Chapter 2

Theoretical Formalism

2.1 Introduction

Unlike the atomic interaction, where the interacting potential is well known electromagnetic interaction, the nature of the nuclear interaction potential is unfamiliar to us and that makes the nucleus complex and opaque to our understanding. Like atomic theory, nuclear theory is also based on models with approximate assumption accounting for the nature of the nuclear potential. History of the nuclear theory stretches back to 1911 when "Rutherford Atomic Model" was put forward. Since then different models have been proposed to describe different aspects of the nucleus. Apparently, a single theory still does not describe all the aspects of nucleus.

Since nucleus is a quantum system, to describe the nucleus mathematically, we
often chose the Schrodinger wave equation as a tool and solve it to get the physical information about the nucleus. The common approach for all nuclear reaction models are

(1) to construct a suitable interaction potential \( U \),

(2) to put it in a Schrodinger wave equation,

\[
\frac{d^2\psi}{dr^2} + \frac{2\mu}{\hbar^2} [E - U] \psi = 0 \tag{2.1}
\]

where \( \psi \) is the wave function describing the interacting system, \( \mu \) is the reduced mass and \( E \) is the energy of the system.

(3) to approximate many body problem and reduce it to a fewer body problem which is done by assuming Centre of Mass (CM) frame and considering only the relative motion between the different entities as the interaction potential depends on the relative co-ordinates only. Then the Schrodinger wave equation is solved with appropriate boundary conditions and the cross section for different channels are calculated.

Cross section \( (\sigma) \) is defined as the probability of a particular event to take place; i.e,

\[
\sigma = \frac{\text{Event per nucleus per sec}}{\text{incident flux}}
\]

Angular dependence is obtained by measuring differential cross section which is defined as,
Figure 2.1: Scattering from a nuclear potential. The Phase shift is measured in the region beyond the influence of the potential.

\[
\frac{d\sigma}{d\Omega}(\theta) = \frac{\text{No. of particles generated in the particular exit channel}}{(\text{No. of incident particles/area})(\text{No. of target nuclei within the beam})}
\]

The shape of the angular distribution of the cross section gives vital information about the interaction.
Our work is mainly concerned with the elastic scattering and the charge exchange reaction. The elastic scattering is a potential scattering and hence carries the information about the nuclear potential. In terms of wave theory, the elastic process is like reflection in presence of nuclear potential (fig. 2.1) which changes the wavelength inside the nucleus. The phase shift $\delta$ between the incident wave and the scattered wave, which is measured in the asymptotic region, carries the information about the potential and cross section can be calculated from it as,

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{2ik} \sum_{L=0}^{\infty} (2L + 1)(e^{2i\delta_L} - 1)P_L(cos\theta) \right|^2$$  \hspace{1cm} (2.2)

Where the term under modulus is called the scattering amplitude. Here $k$ is the wave number, $L$ is the angular momentum carried by the partial wave and $\delta_L$ is the phase shift of the $L^{th}$ partial wave.

The next two sections are on elastic scattering, describing the interaction potential in the light of Optical model (OM) and the formal theory of scattering in terms of quantum mechanics respectively. The theoretical codes used for analysis of the present work are described in the last section. In this section the charge exchange reaction is explicitly treated along with the elastic channel.

### 2.2 Optical Model:

In any nuclear interaction with a given incident flux, part of the flux goes to the elastic channel and rest goes to the other non-elastic channels, depending on the
energy and the structure of the interacting nuclei. In treating the elastic channel explicitly, while taking into consideration of the loss of flux from the elastic channel, a simple model called Optical model has been widely used. This was initially used to analyse the nucleon-nucleus elastic scattering and later extended for heavy ion elastic scattering. In Optical model, a nucleus is assumed to be a semi-transparent sphere with a refractive (real) and an absorptive (imaginary) potential,

\[ V(r) = U_0(r) + iW(r) \quad (2.3) \]

similar to the representation of semi-transparent medium in Optics with a real and imaginary refractive indices which account for the refraction and absorption of the light in the medium.

In the complex potential \( V(r) \) in eqn.(2.3), the real part \( U_0(r) \) represents the potential which causes the elastic scattering and the imaginary part \( W(r) \), called absorptive potential, takes care of all the non-elastic processes and accounts for the loss of flux from the elastic channel. Depending upon the energy and structure of the interacting nuclei, the absorption may be of surface type or volume type or a combination of the both. Again, spin-orbit interaction between the nuclei also sometimes plays a significant role in scattering or absorption. A general optical model potential, taking into account the Coulomb interaction, surface and volume absorption and the spin orbit coupling effect, is usually written as,

\[ V(r) = V_c(r) - U_0(r)f(r) - iW_v f(r) + 4aiW_D f'(r) - (U_s + iW_s)h(r)\vec{E} \cdot \vec{\sigma} \quad (2.4) \]
Where,

\[
V_c(r) = \begin{cases} 
\frac{Z_1 Z_2 e^2}{2R_c} [3 - (r/R_c)^2], & \text{for } r < R_c = \frac{\tau_0 (A_1^{1/3} + A_2^{1/3})}{2} \\
\frac{Z_1 Z_2 e^2}{r}, & \text{for } r > R_c 
\end{cases}
\]

(2.5)
is the Coulomb potential for a uniformly charged sphere of radius \(R_c\). \(f(r)\) is the radial form factor, usually taken as the Wood-Saxon type:

\[
f(r) = \frac{1}{1 + \exp[(r - R)/a]}
\]

(2.6)

Where, \(R = \tau_0 (A_1^{1/3} + A_2^{1/3})\) and ‘a’ is called the diffuseness parameter. \(f'(r)\) is the derivative of \(f(r)\). \(W_D\) and \(W_v\) in eqn.(2.4) are the surface and volume absorption parts of the absorptive potential.

The effect of spin-orbit interaction is included by the term

\[
V_{so}(r) = (U_s + iW_s)h(r)\vec{\ell}\vec{\sigma}
\]

where \(U_s\) and \(W_s\) are the real and imaginary parts of the spin-orbit potential and \(\vec{\ell}\) and \(\vec{\sigma}\) are the orbital angular momentum and spin of the projectile respectively. \(\vec{\ell}\) and \(\vec{\sigma}\) are the orbital angular momentum and spin of the projectile respectively, and, \(\h(r) = -\frac{1}{r}f'(r)\).

As the energy of the system increases, more and more non-elastic channels open up, taking more and more flux from the elastic channel to other channels. Thus, OM potential depends on the energy of the system. Isospin dependence of the OM potential, which is already stated in the first chapter, is now a well established
fact. The potential is also different for charged and uncharged particles, even if we include the coulomb interaction separately in the total interaction potential. This was first observed in proton case and now known as "proton potential anomaly" [9, 10]. A Coulomb correction to the OM potential could solve the anomaly. Taking into account of all these, the real part of the nuclear potential can be written as,

$$U(r) = U_0 - \beta E \pm 4U_1(T_{tot})[t/T/A] + \alpha \frac{Z_1Z_2}{A_1^{1/3} + A_2^{1/3}}$$  \hspace{1cm} (2.6)

Here the first term $U_0$ is similar to the shell model potential (for single nucleon projectile); the second term $\beta E$ clearly indicates the energy dependence of the potential which decreases with the increasing energy of the projectile. However, this non-locality of the potential can also be represented by an equivalent local potential [9]. When a charged particle enters into the potential field of the target, the kinetic energy of the particle decreases due to the Coulomb repulsion and thus potential felt by the particle becomes more attractive than if it were not a charged one. The last term in eqn.(2.6) stands for the Coulomb correction for a charged particle [12, 9].

The third term in eqn.(2.6), which is of our interest, is the representation of the isospin dependence of the optical model potential. Here, $t$ is the isospin of the projectile, $T$ is the isospin of the target and $T_{tot}$ is the total isospin of the system which is the algebraic sum of the isospin of the projectile and the target.
2.2.1 The Isobaric Spin Dependent Term:

In the concluding remark of his famous paper, where he introduced 'the isobaric spin dependent' term in the background of a proton scattered by a target, A. M. Lane writes: “The introduction of isobaric spin dependence into the optical potential has important consequences. Proton data are modified by the splitting of particle levels. Associated with this is the occurrence of proton scattering with charge loss i.e, the incident proton changes into neutron while the target state is converted into its isobaric counterpart. We refer to this as quasi-elastic (p,n) process” [1].

Taking the simplest case of a proton (t_z = -1/2) incident on a target nucleus of isospin T, having a uniform nuclear density \( \rho \), the nuclear part of the potential felt by the proton can be written as,

\[
U_p = \left( \sum_i v_{ip} \right) = \left( \frac{N}{A} \right) \rho v_{np} + \left( \frac{Z}{A} \right) \rho v_{pp} = U_0 - U_1 \epsilon \tag{2.7}
\]

and

\[
U_n = \left( \sum_i v_{in} \right) = \left( \frac{Z}{A} \right) \rho v_{np} + \left( \frac{N}{A} \right) \rho v_{nn} = U_0 + U_1 \epsilon \tag{2.8}
\]

Where \( U_0 = \frac{1}{2} \rho (v_{pp} + v_{np}) \) and \( U_1 = \frac{1}{2} \rho (v_{pp} - v_{np}) \) and \( v_{pp} = v_{nn} \); \( \epsilon = (N-Z)/A \) is the asymmetry parameter.

In the Lane equation

\[
U(r) = U_0(r) + 4U_1(r)t_T/A \tag{2.9}
\]
the $t_T$ term can be expressed as

$$t_T = \frac{1}{2}(t_+T_+ + t_-T_-) + t_zT_z$$

(2.10)

where $t_\pm = t_x \pm i t_y$ and $T_\pm = T_x \pm i T_y$. Isospin operators follow the same algebra like spin operators.

Taking the simple case of $t=\frac{1}{2}$, the potential equation in terms of isospin representation can be written as,

$$\langle t_z = \pm \frac{1}{2}, T_z = \frac{1}{2}(N - Z) \mid U \mid t_z = \pm \frac{1}{2}, T_z = \frac{1}{2}(N - Z) \rangle \equiv U_0 \pm U_1 \epsilon$$

(2.11)

This can be shown that the diagonal matrix elements of eqn.(2.11) is related to $U_0$ and the off diagonal part $t_+T_-$ (+ for the raising operator, - for the lowering operator) induces the occurrence of $(p,n)$ reaction and is related to the asymmetry potential $U_1$. The off diagonal matrix elements of eqn.(2.11) can be represented by,

$$\langle t_z = + \frac{1}{2}, T_z = \frac{1}{2}(N - Z) - 1 \mid U \mid t_z = - \frac{1}{2}, T_z = \frac{1}{2}(N - Z) \rangle = 2U_1(\epsilon/A)^{1/2}$$

(2.12)

when the target isospin has its minimum value ($T = T_z = \frac{1}{2}A\epsilon$)

The single particle state of the projectile may split into multi-states as the total
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Isospin of the system (projectile + target) is $T_{\text{tot}} = T \pm t$ [for $t = 1/2$, the difference between the two states is $2(2T + 1) \frac{V_t}{A}$]. This is another consequence of isospin effect.

Not only the real part of the optical potential but the imaginary part of the same may also depend on the isospin of both the projectile and the target in the same way as does the real part, i.e;

$$W = W_0 \pm W_1(T_{\text{tot}})T/T/A$$

The immediate consequence of this is that the isovector part of $W$ may also contribute to the charge exchange transition between analog states.

Experimental evidences show that the isospin part of the real potential $V_1$ is less than $V_0$ [9]. The imaginary isospin part $W_1$ is still less than $V_1$. Thus a very sophisticated measurement is required to determine the strength of these potentials.

\subsection{Ambiguities in Optical Model}

Optical model, inspite of its great success in fitting the experimental data well over a large number of projectile-target combinations has some ambiguities in itself like normalisation ambiguity, valley ambiguity, potential depth ambiguity, form factor ambiguity [11]. The ambiguities arise mainly due to the fact that the elastic cross section depends only on the scattering phase shift and not on the wave function. Because of this, a several sets of OM potential parameters can produce the same phase shift and hence can fit the same experimental data well. However, physical
acceptability of the parameters can reduce the number of such sets.

2.3 Scattering theory:

Nucleus with its tiny dimension of a few Fermi falls in the domain of quantum physics, i.e., it exhibits both wave and particle features. But depending upon the energy and size of the nucleus, i.e., depending upon the de Broglie wavelength \( \lambda = \frac{h}{mv} \), it can be treated in a classical or quantum mechanical manner.

2.3.1 Nuclear scattering: Quantum Mechanical approach

The general approach for nuclear scattering theory is by using quantum mechanics. However, for heavy ion reactions at high energy, when the de Broglie wave length becomes smaller than the size of the nucleus, semi-classical approach may be used (i.e., classical approach for scattering and quantum for the discrete bound states). But for light and light-medium ions at low energy, the proper approach is always quantum mechanical. (In this chapter the notations and formulae are mostly taken from ref 6 and 8.)

In quantum mechanical treatment, a projectile is considered to be a plane wave and represented by,

\[ \psi_i = e^{ikz} \]  

(2.13)
where $k$ is the wave vector ($= \frac{2\pi}{\lambda}$) and $z$ is the distance along the direction of propagation. The projectile should actually be treated as wave packet. However, the final results in scattering theory, turns out to be the same, even if the plane waves are used.

After the plane wave interacts with the target, the part of the plane wave experiencing nuclear potential produces secondary spherical wavelets while most of the original waves remain unscattered (reference to fig. 2.2). Thus the wave function at the exit channel consists of both plane wave and the spherical wave which dies out with distance.

Assuming a pure elastic scattering, the wave equation for the scattering, far from the scattering centre can be written as
\[ \psi_i = e^{ikr} + f(\theta) \frac{e^{ikr}}{r} \quad (2.14) \]

where the first term on R.H.S. represents the unaffected plane wave, while the second term represents the spherical wave generated during interaction in the nuclear field of the target whose amplitude reduces with the radial distance \( r \). The coefficient \( f(\theta) \) in the second term measures the strength of interaction and is called the scattering amplitude. The square of the scattering amplitude is proportional to the scattering cross section:

\[ \frac{d\sigma}{d\Omega} = \left( \frac{v_\beta}{v_\alpha} \right) \left| f_{\alpha\beta}(\theta) \right|^2 \quad (2.15) \]

where \( v_\beta \) and \( v_\alpha \) are the velocities in the entrance channel \( \alpha \) and the exit channel \( \beta \). For elastic scattering \( v_\beta = v_\alpha \), and hence

\[ \frac{d\sigma}{d\Omega} = \left| f(\theta) \right|^2 \quad (2.16) \]

The scattering amplitude is also related to transition amplitude \( T_{\alpha\beta} \) as,

\[ T_{\alpha\beta} = -\frac{2\pi \hbar^2}{\mu_\beta} f_{\alpha\beta} \quad (2.17) \]

Here \( \mu_\beta \) is the reduced mass in the channel \( \beta \) and \( f_{\alpha\beta} \) is the scattering amplitude from channel \( \alpha \) to \( \beta \).

The charged particle interaction involves Coulomb force, which is so far ne-
glected. The effect of electromagnetic interaction between the charged particle projectile and the target is included by including the Coulomb potential in the total interaction potential. Coulomb potential also modifies the total wave function which can be done by replacing in eqn.(2.14)

\[(k\cdot r) \rightarrow [k\cdot r + \eta \ln(kr - r\cdot r)]\]

\[(kr) \rightarrow (kr - \eta \ln(2kr))\]

where \(\eta = \frac{Z_\beta e^2}{\hbar c}\) is the Sommerfeld parameter for exit channel \(\beta = b + B\).

For \(\alpha \rightarrow \beta\), through an interaction \(V_\beta\) the exact expression for transition amplitude can be written as

\[T_{\alpha\beta}(k_\beta, k_\alpha) = \langle e^{ik_\beta r_\beta} \psi_\beta | V_\beta | \Psi_\alpha^{(+)}(k_\alpha) \rangle\] (2.18)

Where \(\Psi_\alpha^{(+)}\) is the total wave function [6, 8], with outgoing wave boundary condition, defined as,

\[\Psi_\alpha^{(+)} = \sum_\gamma \xi_\gamma \psi_\gamma\] (2.19)

here, \(\xi_\gamma\) is the relative wave function and \(\psi_\gamma\) is the internal wave functions in channel \(\gamma\)
Partial wave analysis of scattering

In partial wave analysis the incident wave is considered to be composed of several partial waves denoted by $L=0,1,2,...$ and each partial wave contributes towards scattering amplitude and hence towards cross section. The total scattering amplitude at a particular angle $\theta$ due to all partial wave is,

$$f(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L + 1)(e^{2i\delta_L} - 1)P_L(\cos \theta)$$  \hspace{1cm} (2.20)

where $\delta_L$ is the phase shift for the $L^{th}$ partial wave.

2.4 Codes for theoretical Analysis

For the optical Model analysis we have used the optical model code 'Snoopy' [15] while for the charge exchange reaction analysis we have used the isospin coupled code called 'TWAR' [12] which was originally written by S. Cotanch [4] and modified locally by us. Here, t.T potential (Lane potential) is used in the potential part. Apart from that the real and imaginary potential of the OM potential are also considered to be isospin dependent.
2.4.1 Snoopy

It is an optical model code for elastic scattering analysis written by P. Schwandt, Dept. of Physics, Indiana University, Bloomington. Snoopy calculates the elastic differential and reaction cross sections for a projectile of spin 0, 1/2 or 1 and a target whose spin is ignored. It also performs the calculation of the vector and tensor analyzing powers and spin rotation parameters. Snoopy does not incorporate the isospin dependent part of the OM potential.

The potential used in the Schrodinger equation consists of Coulomb potential \( V_c \), real nuclear potential \( U_R \), the imaginary volume and surface potentials \( W_v \) and \( W_D \) and a complex spin-orbit potential. Snoopy has a search routine inbuilt for automatic search of the OM parameters to minimise the \( \chi^2 \) value [ Here, \( \chi^2(A) \) for each experimental quantity A is defined as \( \chi^2(A) = \sum \frac{(A_{\text{scale}}-A_{\text{meas}})^2}{A_{\text{A}}} \), where sum is over all appropriate data points ]. The Schrodinger wave equation for partial wave \( f_{lj} \) is solved numerically by Fox-Goodwin method [15, 16] to get the scattering amplitude \( \alpha_{lj} \) and hence the cross section. \( l \) and \( j \) represent the partial wave and the total angular momentum respectively.

2.4.2 TWAVE

This program was originally written in 1973 specially including the t.T potential, for charge exchange reaction of the type \((p,n)\), \((t,{^3He})\) etc. In this code real and imaginary parts of OM potential are also considered to be dependent on isospin.

The following is a brief description of the formalism used in TWAVE.
Isospin plays a significant role in the nuclear interaction potential and affects the elastic and non-elastic channels. So both real and imaginary potentials can be considered as function of isospin \([4]\). If \(t_a\) and \(T_A\) are the isospins of the projectile and the target respectively, there are two possible values of the total isospin \(T\),

\[
T > = T_A + t_a \quad (T \text{ upper state})
\]

\[
T <= T_A - t_a \quad (T \text{ lower state})
\]

The coupled equations formed explicitly for elastic and charge exchange channels are,

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2\mu}{\hbar^2} (E_p - U_{ip}^p(r)) \right] f_{ij}^p(r) = \frac{2\mu}{\hbar^2} U_{pn}(r) f_{ij}^n(r)
\] (2.21)

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2\mu}{\hbar^2} (E_n - U_{in}^n(r)) \right] f_{ij}^n(r) = \frac{2\mu}{\hbar^2} U_{np}(r) f_{ij}^p(r)
\] (2.22)

with

\[
U_{ip}^p(r) = \alpha U_<(r) + \beta U_>(r) + V_{s.o.}(r) a^V + V_{Coul}(N_A, -1/2)
\] (2.23)

\[
U_{in}^n(r) = \beta U_<(r) + \alpha U_>(r) + V_{s.o.}(r) a^V + V_{Coul}(N_A - 1, 1/2)
\] (2.24)

\[
U_{pn}(r) = U_{np}(r) = (\alpha \beta)(U_> - U_<)
\] (2.25)
where,
\[
\alpha = \frac{(T_A + N_A)}{(2T_A + 1)} \quad (2.26)
\]
\[
\beta = \frac{(T_A - N_A + 1)}{(2T_A + 1)} \quad (2.27)
\]
\[
a^{ij} = 1/2(j(j + 1) - l(l + 1) - 3/4) \quad (2.28)
\]

Here, \(N_A\) is the Z-component of isospin of the target. The \(U<(r)\) and \(U>(r)\) are defined as
\[
U>(r) = f>(r) + \frac{T_A}{2A} U_1(T>) g>(r) \quad (2.29)
\]
\[
U<(r) = f<(r) + \frac{(T_A + 1)}{2A} U_1(T<) g<(r) \quad (2.30)
\]

where \(f(r)\) and \(g(r)\) are the radial form factors. The relation between \(E_p\) and \(E_n\) is
\[
E_n = E_p - \Delta_c \quad (2.31)
\]

and
\[
\Delta_c = \Delta E_c(T_A, N_A) + \Delta E_c(t, n) \quad (2.32)
\]

\(\Delta E_c\) represents the Coulomb displacement energy [17]

Similar are the form of the potentials for imaginary part. However, \(|W<|>|W>|\) according to the isospin conservation. Again from the symmetry energy \((E>-E<)\) and Pauli exclusion principle, \(|U>|>|U<|\).
The Coupled equations (2.21) and (2.22) are solved numerically. The method of solutions can be found in ref. [16].

TWAVE has been modified to incorporate the symmetric system like $^7\text{Li}+^7\text{Li} = ^7\text{Li} + ^7\text{Li}$ and also the asymmetric system like $^7\text{Be}+^7\text{Li} = ^7\text{Be} + ^7\text{Li}$. In the former case, the indistinguishability of the final products in the elastic channel from the scattered and recoiled particles and in the later case, indistinguishability of the final products in the elastic channel from direct elastic and charge exchange reaction are taken into account.

For identical particle scattering like $^7\text{Li}+^7\text{Li} = ^7\text{Li} + ^7\text{Li}$, the particles in the exit channels are indistinguishable with respect to the ejectiles or the recoiled from the target and the scattering amplitude (or cross section) is the coherent superposition of the scattering amplitude (or cross section) from both ejectiles and recoiled particles at any given angle. Mathematically,

$$f_s(\theta) = f(\theta) + f(\pi - \theta) \quad (2.33)$$

and so

$$\frac{d\sigma}{d\Omega} = |f(\theta) \pm f(\pi - \theta)|^2 \quad (2.34)$$

where + sign is for bosons (no spin) and - sign is for fermions (with spins). In partial wave expansion of eqn.(2.20) and using the property

$$P_L(\cos(\pi - \theta)) = (-)^LP_L(\cos\theta) \quad (2.35)$$
one can see that only the even partial waves contribute to the symmetric scattering amplitude $f_s(\theta)$

![Diagram](image)

**DIRECT SCATTERING**  
**ELASTIC CHARGE EXCHANGE**

Figure 2.3: Direct scattering and charge exchange reaction

On the other hand, for the scattering of the asymmetric system like $^7\text{Be} + ^7\text{Li} = ^7\text{Be} + ^7\text{Li}$, presence of charge exchange channel again can contribute to the scattering cross section. The exit channel scattering cross section is indistinguishable with respect to the direct elastic cross section or charge exchange cross section. In fig. 2.3, A and B are two mirror nuclei, with inert core ($N=Z$) (shown in CM frame) undergoing direct elastic scattering and charge exchange reaction. By charge exchange each nucleus inverts its identity, that is A becomes B and vice versa. [in fig. 2.3, $A=A'$ and $B=B'$] The cross section at any angle $\theta$ is contributed from the coherent superposition from both elastic and charge exchange channels but can not be distinguished whether A (or B) is from direct elastic channel or charge exchange channel. Thus,

$$f_{\text{asy}}(\theta) = f_d(\theta) + f_{ce}(\pi - \theta) \quad (2.36)$$
and

\[ \frac{d\sigma}{d\Omega} = |f_d(\theta) \pm f_{ce}(\pi - \theta)|^2 \]  \hspace{1cm} (2.37)

Here, \( f_d \) is the direct scattering amplitude, \( f_{ce} \) is the exchange amplitude and \( f_{asym}(\theta) \) is the asymmetric amplitude.
Bibliography


