex number, $\text{Re} \, \delta > 0$, and $\lambda$ a complex number, $|\lambda| < 1$, and $h \in A$. If
\[
\frac{1 - |z|^{2\text{Re} \, \delta}}{\text{Re} \, \delta} \frac{|zh''(z)|}{|h'(z)|} \leq 1 - |\lambda|
\]
for all $z \in U$, then for any complex number $\eta$, $\text{Re} \, \eta \geq \text{Re} \, \delta$, the function
\[
F_\eta(z) = \left(\eta \int_0^z t^{\eta-1}h'(t)dt\right)^{1/\eta}
\]
is in the class $S$” (67, Pescar, 2003).

**Lemma 5.1.5.** “Let $p(z)$ be analytic in $U$, $p(0) = 1$, $p(z) \neq 0$ in $U$ and suppose that there exists a point $z_0 \in U$ such that
\[
|\arg(p(z))| < \frac{\pi}{2} \alpha, \quad \text{for} \quad |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2} \alpha,
\]
where $0 < \alpha \leq 1$, then we have
\[
\frac{z_0p'(z_0)}{p(z_0)} = ika,
\]
where
\[
k \geq \frac{1}{2} \left(a + \frac{1}{a}\right) \geq 1 \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi}{2} \alpha,
\]
\[
k \leq -\frac{1}{2} \left(a + \frac{1}{a}\right) \leq -1 \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi}{2} \alpha,
\]
\[
p(z_0)^{1/\alpha} = \pm ai,
\]
($a > 0$)” (54, Nunokawa, 1993).
5.2 SUFFICIENT CONDITIONS FOR THE CLASS $B(\gamma, \beta, p, \alpha)$

By using Lemma 5.1.2, we first prove the following theorem.

**Theorem 5.2.1.** Let $f \in A(p)$. Define a function $G_{\beta, \gamma}$ by

$$G_{\beta, \gamma}(z) := \beta \gamma z^2 f''(z) + (\beta - \gamma)zf'(z) + (1 - \beta + \gamma)f(z),$$

$$(0 \leq \gamma \leq \beta \leq 1; z \in U),$$

and if $G_{\beta, \gamma}(z)$ satisfies

$$\text{Re} \left\{ \frac{zG'_{\beta, \gamma}(z)}{G_{\beta, \gamma}(z)} \left( 2 + \frac{zG''_{\beta, \gamma}(z)}{G'_{\beta, \gamma}(z)} - \frac{zG'_\gamma(z)}{G_{\beta, \gamma}(z)} \right) \right\} > p \left( 1 - \frac{n}{2} \right) + \frac{n}{2} \alpha$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p, n \in \mathbb{N}),$$

then $f(z) \in B(\gamma, \beta, p, \alpha)$.

**Proof.** Let $f \in A(p)$. Define a function $w(z)$ in $U$ by

$$\frac{zG'_{\beta, \gamma}(z)}{G_{\beta, \gamma}(z)} = p + (p - \alpha)w(z), \quad (0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}), \quad (5.3)$$

then the function $w(z)$ is analytic in $U$, and $w(0) = 0$.

A computation using (5.3) shows that

$$\frac{zG'_{\beta, \gamma}(z)}{G_{\beta, \gamma}(z)} \left( 2 + \frac{zG''_{\beta, \gamma}(z)}{G'_{\beta, \gamma}(z)} - \frac{zG'_\gamma(z)}{G_{\beta, \gamma}(z)} \right) = (p - \alpha)[zw'(z) + w(z)] + p$$

$$= \Phi(w(z), zw'(z); z),$$

where $\Phi(r, s; z) = (p - \alpha)[s + r] + p$.

For all real $x, y$ satisfying $y \leq -n(1 + x^2)/2$, we have

$$\text{Re} \Phi(ix, y; z) = \text{Re} \left\{ (p - \alpha)[y + ix] + p \right\}$$

$$\leq -\frac{n}{2}(p - \alpha)(1 + x^2) + p$$

$$\leq -\frac{n}{2}(p - \alpha) + p$$

$$= p \left( 1 - \frac{n}{2} \right) + \frac{n}{2} \alpha.$$

Let $\Omega = \{ w : \text{Re} w > p(1 - \frac{n}{2}) + \frac{n}{2} \alpha \}$. Then $\Phi(w(z), zw'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real $x$ and $y \leq -n(1 + x^2)/2$, $z \in U$. 
By using Lemma 5.1.2, we have \( \text{Re} \, w(z) > 0 \), which implies that
\[
\text{Re} \left\{ \frac{zG_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \right\} > \alpha, \quad (0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),
\]
and hence \( f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha) \).

By setting \( \gamma = \beta = 0 \) in Theorem 5.2.1, we have following corollary.

**Corollary 5.2.1.** If \( f \in A(p) \) satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( 2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right\} > p \left( 1 - \frac{n}{2} \right) + \frac{n}{2} \alpha
\]
\[
(0 \leq \alpha < p; p, n \in \mathbb{N}),
\]
then \( f(z) \in S^*(p, \alpha) \). That is, \( f \) is \( p \)-valently starlike of order \( \alpha \).

Its further case when \( \alpha = 0 \) and \( p = 1 \), Corollary 5.2.1 reduces to Corollary 5.2.2.

**Corollary 5.2.2.** If \( f \in A \) satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( 2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right\} > 1 - \frac{n}{2}, \quad (n \in \mathbb{N}),
\]
then \( f(z) \in S^* \).

By taking \( \gamma = 0, \beta = 1 \) in Theorem 5.2.1, we get the following corollary.

**Corollary 5.2.3.** If \( f \in A(p) \) satisfies
\[
\text{Re} \left\{ \frac{(zf'(z))^2}{f'(z)} \left( 1 + \frac{zf'''(z)}{zf''(z) + f'(z)} - \frac{zf''(z)}{f'(z)} \right) \right\} > p \left( 1 - \frac{n}{2} \right) + \frac{n}{2} \alpha
\]
\[
(0 \leq \alpha < p; p, n \in \mathbb{N}),
\]
then \( f(z) \in C(p, \alpha) \). That is, \( f \) is \( p \)-valently convex of order \( \alpha \).

A further case of Corollary 5.2.3, when \( \alpha = 0, p = 1 \) gives the following corollary.
Corollary 5.2.4. If $f \in A$ satisfies
\[
\text{Re} \left\{ \frac{(zf'(z))'}{f'(z)} \left( 1 + \frac{z^2f'''(z) + 2zf''(z)}{zf'(z) + f'(z)} \right) \right\} > 1 - \frac{n}{2}, \quad (n \in \mathbb{N}),
\]
then $f(z) \in C$. i.e. $f$ is convex.

Theorem 5.2.2. Let $-1 < b < a \leq 1$, $0 \leq a < p$, $p \in \mathbb{N}$ such that $p(1 + a) + a \leq 2p(p - b) + b$. If $G_{\beta,\gamma}(z)$ satisfies the inequality
\[
\left| 1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \right| < \frac{p(a + b)}{(p + a)(p - b)} \quad (z \in \mathbb{U}),
\]
then $f(z) \in B(\gamma, \beta, p, a)$.

Proof. Define a function $w(z)$ by
\[
\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{p + aw(z)}{p - bw(z)} \quad (z \in \mathbb{U}).\tag{5.5}
\]
Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. By the logarithmic differentiation of (5.5), we get
\[
1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{p(a + b)zw'(z)}{(p + aw(z))(p - bw(z))}.	ag{5.6}
\]
Now suppose that there exists $z_0 \in \mathbb{U}$ such that
\[
\max_{|z| < |z_0|} |w(z)| = |w(z_0)| = 1,
\]
then from Lemma 5.1.1, we have (5.2). Letting $w(z_0) = e^{i\theta}$, from (5.6), we have
\[
\left| 1 + \frac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - \frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} \right| = \left| \frac{p(a + b)ke^{i\theta}}{(p + ae^{i\theta})(p - be^{i\theta})} \right| \geq \frac{p(a + b)}{(p + a)(p - b)}.
\]
This contradicts our assumption (5.4). Therefore $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$. Thus we conclude from (5.5) that
\[
\left| \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p \right| = \left| \frac{p + aw(z)}{p - bw(z)} - p \right| < \frac{p + a - p(p - b)}{p - b} \leq p - a \quad (z \in \mathbb{U}),
\]
which implies that $f(z) \in B(\gamma, \beta, p, a)$.
Theorem 5.2.3. Let \( f \in A(p) \). If \( G_{\beta,\gamma}(z) \) satisfies anyone of the following conditions:

\[
\left| 1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \right| < \frac{p - \alpha}{2p - \alpha'}, \tag{5.7}
\]

\[
\left| \frac{zG''_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \left( 1 + \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \right) \right| < p - \alpha, \tag{5.8}
\]

\[
\left| 1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - 1 \right| < \frac{p - \alpha}{(2p - \alpha)z}, \tag{5.9}
\]

\[
\left| \frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p} - 1 \right| < \frac{1}{(2p - \alpha)}, \tag{5.10}
\]

\[
\text{Re} \left\{ \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \left( 1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p \right) \right\} < 1, \tag{5.11}
\]

then \( f(z) \in B(\gamma, \beta, p, \alpha) \).

Proof. Let \( f \in A(p) \). Define a function \( w(z) \) in \( U \) by

\[
\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = p + (p - \alpha)w(z), \quad (0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}), \tag{5.12}
\]

then the function \( w(z) \) is analytic in \( U \), and \( w(0) = 0 \).

It follows from (5.12) that

\[
1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{(p - \alpha)zw'(z)}{p + (p - \alpha)w(z)}. \tag{5.13}
\]
Hence, from (5.12) and (5.13), we have
\[
\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} \left(1 + \frac{zG_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right) = (p - \alpha)zw'(z),
\]
(5.14)

\[
1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - 1 = \frac{(p - \alpha)zw'(z)}{[p + (p - \alpha)w(z)]^2},
\]
(5.15)

\[
1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p
\]
\[
\frac{zG_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z) - p} - 1 = \frac{zw'(z) - 1}{w(z)} \frac{1}{p + (p - \alpha)w(z)},
\]
(5.16)

and
\[
\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \left(1 + \frac{zG_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right)
\]
\[
\frac{zG_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z) - p}
\]

\[
= \frac{zw'(z)}{w(z)}.
\]
(5.17)

Now, suppose there exists \(z_0 \in U\) such that
\[
\max_{|z| < |z_0|} |w(z)| = |w(z_0)| = 1,
\]
then from Lemma 5.1.1, we have (5.2). Therefore, letting \(w(z_0) = e^{i\theta}\) in each of (5.13)-(5.17), we obtain that
\[
\left|1 + \frac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - \frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}\right| = \left|\frac{(p - \alpha)ke^{i\theta}}{p + (p - \alpha)e^{i\theta}}\right| \geq \frac{p - \alpha}{2p - \alpha},
\]
\[
\left|\frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} \left(1 + \frac{z_0G_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - \frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}\right)\right| = \left|\frac{(p - \alpha)ke^{i\theta}}{p + (p - \alpha)e^{i\theta}}\right| \geq (p - \alpha),
\]
\[
\left|1 + \frac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - 1\right| = \left|\frac{(p - \alpha)ke^{i\theta}}{(p + (p - \alpha)e^{i\theta})^2}\right| \geq \frac{p - \alpha}{(2p - \alpha)^2},
\]

\[ \left| 1 + \frac{z_0 G''_{\beta, \gamma}(z_0)}{G'_{\beta, \gamma}(z_0)} - p \right| = \left| \frac{k}{p + (p - \alpha)e^{i\theta}} \right| = \frac{1}{(2p - \alpha)}, \]

which contradict our assumption (5.7)-(5.11), respectively. Therefore \(|w(z)| < 1\) holds true for all \(z \in U\). From (5.12), we have

\[ \left| \frac{z G'_{\beta, \gamma}(z)}{G_{\beta, \gamma}(z)} - p \right| = |(p - \alpha)w(z)| < (p - \alpha), \]

(0 \(\leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}\), which implies that

\[ \Re \left\{ \frac{z G'_{\beta, \gamma}(z)}{G_{\beta, \gamma}(z)} \right\} = \Re \left\{ \frac{\beta \gamma z^3 f'''(z) + (2\beta \gamma + \beta - \gamma)z^2 f''(z) + zf'(z)}{\beta \gamma z^2 f''(z) + (\beta - \gamma)zf'(z) + (1 - \beta + \gamma)f(z)} \right\} > \alpha, \]

\(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}\),

and hence \(f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)\).

\[ \text{Remark 5.2.1. By taking } \gamma = 0; \gamma = \beta = 0; \gamma = 0 \text{ and } \beta = 1; \gamma = \beta = \alpha = 0 \]

and \(p = 1; \gamma = \alpha = 0 \text{ and } \beta = p = 1\) in Theorem 5.2.3, we get the results of Irmak and Raina (23, Theorem 1, Corollary 1-4).

### 5.3 SUFFICIENT CONDITIONS FOR UNIVALENT

**Theorem 5.3.1.** Let 0 \(\leq \gamma \leq \beta \leq 1; 0 \leq \alpha < 1, \delta \) be a complex number, \(\Re \delta \geq \frac{3 - 2\alpha}{2 - \alpha}\) and \(\lambda \) be a complex number which satisfies the inequality

\[ |\lambda| \leq 1 - \frac{\frac{3 - 2\alpha}{\Re \delta(2 - \alpha)}}. \]

(5.18)
If $F_{\beta, \gamma}(z) := \frac{zG'_{\beta, \gamma}(z)}{G_{\beta, \gamma}(z)}$ is regular in $U$ and
\[
\left| 1 + \frac{zG''_{\beta, \gamma}(z)}{G'_{\beta, \gamma}(z)} - \frac{zG'_{\beta, \gamma}(z)}{G_{\beta, \gamma}(z)} \right| \leq \frac{1 - \alpha}{2 - \alpha} \quad (z \in U),
\] (5.19)
then the function
\[
F(z) = \left( \delta \int_0^z t^{\delta - 1} \frac{G'_{\beta, \gamma}(t)}{G_{\beta, \gamma}(t)} \, dt \right)^{1/\delta}
\] (5.20)
is univalent in $U$.

**Proof.** Define a function
\[
h(z) = \int_0^z \frac{F_{\beta, \gamma}(t)}{t} \, dt,
\]
then we have $h(0) = h'(0) - 1 = 0$. Also a simple computation yields $h'(z) = \frac{F_{\beta, \gamma}(z)}{z}$ and
\[
\frac{zh''(z)}{h'(z)} = \frac{zF'_{\beta, \gamma}(z)}{F_{\beta, \gamma}(z)} - 1.
\] (5.21)
From (5.21), we have
\[
\left| \frac{zh''(z)}{h'(z)} \right| \leq \left| \frac{zF'_{\beta, \gamma}(z)}{F_{\beta, \gamma}(z)} \right| + 1
\]
\[
= \left| 1 + \frac{zG''_{\beta, \gamma}(z)}{G'_{\beta, \gamma}(z)} - \frac{zG'_{\beta, \gamma}(z)}{G_{\beta, \gamma}(z)} \right| + 1.
\] (5.22)
Hence, from (5.19) and (5.22), we have
\[
\left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{3 - 2\alpha}{2 - \alpha}.
\] (5.23)
Using (5.23), we have
\[
\left| \frac{z}{|z|^{2\delta}} + \left( 1 - |z|^{2\delta} \right) \frac{zh''(z)}{\delta h'(z)} \right| \leq \left| \frac{zh''(z)}{\delta h'(z)} \right| \leq \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{3 - 2\alpha}{2 - \alpha}.
\]
Again using (5.18), we have
\[
\left| \frac{z}{|z|^{2\delta}} + \left( 1 - |z|^{2\delta} \right) \frac{zh''(z)}{\delta h'(z)} \right| \leq 1.
\]
Applying Lemma 5.1.3, we obtain that the function $F(z)$ defined by (5.20) is univalent in $U$. \qed
We obtain Theorem 5.3.2 below, by using Lemma 5.1.4 and the same techniques as in the proof of Theorem 5.3.1.

**Theorem 5.3.2.** Let $\delta$ be a complex number, $\Re \delta > 0$, $\lambda$ a complex number, $|\lambda| < 1$, and $f \in \mathcal{A}$. If

\[ \left| 1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \right| \leq \frac{1 - \alpha}{2 - \alpha} \quad (z \in U; 0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < 1), \]

then for any complex number $\eta$,

\[ \Re \eta \geq \Re \delta \geq \frac{3 - 2\alpha}{(1 - |\lambda|)(2 - \alpha)}, \]

the integral operator

\[ F_{\eta}(z) = \left( \eta \int_0^z t^{\eta - 1} \frac{G'_{\beta,\gamma}(t)}{G_{\beta,\gamma}(t)} dt \right)^{1/\eta} \]

is in the class $S$.

**Theorem 5.3.3.** Let $p(z)$ be an analytic function in $U$, $p(z) \neq 0$ in $U$ and suppose that

\[ \left| \arg \left( p(z) + \frac{z^2 G'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} p'(z) \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U), \] (5.24)

where $0 < \alpha < p, 0 \leq \gamma \leq \beta \leq 1$ and $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$, then we have

\[ |\arg(p(z))| < \frac{\pi}{2} \alpha \quad (z \in U). \]

**Proof.** Suppose there exists a point $z_0 \in U$ such that

\[ |\arg(p(z))| < \frac{\pi}{2} \alpha, \quad \text{for} \quad |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2} \alpha. \]

Then, applying Lemma 5.1.4, we have

\[ \arg \left( p(z_0) + \frac{z_0^2 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} p'(z_0) \right) = \arg \left( p(z_0) \left( 1 + \frac{z_0 G'_{\beta,\gamma}(z_0) z_0 p'(z_0)}{G_{\beta,\gamma}(z_0) p(z_0)} \right) \right) \]

\[ = \arg(p(z_0)) + \arg \left( 1 + i \frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0) \alpha} \right). \] (5.25)
When \( \arg(p(z_0)) = \pi \alpha/2 \), since
\[
\Re\left( \frac{z_0 G'_{\beta, \gamma}(z_0)}{G_{\beta, \gamma}(z_0)} \alpha \right) > 0 \Rightarrow \arg\left( 1 + i \frac{z_0 G'_{\beta, \gamma}(z_0)}{G_{\beta, \gamma}(z_0)} \alpha \right) > 0,
\]
Eq. (5.25) becomes
\[
\arg \left( p(z_0) + \frac{z_0^2 G'_{\beta, \gamma}(z_0)}{G_{\beta, \gamma}(z_0)} p'(z_0) \right) > \frac{\pi}{2} \alpha. \tag{5.26}
\]
Similarly, if \( \arg(p(z_0)) = -\pi \alpha/2 \), since
\[
\Re\left( \frac{z_0 G'_{\beta, \gamma}(z_0)}{G_{\beta, \gamma}(z_0)} \alpha \right) < 0 \Rightarrow \arg\left( 1 + i \frac{z_0 G'_{\beta, \gamma}(z_0)}{G_{\beta, \gamma}(z_0)} \alpha \right) < 0,
\]
we obtain that
\[
\arg \left( p(z_0) + \frac{z_0^2 G'_{\beta, \gamma}(z_0)}{G_{\beta, \gamma}(z_0)} p'(z_0) \right) = \arg(p(z_0)) + \arg \left( 1 + i \frac{z_0 G'_{\beta, \gamma}(z_0)}{G_{\beta, \gamma}(z_0)} \alpha \right) < -\frac{\pi}{2} \alpha. \tag{5.27}
\]
Thus, we see that (5.26) and (5.27) contradict our assumption (5.24). Consequently, we conclude that
\[
|\arg(p(z))| < \frac{\pi}{2} \alpha \quad (z \in \mathbb{U}).
\]

\[\square\]

The results obtained in this chapter are published in the journal "Global Journal of Science Frontier Research (F)", Volume 13 Issue 3 Version 1.0, 2013, 15-25.

In the next chapter, a new class of \( p \)-valent analytic functions in terms of the generalized multiplier transformation is introduced and studied. Several fascinating subordination properties and a sharp containment relation for this subclass is discussed.