CHAPTER V

A GENERALIZATION AND APPLICATIONS OF GEOMETRIC INFINITE DIVISIBILITY

5.1 INTRODUCTION

It is a natural question to ask, "why the interest is restricted to geometric compounding alone? This led us to a generalization of infinite divisibility of random variables. Klevanov, et al. (1985) have developed the notion of $\nu_p$-infinite divisibility and $\nu_p$ - strictly stable laws. Here, these concepts are further improved for nonnegative random variables. Section 5.2 is devoted to such a study.

In Chapter III we have mentioned some practical applications of $\nu$-thinning of renewal processes of Cox processes. Apart from this, in Section 5.2, some extensions of theorems concerning the complete monotonicity of density functions which have been discussed in Chapter IV. Section 5.4 deals with the connection between log convexity and geometric infinite divisibility. In section 5.5 we deal with some developments in reliability theory connected with geometric infinite divisibility. Section 5.6 deals with the superposition of renewal processes and geometric infinite divisibility.

5.2 $\nu$-INFINITE DIVISIBILITY

We saw that a random variable $X$ is said to be geometrically infinitely divisible if for every $p$: (5.1) it can be expressed as

$$X \equiv x P(X|P) \quad \text{for } \quad j = 1$$
CHAPTER V

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5.1 INTRODUCTION

It is a natural question to ask, "why the interest is restricted to geometric compounding alone." This led us to a generalization—\( \nu \)-infinite divisibility of random variables. Klevanov, et al. (1985) have developed the notion of \( \nu_p \)-infinite divisibility and \( \nu_p \) - strictly stable laws. Here these concepts are further improved for nonnegative random variables. Section 5.2 is devoted to such a study.

In Chapter III we have mentioned some practical applications of \( p \)-thinning of renewal processes and also of Cox processes. Apart from this, in section 5.3 we see some applications of theorems concerning the complete monotonicity of density functions which have been discussed in chapter IV. Section 5.4 deals with the connection between log convexity and geometric infinite divisibility. In section 5.5 we deal with some developments in reliability theory connected with geometric infinite divisibility. Section 5.6 deals with the superposition of renewal processes and geometric infinite divisibility.

5.2 \( \nu \)-INFINITE DIVISIBILITY

We saw that a random variable \( X \) is said to be geometrically infinitely divisible if for every \( p \in (0,1) \) it can be expressed as

\[
X \overset{d}{=} \sum_{j=1}^{N} \xi_j^{(p)}
\]

where the \( \xi_j^{(p)} \) are independently identically distributed for \( j = 1, 2, \ldots \) (Definition 5.3.1).
where \( P \{ N_p = k \} = p (1-p)^{k-1}, \ k = 1,2, \ldots, N_p \) and \( X_j^{(P)} \) are independent and \( X_j^{(P)} \) are independently identically distributed for \( j = 1,2, \ldots \) (Definition 1.6.1).

Now, at this stage, instead of \( N_p \), the geometric random variable, we consider \( \nu \), which is an integer valued random variable with generating function

\[
 f(s) = \sum_{k=0}^{\infty} s^k P \{ \nu = k \} .
\]

Also we define \( f_0(s) = s \), \( f_1(s) = f(s) \) and

\[
 f_{n+1}(s) = f(f_n(s)), \ n \geq 1 \quad (5.2.1)
\]

and define \( \nu \) - infinite divisibility as follows.

**DEFINITION 5.2.1**

A nonnegative random variable \( X \) is defined to be \( \nu \) -infinitely divisible if there exists a sequence of random variables \( X_j \)'s such that

\[
 X \overset{d}{=} \frac{1}{m} \sum_{j=1}^{\nu} X_j \quad (5.2.2)
\]

where the random variables \( X_j, j = 1,2, \ldots \) are independently identically distributed as \( X \) and also independent of \( \nu \) and \( \nu \) has a generating function \( f_n(s) \) of \( (5.2.1) \) and \( E(\nu) = m > 1 \).
Taking Laplace transforms on both sides of (5.2.2) we get,
\[ \phi(\lambda) = f(\phi\left(\frac{\lambda}{m}\right)) \] (5.2.3)

Iterating,
\[ \phi(\lambda) = f_n(\phi\left(\frac{\lambda}{m}\right)) \text{ for all } n \geq 1. \] (5.2.4)

**REMARK 5.2.1**

The notion of infinite divisibility occurs from the fact that equation (5.2.2) is equivalent to

\[ X \overset{d}{=} \frac{1}{m^n} \sum_{j=1}^{v_n} X_{nj}, \]

for every \( n \geq 1 \) as evidenced from equation (5.2.4).

Now, let us find the connection of \( \nu \)-infinite divisibility with branching process. In Galton-Watson branching process with \( f(s) \) as the generating function of the probability distribution of the number of progenies in each generation, the number in the \( n \)th generation \( Z_n \) is given by

\[ Z_n = \sum_{j=1}^{Z_{n-1}} \xi_j. \]

And this random variable \( Z_n \) has the property that \( \frac{Z_n}{m^n} \) tends to a random variable \( W \) whose Laplace transform satisfies the Abel's functional equation with respect to the generating function \( f \).
\[ \phi(\lambda) = f(\phi(\frac{\lambda}{m})), \quad m = E(Z_1) \]

It has been shown in Arthreya and Ney (1971), page 10 that when
\[ E(Z_1) \leq 1, \quad W_n = \frac{Z_n}{m_n} \rightarrow W, \quad \text{where } P(W = 0) = 1; \text{ equivalently, } f_n(s) \rightarrow 1. \]
Therefore, we consider the case \( E(Z_1) > 1 \).

**EXAMPLE 5.2.1**

The exponential distribution with \( \phi(\lambda) = \frac{1}{1+\lambda} \) is \( \nu \)-infinitely divisible with respect to geometric distribution with

\[ f(s) = \frac{ps}{1-qs} \]

\[ f'(1) = \frac{1}{p} > 1. \]

Consider
\[ f(\phi(\frac{\lambda}{m})) = \frac{p\phi(\frac{\lambda}{m})}{1-q\phi(\frac{\lambda}{m})} \]

\[ = \frac{p}{1+\lambda} \frac{1}{p} \frac{1}{1-q \frac{1}{1+\lambda}} \]

\[ = \phi(\lambda) \]

Hence the result.

**RESULT 5.2.1**

\( \nu \)-infinite divisibility coincides with geometric infinite divisibility when \( \nu \) is geometric.
When $\phi(\lambda)$ is $\nu$-infinitely divisible, $\phi(\lambda)$ satisfies,

$$\phi(\lambda) = f(\phi\left(\frac{\lambda}{m}\right))$$

Therefore,

$$f(s) = \frac{ps}{1-qs}$$

Hence,

$$\phi(\lambda) = \frac{p\phi(\lambda)}{1-q\phi(\lambda)}$$

Writing $\phi(\lambda) = \frac{1}{1+\psi(\lambda)}$, $\psi(\lambda)$ satisfies $\psi(\lambda) = \frac{1}{p}\psi(p\lambda)$ (5.2.5)

(5.2.5) shows that $\phi(\lambda)$ is geometrically infinitely divisible.

Now we show that $\nu$-infinite divisibility does not hold for every discrete random variable.

**THEOREM 5.2.1**

No random variable is Poisson (a) infinitely divisible for $a > 1$.

**PROOF**

If $\phi$ is $\nu$-infinitely divisible with respect to Poisson (a) then $\phi$ satisfies,

$$\phi(\lambda) = f(\phi\left(\frac{\lambda}{m}\right)), \quad \frac{1}{\phi(\lambda)} = \frac{1}{\phi(0)}$$
where \( f(s) = e^{-a(1-s)} \).

Therefore,

\[ \phi'(\lambda) = e^{-a(1-\phi(\lambda/a))} \]

Hence, the theorem.

\[ 1 - \phi(\lambda/a) = -\frac{\log \phi(\lambda)}{a} \]

Now, we seek a necessary and sufficient condition that \( \phi(\lambda/a) \) exists.

\[ \phi(\lambda/a) = 1 + \frac{\log \phi(\lambda)}{a} \]

Differentiating, we get

\[ \phi'(\lambda/a) = \frac{\phi'(\lambda)}{\phi(\lambda)} \]

(or) \[ \frac{1}{\phi(\lambda)} = \frac{\phi'(\lambda/a)}{\phi'(\lambda)} \]

Iterating,

\[ \frac{1}{\phi(\lambda/a^2)} = \frac{\phi'(\lambda/a^2)}{\phi'(\lambda/a)} \]

Then \( \phi(m \phi^{-1}(s)) = f(s) \phi(m \phi^{-1}(s)) \)

\[ = f(s) \phi^{-1}(s) \]

Multiplying all these equations we get,

\[ \frac{1}{\phi'(\lambda)} = \frac{1}{\phi(\lambda)} \frac{1}{\phi(\lambda/a)} \]
The infinite product on the right hand side exists for $a > 1$. On the left hand side we have a negative quantity, since $\phi(\lambda)$ is completely monotone and on the right hand side we have an infinite product of positive quantities. So no solution exists.

Hence the theorem.

Now, we seek a necessary and sufficient condition that a random variable $X$ is $\nu$ infinitely divisible.

**THEOREM 5.2.2**

A necessary and sufficient condition that $X$ is $\nu$-infinitely divisible is that

$$\phi(m \phi^{-1}(s))$$

is a generating function, where $\phi$ is the Laplace transform of $X$. In that case, $\phi(m \phi^{-1}(s))$ is its generating function.

**PROOF**

Suppose that $X$ is $\nu$-infinitely divisible and $f(s)$ is the generating function of $\nu$. Then $\phi(\lambda)$ satisfies

$$\phi(\lambda) = f\left(\frac{\lambda}{m}\right)$$

Then

$$\phi(m \phi^{-1}(s)) = f \left(\phi\left(\frac{m \phi^{-1}}{m}(s)\right)\right)$$

where $k > 0$ and $k$ is a positive integer.

Therefore,

$$\phi(\lambda) = f(s)$$
Now, suppose that \( \phi(m\phi^{-1}(s)) \) is a generating function, say \( g \). We have to prove that \( g \) satisfies (5.2.2)

\[
g \left( \phi \left( \frac{\lambda}{m} \right) \right) = \phi \left( m \phi^{-1} \left( \phi \left( \frac{\lambda}{m} \right) \right) \right)
\]

\[
= \phi \left( \frac{m\lambda}{m} \right)
\]

\[
= \phi(\lambda)
\]

**REMARK 5.2.2**

If \( X \) is \( \nu^1 \) infinitely divisible and \( \nu^2 \) infinitely divisible, then \( \nu^1 \models \nu^2 \) since \( \phi \) and \( f \) determine each other uniquely by virtue of equation (5.2.3).

Now let us see some examples of \( \nu \)-infinite divisibility other than the geometric case.

**EXAMPLE 5.2.2** (Harris (1948))

Consider \( f(s) = 1 - \frac{m}{\lambda} + \frac{m}{\lambda} \left( \frac{1}{\lambda + \lambda - g\lambda} \right) \), \( m > 1 \)

Let

\[
\phi(\lambda) = 1 - \frac{(m-1)\lambda}{(m-1)\lambda} \left( \frac{1}{\lambda + \lambda - g\lambda} \right)
\]

Take \( \phi(\lambda) = \frac{k^{-1/k}}{\Gamma(1/k)} \int_0^\infty e^{-\lambda x} x^{-x/k} x^{1/k-1} dx \),

where \( k > 0 \) and \( k \) is a positive integer.

Therefore, \( \phi(\lambda) = \frac{1}{(1+k\lambda)^{1/k}} \).
\[ f(\frac{\lambda}{m}) = \frac{(1 + \frac{k\lambda}{m}) - \frac{1}{k}}{\{ m-(m-1)((1+ \frac{k\lambda}{m})^{-1/k})\}^{1/k}} \]

\[ = \frac{1}{(m+k\lambda-m+1)^{1/k}} \]

Now we shall define \( \psi \)-stable and \( \psi \)-semi-stable laws, as we have geometric stable and geometric semi-stable laws.

\[ \frac{1}{(1 + k\lambda)^{1/k}} \]

Suppose that the monotone function \( \phi(\lambda) \) satisfies

\[ f(s) = g(w) \phi^{-1}(s), \quad (5.2.6) \]

for some strictly monotone function and \( \phi(\lambda) \) satisfies

\[ \phi(\lambda) = 1 - \frac{(m-1)\lambda}{(m-1)-\alpha\lambda} \quad (5.2.7) \]

EXAMPLE 5.2.3 (Harris (1948))

Consider \( f(s) = 1 - \frac{m}{\alpha} + \frac{m}{\alpha} \left( \frac{1}{1 + \alpha - \alpha s} \right) \)

Let \( \phi(\lambda) = 1 - \frac{(m-1)\lambda}{(m-1)-\alpha\lambda} \)

\[ \phi(\frac{\lambda}{m}) = 1 + \frac{(m-1)\lambda}{(m-1)-\frac{\alpha\lambda}{m}} \]

Therefore, \( f(\psi) = f(\phi) \) gives,

\[ f(\phi(\frac{\lambda}{m})) = 1 - \frac{m}{\alpha} + \frac{m}{\alpha} \left( \frac{1}{1 + \alpha - \alpha \phi(\frac{\lambda}{m})} \right) \]
Simplifying we get,
\[ f(\phi\left(\frac{\lambda}{m}\right)) = 1 + \frac{\alpha \lambda (m-1)}{(m-1) - \alpha \lambda} \]

Hence we have,
\[ = 1 + \frac{(m-1) \lambda}{m-1 - \alpha \lambda} \]
\[ = \phi(\lambda) \]

Now we shall define \( \nu \)-stable and \( \nu \)-semi stable laws, as we have geometric stable and geometric semi stable laws.

Suppose that the generating function of \( \nu \) has the property,
\[ f(s) = g(m g^{-1}(s)), \quad (5.2.6) \]

for some strictly monotone function and \( \phi(\lambda) \) satisfies,
\[ \phi(\lambda) = f(\phi(b \lambda)) \quad \text{for} \quad m b^\alpha = 1, \quad 0 < \alpha < 1. \]

Then we have,
\[ \phi(\lambda) = g(m g^{-1}(\phi(b \lambda))) \quad (5.2.7) \]

Define \( \Psi = g^{-1}(\phi) \)

Therefore, \( (5.2.7) \) in terms of \( \Psi \) gives,
\[ \Psi(\lambda) = m \Psi(b \lambda), \quad m > 1 > b > 0 \]

\[ \psi(\lambda) = \lambda^\alpha h(\lambda) \], where \( h(\lambda) \) is periodic in \(-\log \lambda\) and \( \alpha \) is the unique solution of \( mb^\alpha = 1 \).

Then \( \phi(\lambda) = g(\psi(\lambda)) \).

Hence we have,

**DEFINITION 5.2.3**

A distribution is \( \psi \)-stable if \( \psi \) has a generating function as in (5.2.6), and \( g^{-1}(\phi(\lambda)) = \lambda^\alpha \).

**DEFINITION 5.2.4**

A distribution is \( \psi \)-semi stable if \( \psi \) has a generating function as in (5.2.6), where \( g^{-1}(\phi) = \psi \) satisfies \( \psi(\lambda) = \lambda^\alpha h(\lambda) \).

**THEOREM 5.2.3**

If \( \phi(\lambda) \) satisfies \( \phi(\lambda) = f(\phi(\lambda b_1)) = f(\phi(\lambda b_2)) \), where \( \log b_1 \)

\begin{align*}
\log b_2
\end{align*}

is irrational, then \( \phi(\lambda) \) is \( \psi \)-stable.

**PROOF**

Follows from Theorem 5 of Pillai (1985), where the same property of \( \psi(\lambda) \) is used.

**5.3 COMPLETE MONOTONICITY AND GEOMETRIC INFINITE DIVISIBILITY**

Pillai and Sandhya (1990 b) proved that a distribution function (with positive support) with completely monotone derivative is geometrically infinitely divisible or in otherwords, completely monotone.
densities on \((0, \infty)\) are geometrically infinitely divisible. This opens up a larger field of geometrically infinitely divisible distributions. Steutel (1969), where he proves that all completely monotone densities are infinitely divisible says, "the complete monotone criterion is useful, because it is much easier to verify that a function is completely monotone than to prove (directly) that it is a mixture of exponential densities".

The following two examples directly follow.

(i) Consider Pareto distribution, as considered by Thorin (1977a) with \(F(x) = 1 - \frac{1}{(1 + x)^\alpha}, \ x > 0\).

\[
f(x) = \frac{\alpha}{(1 + x)^{\alpha + 1}}\]

\(f(x)\) is completely monotone and the Pareto distribution is geometrically infinitely divisible.

(ii) Gamma \((\alpha), \ 0 < \alpha \leq 1\).

\[
f(x) = \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha - 1}, \ x > 0.
\]

\(e^{-x}\) is completely monotone, \(x^{\alpha - 1}\) is completely monotone for \(0 < \alpha \leq 1\).

Hence \(f(x)\) is completely monotone and hence geometrically infinitely divisible.
Also Steutel (1969) cites some more examples where the densities are proportional to the following functions.

(iii) \( x^{-2} \exp(x^{-1}), \quad x > 0 \)

(iv) \( \exp(-x^\alpha), \quad 0 < \alpha \leq 1, \quad x > 0 \)

Thorin (1977 a,b) introduced the generalized gamma convolutions, which are infinitely divisible and proved that both the Pareto and the log normal distributions are generalized gamma convolutions. Bondesson (1979a) discusses the infinite divisibility of a large class of probability densities on \((0, \infty)\). Bondesson (1979 b) found that \( \mathcal{G} \), the class of nondegenerate generalized gamma convolutions coincides with the class of distributions with densities of the form

\[
  f(x) = c x^{\beta-1} \prod_{j=1}^{M} (1 + c_j x)^{-\gamma j}, \quad x > 0 \tag{5.3.1}
\]

and their nondegenerate weak limits. When \(0 < \beta \leq 1\), \( f(x) \) is completely monotone and in that case \( f(x) \) is geometrically infinitely divisible. So when \( \beta \leq 1 \), the generalized gamma convolutions characterized by (5.3.1) are geometrically infinitely divisible. Also Bondesson's (1979 b) family of densities of the form

\[
  f(x) = c x^{\beta-1} \prod_{j=1}^{M} (1 + \sum_{k=1}^{N} c_{jk} x^{\alpha_{jk}})^{-\gamma j}, \quad x > 0,
\]

where all parameters are strictly positive and the \( \alpha_{jk} \)'s, are less
than or equal to 1 becomes geometrically infinitely divisible when \( \beta \leq 1 \) and so are their weak limits.

Bondesson (1979 b) also considers densities of the form

\[
f(x) = cx^{\beta-1} \exp \left(- \sum_{k=1}^{N} c_k x^k \right)
\]

\( x > 0, \ c > 0 \) and \( |a_k| \leq 1 \). All densities of the form (5.3.2) becomes completely monotone when \( 0 < \beta \leq 1 \) and \( 0 < a_k \leq 1 \) and hence geometrically infinitely divisible and so are their weak limits.

Well-known densities like the ratio of two gamma variables, in particular the \( F \) distribution, the so-called Burr distribution of type XII, etc. are special cases of the density

\[
f_1(x) = C x^{\beta-1} (1+cx^\alpha)^{-\gamma}
\]

which can be proved to be geometrically infinitely divisible for \( \beta \leq 1 \) and \( \alpha \leq 1 \). The density,

\[
f_2(x) = C x^{\beta-1} \exp(-cx^\alpha), \quad 0 < \alpha < 1
\]

is completely monotone for \( 0 < \beta \leq 1 \) and hence geometrically infinitely divisible. \( f_2(x) \) corresponds to a power with exponent \( \frac{1}{\alpha} \), \( 0 < \alpha < 1 \) of a gamma variable. Common names of this distribution for \( \alpha > 0 \) are generalized gamma distribution and Stacy's distribution. The Weibull distribution appears as a special case.

In \( f_2(x) \), put \( \beta = \sigma^{-2} (1+\alpha^{-1}) \) and
where \( 0 < \alpha \leq 1, \sigma > 0 \) and restrict \( \beta \) in the interval \((0,1]\). Letting \( \alpha \to 0 \) we get the log normal density (Bondesson (1979b)),

\[
f_3(x) = C x^{-1} \exp \left\{ \frac{-\left( \log x - \mu \right)^2}{2 \sigma^2} \right\}, \quad x > 0.
\]

Hence the log normal distribution for \( 0 < \mu + \frac{1}{\alpha} \leq \sigma^2 \) becomes geometrically infinitely divisible, since the weak limit of geometrically infinitely divisible densities is again geometrically infinitely divisible.

According to Bondesson (1990), the class of completely monotone densities is closed with respect to mixing, which implies that they are in particular closed with respect to multiplication and division of random variables. But they are not closed with respect to the operation \( X \to \frac{1}{X} \) and contain only decreasing densities. The completely monotone densities have interpretations as first passage time densities for skip-free Markov processes (Steutel (1973)). Also from Cox (1981) every density in the class of nondegenerate gamma convolutions as in (5.3.1) with \( 0 < \beta \leq 1 \) is the marginal density for a stationary AR(1) process.

### 5.4 Log Convexity and Geometric Infinite Divisibility

Steutel (1991) proves that all log convex lattice distributions are geometrically infinitely divisible and hence all log convex densities on \((0, \infty)\), using a simple approximation argument together with the fact that the set of geometrically infinitely divisible distributions is closed under weak convergence. Since completely monotone densities
are log convex he gets the result that all completely monotone densities on \((0, \infty)\) are geometrically infinitely divisible. Also since not all log convex densities are completely monotone, not all geometrically infinitely divisible distributions have completely monotone densities.

Also we have (see Johnson et al. (1983), pages 422-423) the following result: Every Laplace transform of a measure on the non-negative half line is log convex. The Laplace transform of a probability measure on \((0, \infty)\) is a survival function because it can be thought of as a mixture of exponential survival functions and the associated densities are completely monotone. From Bondesson (1990), for a log convex density \(f\), the survival function \(\tilde{F}(x) = 1 - F(x)\) is log convex, which means that the hazard rate \(\frac{f(x)}{\tilde{F}(x)}\) is decreasing. Let now \(f\) be a decreasing density in the class of generalized gamma convolutions. It is completely monotone and hence log convex. From the proof of Theorem 6.2, of Bondesson (1990) it follows that \(\tilde{F}(uv)x\) \(\tilde{F}(u/v)\) is completely monotone in \(w = v + \frac{1}{v}\) and hence \(\tilde{F}\) is the Laplace transform of a generalized gamma convolution. Thus the hazard rate \(\frac{f(x)}{\tilde{F}(x)}\) is a Stieltjes transform of a nonnegative measure and the density \(f\) corresponds essentially to a random variable of the type \(X = \min X_i\) where \(X_i\)'s are independent and Pareto distributed.

5.5 STATISTICAL RELIABILITY AND GEOMETRIC INFINITE DIVISIBILITY—SOME RECENT APPLICATIONS.

The discussion of this section is based on Pillai (1990b).

Consider a fast repair model. We consider a system consisting
of two identical blocks 1 and 2 and a repair facility. The system starts operating with block 1 working. When it fails block 1 goes to repair and block 2 takes its place and the repair time lasts a random time $\eta$. With a small probability $p$, the repair of the failed block may last longer than the failure free operation of block 2. If this happens the system is considered as failed. With probability $(1-p)$, the block 1 will be renewed before block 2 fails. The block 1 takes the place of block 2 and block 2 returns to the stand by. Let us call a cycle the time interval which starts by putting block 1 into operation and which ends with the completion of its repair before operating block fails. Denote by $\xi$ the length of the interval, $\xi_1$, its independent copies. let $\bar{\eta}$ be the length of the interval, which starts by putting block 1 into operation and terminates by the system failure and does not contain any cycle. The time to system failure has the representation,

$$T = \xi_1 + \xi_2 + \ldots + \xi_N + \bar{\eta},$$

where $N_p$ is geometric random variable $G_p$ with

$$P(N_p=k) = p(1-p)^{k-1}, \quad k \geq 1.$$ If we set

$$T' = \xi_1 + \xi_2 + \ldots + \xi_N,$$

then $T$ is almost equal to $T'$, since the contribution of $\bar{\eta}$ is negligible.

More generally, let $\{\xi_i, \ i \geq 1\}$ be a regenerative process where $\xi_i = T_i - T_{i-1}, \ i = 1, 2, \ldots$ being instants of regeneration. Assume that the event $A$ can happen with probability $p$. Set
\[ Y = \xi_1 + \ldots + \xi_N, \] where \( N_p \sim G_p \). Then \( Y \) represents the duration of a regenerative process which is stopped at the end of that period on which the event \( A \) had happened for the first time. Here we see the conditions under which time to failure becomes geometric sum.

The hazard rate function is defined by

\[ h(x) = \frac{f(x)}{1-F(x)}, \] where \( F(x) \) is the lifetime distribution function and \( f(x) \) is its density.

Pillai (1990b) proves the following result.

If the hazard rate is completely monotone, then the lifetime is geometrically infinitely divisible. As an example he considers the Weibull distribution with \( \alpha \leq 1 \), \( h(x) = c \alpha x^{\alpha-1} \), which is completely monotone. Therefore, Weibull distribution is geometrically infinitely divisible.

A condition for the hazard rate to be completely monotone is provided by the following result. If \( \frac{F(x)}{\bar{F}(x)} \), where \( \bar{F}(x) = 1-F(x) \) has completely monotone derivative, then the hazard rate is completely monotone and hence the lifetime is geometrically infinitely divisible. For example, Pareto type I and III for \( \alpha \leq 1 \) are geometrically infinitely divisible. Pareto type I has \( \frac{F(x)}{\bar{F}(x)} = (1+c\alpha)^{-1} \), which is completely monotone for \( \alpha \leq 1 \). Pareto type III has \( \frac{F(x)}{\bar{F}(x)} = c\alpha \), which is completely monotone for \( \alpha \leq 1 \).
A random variable $X$ is a geometric minimum $(F,p)$ if $X = \text{Min} (X_1^p, \ldots, X_N^p)$, where $X_i$'s are independently identically distributed as $F$ and $N$ follows a geometric distribution with parameter $p$. A random variable $X$ is a universal geometric minimum if $X = \text{Min} (X_1^p, \ldots, X_N^p)$, where $X_i$'s are independently identically distributed as $F_p$ and $N_p$ follows a geometric distribution with parameter $p$ for all $p \in (0,1)$. Also, $F(x)$ is semi Pareto (type III) if

$$F(x) = \frac{1}{1 + \psi(x)}$$

where $\psi(x) = h(x) x^\alpha$, where $h(x)$ is periodic in $\log x$.

The following results hold.

(i) If $\frac{F'}{F}$ has completely monotone derivative, then the corresponding lifetime is a universal geometric minimum.

(ii) A geometric minimum $(F,p)$ is distributed as the same type as $F$ if and only if it is semi Pareto.

(iii) If $\frac{F'}{F}$ is completely monotone, then $F$ is the limit distribution of geometric minimum.

As an example of (iii), Pareto type III is the limit of a sequence of geometric sums of Weibull and Pareto type I is the limit of a sequence of geometric sums of truncated Weibull.

For some other results in this direction, see Harris (1983).
5.6 SUPERPOSITION OF RENEWAL PROCESSES AND GEOMETRIC INFINITE DIVISIBILITY

The superposition of two point processes $N_1$ and $N_2$ is the point process $N$ defined by

$$N(t) = N_1(t) + N_2(t), \quad t \geq 0$$

or we write,

$$N = N_1 + N_2 \quad \text{(Cinlar (1972))}.$$ 

We assume that no two events of $N_1$ and $N_2$ occur at the same time, that is there are no multiplicities.

The superposition of two renewal processes may be thought of as the process obtained by considering the renewal points of the two processes in the order in which they occur. It is not necessary that the superposition of two renewal process is renewal.

The earliest result in this direction is due to Mc Fadden (1962), which is given below.

**THEOREM 5.6.1**

Suppose that $N_1$ is a stationary renewal process with finite variance for the interrenewal times and let $N_2$ be a Poisson process independent of $N_1$. If $N = N_1 + N_2$ is a stationary renewal process, then $N_1$ is a Poisson process.

Many authors have treated the problem by assuming that the component processes are independent. But Cinlar and Agnew (1968)
considers the superposition of nonindependent processes. They use an indicator process which labels the points of the superposition process according to the components to which they belong. Let $N_1$, $N_2$ be two point processes and let $N = N_1 + N_2$. If $T_1, T_2, \ldots$ are the points of $N$, we define

$$X_n = k \text{ if and only if } T_n \in N_k, \quad k = 1, 2.$$ 

The process $X = \{X_1, X_2, \ldots\}$ is called the indicator process for $(N_1, N_2)$. The following result is known.

THEOREM 5.6.2

Suppose that $X$ is a Bernoulli process independent of $N$. Then, $N_1$ and $N_2$ are independent only if $N$ is a Poisson process (possibly non-stationary).

Here we are observing the problem in a different manner. We have $X$ which is distributed as Bernoulli ($p$) and $N$, a renewal process. Given $N$, the process $N_1$ ($N_2$) may be thought of as those process such that $N_1$ ($N_2$) is the $p$-thinned ($q$-thinned) processes of $N$.

We can relax the stationarity assumption and also that of finite variances and we consider a case other than Poisson.

Let us consider $N$ as a renewal process where interarrival times are distributed as Mittag-Leffler ($\alpha$), $0 < \alpha < 1$.

That is $\phi(\lambda) = \frac{1}{1 + \lambda^\alpha}$.
where \( \phi_p(\lambda) \) is the Laplace transform of the interarrival time distribution of the \( p \)-thinned process and \( \phi_p(\lambda) \) is also Mittag-Leffler (with a scale change). Since \( \phi(\lambda) \) is geometrically infinitely divisible it serves as the Laplace transform of the interarrival time distribution of a Cox and renewal process and this implies that \( \phi_p(\lambda) \) \( (\phi_q(\lambda)) \) also corresponds to renewal processes and they are also Mittag-Leffler. Hence we have,

**Theorem 5.6.3**

Suppose that \( X \) is Bernoulli, independent of \( N \) where \( N \) is renewal. If \( N \) has interarrival time distribution Mittag-Leffler, then \( N_1 \) and \( N_2 \) are Mittag-Leffler (with a scale change) and \( N_1 \) and \( N_2 \) are renewal.

Also we have,

**Theorem 5.6.4**

Suppose that \( N \) is a point process and \( I_p \) is the indicator process which is Bernoulli. Then \( N_1(p) \) and \( N_2(p) \) are renewal with respect to \( I_p \) for all \( p \in (0, 1) \) if and only if \( N \) has an interarrival time distribution which is geometrically infinitely divisible.

**Proof**

Suppose that \( N \) has a geometrically infinitely divisible interarrival time with Laplace transform \( \phi(\lambda) \).
Then \( N_1(p) \) has an interarrival time distribution with Laplace transform,

\[
\phi_1(\lambda) = \frac{p \phi(\lambda)}{1-q \phi(\lambda)} \quad \text{for all } p \in (0, 1).
\]

Hence \( \phi_1(\lambda) (\phi_2(\lambda)) \) is the Laplace transform of the interarrival time distribution of a Cox and Renewal process.

Now suppose that \( N, N_1 \) and \( N_2 \) are renewal with respect to \( I_p \) for all \( p \in (0,1) \).

Hence

\[
\phi_1(\lambda) = \frac{p \phi(\lambda)}{1-q \phi(\lambda)} \quad \text{for all } p \in (0,1)
\]

\[
\phi_2(\lambda) = \frac{q \phi(\lambda)}{1-p \phi(\lambda)} \quad \text{for all } q \in (0,1)
\]

Hence \( \phi(\lambda) \) is geometrically infinitely divisible.