CHAPTER IX
EFFECTS OF MHD MIXED CONVECTIVE FLOW WITH OHMIC HEATING AND VISCOS DISSIPATION OVER A CONTINUOUSLY MOVING INCLINED PLATE

9.1 INTRODUCTION

Interest in the study of mixed convection flow in porous and non-porous vertical channels, is motivated by the importance of it in a wide range of engineering applications, such as in many situations like moisture migration through air which is contained in fibrous insulations, cooling of electronic instruments and devices, electrochemical processes, and in solar energy collector. On the other hand, natural convection flow occurs frequently in nature. It occurs due to temperature differences, as well as due to concentration differences or the combination of these two, for example in atmospheric flows, there exists differences in water concentration and hence the flow is influenced by such concentration difference.

The effects of free-stream oscillations on the flow past a semi-infinite plate were first studied by Lin (1957) for finite amplitude and by Lighthill (1974) for small amplitude oscillations. Lighthill has studied this problem by employing momentum integral method. Hill and Stenning (1960) have verified these results experimentally. Free convection effects on the oscillatory flow past an infinite vertical porous plate with constant suction was initiated by Soundalgekar (19731, 19732). In both the papers, suction was assumed to be constant. In all these studies, the plate temperature was assumed to be constant and hence isothermal.

Unsteady free convection flows through a porous medium have been studied by Raptis (1983). Oscillatory flow through a porous medium in the presence of free convective flow has been analyzed by Raptis and Perdikis (1985). Hossain (1992) has computed viscous and Joule heating effects on MHD free convection flow with variable
plate temperature. Hopper et al. (1994) have measured mixed convection along an isothermal vertical plate in porous medium with injection and suction. Soundalgekar et al. (1997) have examined free convection effects on MHD flow past an infinite vertical oscillating plate with constant heat flux. MHD unsteady free convection flow past a vertical porous plate has been discussed by Helmy (1998). Hossain et al. (1999) have assessed the effect of radiation on free convection from a porous vertical plate. Jaiswal and Soundalgekar (2001) have investigated unsteady free and forced convection MHD flow past an infinite vertical porous plate with variable suction and oscillating plate temperature. Hydromagnetic unsteady mixed convection flow past an infinite vertical porous plate has been discussed by Sharma et al (2011). Choudhury and Dey (2010) have analyzed free convective visco-elastic flow with heat and mass transfer through a porous medium with periodic permeability.

The effects of viscous dissipation and Ohmic heating are very much necessary to explore the impact of the magnetic field on the thermal transport in the boundary layer. With this awareness, combined heat and mass transfer in MHD free convection from a vertical surface with ohmic heating and viscous dissipation have been measured by Chien-Hsin-Chen (2004). The problem of steady laminar magneto hydrodynamic (MHD) mixed convection heat transfer about a vertical plate is solved numerically by Aydin and Kaya (2009) taking into account the effect of ohmic heating and viscous dissipation. Mixed convective visco-elastic MHD flow with Ohmic heating has been discussed by Choudhury and Das (2013).

The present investigation deals with the effects of ohmic heating and viscous dissipation on elastico-viscous fluid past a vertical plate embedded on a porous medium and subjected to oscillating suction and temperature field.

### 9.2 MATHEMATICAL FORMULATION

We consider the flow of an electrically conducting visco-elastic incompressible fluid over a continuously moving inclined surface through a porous medium bounded by an infinite vertically porous flat plate. The \( \vec{x} \)-axis is taken along the plate, being the vertically upward direction of the flow and \( \vec{y} \)-axis is taken perpendicular to the plate directed into the fluid. The fluid flows with uniform free stream velocity \( U \). Heat is supplied to the plate at the constant rate. A uniform magnetic field \( \vec{B} \) is imposed along
the \( \vec{y} \)- axis. The inclination of the angle is assumed to so small that \( \sin \alpha = 0 \). The physical model of the problem is shown in Figure 9.1.

Since the plate is considered infinite in the \( \vec{x} \)-direction, hence all the fluid properties are independent of \( \vec{x} \). Let \( \vec{u}, \vec{v} \) be the fluid velocities along \( \vec{x}, \vec{y} \)-axes respectively and the plate temperature \( \vec{T} \) is oscillating about a non-zero plate temperature \( \vec{T}_0 \). The induced magnetic field is negligible which is possible on a laboratory scale. The variation of the suction velocity distribution is of the form

\[
\vec{v}(\vec{t}) = -V(1 + e^{i\omega \vec{t}})
\]

(9.2.1)

where \( V > 0 \) is the constant mean velocity and \( < 1 \), the negative sign in equation (9.2.1) indicates that the suction is towards the plate. Then under usual Boussinesq’s approximation, the magnetohydrodynamic flow in the porous medium is governed by the following differential equations

\[
\frac{\partial \vec{v}}{\partial \vec{y}} = 0
\]

(9.2.2)

\[
\frac{\partial \vec{u}}{\partial \vec{t}} + \vec{v} \frac{\partial \vec{u}}{\partial \vec{y}} = \nu \frac{\partial^2 \vec{u}}{\partial \vec{y}^2} + \frac{k_0}{\rho} \left( \frac{\partial^3 \vec{u}}{\partial \vec{t} \partial \vec{y}^2} + \vec{v} \frac{\partial^3 \vec{u}}{\partial \vec{y}^2 \partial \vec{y}^2} \right) + g \beta \cos \alpha (\vec{T} - \vec{T}_0) + g \beta \cos \alpha (\vec{C} - \vec{C}_0)
\]

\[
+ \frac{\sigma B_0^2}{\rho} (U - \vec{u}) + \nu \frac{\nu}{K} (U - \vec{u})
\]

(9.2.3)

\[
\rho c_p \left( \frac{\partial \vec{T}}{\partial \vec{t}} + \vec{v} \frac{\partial \vec{T}}{\partial \vec{y}} \right)
\]

\[
= \kappa \frac{\partial^2 \vec{T}}{\partial \vec{y}^2} + \mu \left( \frac{\partial \vec{u}}{\partial \vec{y}} \right)^2 + \sigma B_0^2 \vec{u}^2 + k_0 \left( \frac{\partial \vec{u}}{\partial \vec{y}} \frac{\partial^2 \vec{u}}{\partial \vec{t} \partial \vec{y}} + \vec{v} \frac{\partial \vec{u}}{\partial \vec{y}} \frac{\partial^2 \vec{u}}{\partial \vec{y}^2 \partial \vec{y}^2} \right)
\]

\[
- \frac{\partial \vec{q}_r}{\partial \vec{y}}
\]

(9.2.4)

\[
\frac{\partial \vec{C}}{\partial \vec{t}} + \vec{v} \frac{\partial \vec{C}}{\partial \vec{y}} = D \frac{\partial^2 \vec{C}}{\partial \vec{y}^2}
\]

(9.2.5)

In these equations \( \rho \) is the density, \( \vec{t} \) is the time, \( \nu \) is the kinematic viscosity, \( g \) is the acceleration due to gravity, \( \beta \) is the coefficient of volume expansion, \( K \) is the permeability of the porous medium, \( \vec{T} \) is the characteristic temperature of the fluid, \( \vec{T}_0 \) is the temperature of the fluid far away from the plate, \( \vec{C} \) is the species concentration, \( \vec{C}_0 \) is the species concentration in the free stream, \( D \) is the co-efficient of chemical molecular diffusivity, \( \sigma \) is the scalar electrical conductivity, \( U \) is the uniform velocity of the fluid in the upward direction, \( C_p \) is the specific heat at constant pressure, \( \kappa \) is the
thermal conductivity, $\mu$ is the coefficient of viscosity, $\bar{q}_r$ is the radiative heat flux. The plate being infinite in length, the flow variables are functions of $\bar{y}$ and $\bar{t}$ only.

The radiative heat flux $q_r$ [Cogley et al. (1968)] is given by

$$\frac{\partial \bar{q}_r}{\partial \bar{y}} = 4(\bar{T} - T_\infty)l$$

where

$$l = \int_0^\infty K_{Aw} \frac{\partial e_{b\lambda}}{\partial T}\,d\lambda,$$

where $K_{Aw}$ is the absorption coefficient at the wall and $e_{b\lambda}$ is Planck’s function.

The initial boundary conditions of the problem are

$$\begin{align*}
\bar{y} = 0 & : \bar{u} = 0, \bar{T} = \bar{T}_w + \varepsilon(\bar{T}_w - \bar{T}_\infty)e^{i\sigma\tau}, \bar{C} = \bar{C}_w + \varepsilon(\bar{C}_w - \bar{C}_\infty)e^{i\sigma\tau} \\
\bar{y} \to \infty & : \bar{u} = U, \bar{T} = \bar{T}_\infty, \bar{C} = \bar{C}_\infty
\end{align*}$$

(9.2.6)

Here $\sigma$ is the frequency of the plate temperature oscillations and $\bar{T}_w$ is the temperature of the plate. The subscripts $w$ and $\infty$ denote physical quantities at the plate and in the free stream respectively.

We introduce the following non-dimensional quantities

$$y = \frac{\bar{y}V}{\nu}, t = \frac{\bar{t}V^2}{\nu}, u = \frac{\bar{u}}{U}, \omega = \frac{\nu}{V^2}, \theta = \frac{\bar{T} - \bar{T}_\infty}{\bar{T}_w - \bar{T}_\infty}, \phi = \frac{\bar{C} - \bar{C}_\infty}{\bar{C}_w - \bar{C}_\infty},$$

$$K = \frac{KV^2}{\nu^2}, M = \frac{\sigma B_0^2 V}{\rho V^2}, Pr = \frac{\mu C_p}{\kappa}, Gr = \frac{\varepsilon g(\bar{T}_w - \bar{T}_\infty)}{UV^2},$$

$$Gm = \frac{\varepsilon g(\bar{C}_w - \bar{C}_\infty)}{UV^2}, Ec = \frac{U^2}{C_p(\bar{T}_w - \bar{T}_\infty)}, Sc = \frac{\nu}{D}, F = \frac{4\nu l}{\rho C_p V^2}, k = \frac{k N V^2}{\rho \eta^2}$$

where $\theta$ is the non-dimensional temperature, $\phi$ is the non-dimensional concentration, $K$ is the permeability parameter, $M$ is the Hartmann number, $Pr$ is the Prandtl number, $Gr$ is the Grashof number for heat transfer, $Gm$ is the Grashof number for mass transfer, $Ec$ is the Eckert number, $Sc$ is the Schmidt number, $F$ is the radiation parameter, $k$ is the visco-elastic parameter; in the equations (9.2.2) to (9.2.5) and the non-dimensional governing equations are given by
\[ \frac{\partial u}{\partial t} - (1 + \varepsilon e^{i\omega t}) \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \left[ \frac{\partial^3 u}{\partial t \partial y^3} - (1 + \varepsilon e^{i\omega t}) \frac{\partial^3 u}{\partial y^3} \right] Gr \cos \theta + Gmc \cos \phi \\
+ M(1 - u) - \frac{1}{K} (1 - u) \]  
(9.2.7)

\[ \frac{\partial \theta}{\partial t} - (1 + \varepsilon e^{i\omega t}) \frac{\partial \theta}{\partial y} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^2} + Ec \left( \frac{\partial u}{\partial y} \right)^2 - Eck \left( \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial t \partial y} - \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \right) - F \theta \\
+ Ec Mu^2 \]  
(9.2.8)

\[ \frac{\partial \phi}{\partial t} - (1 + \varepsilon e^{i\omega t}) \frac{\partial \phi}{\partial y} = \frac{1}{Sc} \frac{\partial^2 \phi}{\partial y^2} \]  
(9.2.9)

subject to boundary conditions:

\[ \begin{align*}
y = 0: u &= 0, \theta = 1 + \varepsilon e^{i\omega t}, \phi = 1 + \varepsilon e^{i\omega t} \\
y \to \infty: u &= 1, \theta = 0, \phi &= 0
\end{align*} \]  
(9.2.10)

9.3 METHOD OF SOLUTION

When the amplitude \( \varepsilon \) is very small i.e. \( \varepsilon \ll 1 \) then in order to solve the coupled non-linear differential equations (9.2.7) to (9.2.9), we assume that the solution in the neighbourhood of the plate that is the unsteady flow is superimposed on the mean steady flow and is represented mathematically in the forms:

\[ u(y,t) = u_0(y) + \varepsilon e^{i\omega t} u_1(y) \]
\[ \theta(y,t) = \theta_0(y) + \varepsilon e^{i\omega t} \theta_1(y) \]
\[ \phi(y,t) = \phi_0(y) + \varepsilon e^{i\omega t} \phi_1(y) \]  
(9.3.1)

Substituting (9.3.1) into equations (9.2.7)-(9.2.9), equating the coefficients of harmonic and non-harmonic terms, neglecting the coefficients of like powers of \( \varepsilon^2 \) and higher powers of \( \varepsilon \), we get

\[ ku_0'''' + u_0'' + u_0' - \left( M + \frac{1}{K} \right) u_0 = -Gr \cos \theta_0 - Gmc \cos \phi_0 - \left( M + \frac{1}{K} \right) \]  
(9.3.2)

\[ \theta_0'' + Pr \theta_0' - FPr \theta_0 = -Ec Pr u_0'' - Ec Pr u_0' u_0'' - Ec Pr Mu_0^2 \]  
(9.3.3)

\[ \phi_0'' + Sc \phi_0' = 0 \]  
(9.3.4)
\[ u_1'' + u_1' + k(u_1''' - i\omega u_1'' + u_0''' - \left( M + \frac{1}{K} + i\omega \right) u_1 \]
\[ = -u_0' - Gr\cos\alpha \theta_1 - Gm\cos\alpha \phi_1 \]
\[ \theta_1'' + Pr\theta_1' - (F + i\omega)Pr\theta_1 \]
\[ = -Pr\theta_0' - 2EcPr u_0' u_1' + EcPrk(u_0'' u_1' - u_0' u_1'' - u_1'' u_0') \]
\[ - 2Pr Ec M u_0 u_1 \]
\[ \phi_1'' + Sc \phi_1' - i\omega Sc \phi_1 = -Sc \phi_0' \]
where primes denote differentiation with respect to \( y \).

The corresponding boundary conditions are:
\[\begin{align*}
  y = 0: & \quad u_0 = 0, u_1 = 0, \theta_0 = 1, \theta_1 = 1, \phi_0 = 0, \phi_1 = 1 \\
  y \to \infty: & \quad u_0 = 1, u_1 = 0, \theta_0 = 0, \theta_1 = 0, \phi_0 = 0, \phi_1 = 0
\end{align*}\]
(9.3.8)

Solving (9.3.4) and (9.3.7) with boundary conditions (9.3.8) we get
\[ \phi_0 = e^{-Sc y} \]
\[ \phi_1 = B_1 e^{-a_1 y} + B_2 e^{-Sc y} \]
(9.3.9) (9.3.10)

Again, the physical variables \( u_0, \theta_0, u_1, \theta_1 \) are expanded in powers of Eckert number \( Ec \) (as \( Ec \ll 1 \) for incompressible fluids)
\[\begin{align*}
  u_0(y) &= u_{00}(y) + Ec u_{01}(y) \\
  u_1(y) &= u_{10}(y) + Ec u_{11}(y) \\
  \theta_0(y) &= \theta_{00}(y) + Ec \theta_{01}(y) \\
  \theta_1(y) &= \theta_{10}(y) + Ec \theta_{11}(y)
\end{align*}\]
(9.3.11)

Substituting (9.3.11) into the equations (9.3.2), (9.3.3), (9.3.5) and (9.3.6) and comparing the coefficients of like terms of \( Ec \), also neglecting the higher order term of \( Ec \) we get

\[ ku_{00}''' + u_{00}'' + u_{00}' - \left( M + \frac{1}{K} \right) u_{00} \]
\[ = -Gr\cos\alpha \theta_{00} - Gm\cos\alpha \phi_0 - \left( M + \frac{1}{K} \right) \]
\[ ku_{10}''' + u_{10}'' + k(u_{10}''' - i\omega u_{10}'' + u_{00}''' - \left( M + \frac{1}{K} + i\omega \right) u_{10} \]
\[ = -u_{00}' - Gr\cos\alpha \theta_{10} - Gm\cos\alpha \phi_1 \]
\[ \theta_{00}'' + Pr\theta_{00}' - FPr\theta_{00} = 0 \]
\[ \theta_{10}'' + Pr\theta_{10}' - (F + i\omega)Pr\theta_{10} = -Pr\theta_{00}' \]
\[ ku_{01}''' + u_{01}'' + u_{01}' - \left( M + \frac{1}{K} \right) u_{01} = -Gr\cos\alpha \theta_{01} \]
(9.3.12) (9.3.13) (9.3.14) (9.3.15) (9.3.16)
\[ u_{11}'' + u_{11}' + k\left(u_{11}''' - i\omega u_{11}'' + u_{01}''\right) - \left(M + \frac{1}{K} + i\omega\right)u_{11} = -u_{01}' - G\cos a\theta_{11} \]  
(9.3.17)

\[ \theta_{01}'' + Pr\theta_{01}' - FPr\theta_{01} = -Pr u_{00}'^2 - Pr ku_{00}'u_{00}'' - PrMu_{00}^2 \]  
(9.3.18)

\[ \theta_{11}'' + Pr\theta_{11}' - (F + i\omega)Pr\theta_{11} = -Pr\theta_{01}' - 2Pr u_{00}'u_{10}' + Pr \left(i\omega u_{00}'u_{10}' - u_{00}'u_{10}'' - u_{10}'u_{00}''\right) - 2PrM u_{00}u_{10} \]  
(9.3.19)

with relevant boundary conditions:

\[ y = 0; u_0 = 0, u_{01} = 0, u_{10} = 0, u_{11} = 0, \theta_{00} = 1, \theta_{01} = 0, \theta_{10} = 1, \theta_{11} = 0 \]

\[ y \to \infty; u_0 = 1, u_{01} = 0, u_{10} = 0, u_{11} = 0, \theta_{00} = 0, \theta_{01} = 0, \theta_{10} = 0, \theta_{11} = 0 \]  
(9.3.20)

Solving (9.3.14) and (9.3.15) with conditions (9.3.20) we get

\[ \theta_{00} = e^{-a_0 y} \]  
(9.3.21)

\[ \theta_{10} = B_0 e^{-a_0 y} + B_1 e^{-a_0 y} \]  
(9.3.22)

Equations (9.3.12), (9.3.13) and (9.3.16)-(9.3.19) are non-linear differential equations and their exact solutions are not possible. So we use the multi-parameter perturbation technique and the velocity and temperature components are expanded in the power of visco-elastic parameter \( k \). Thus the expressions for velocity and temperature components are considered as

\[ u_{00} = u_{000} + ku_{001} \]

\[ u_{10} = u_{100} + ku_{101} \]

\[ u_{01} = u_{010} + ku_{011} \]

\[ u_{11} = u_{110} + ku_{111} \]

\[ \theta_{01} = \theta_{010} + k\theta_{011} \]

\[ \theta_{11} = \theta_{110} + k\theta_{111} \]  
(9.3.23)

Substituting (9.3.23) in (9.3.12), (9.3.13) and (9.3.16)-(9.3.19) and comparing the coefficients of like terms, neglecting the higher order terms we get

\[ u_{000}'' + u_{000}' - \left(M + \frac{1}{K}\right)u_{000} = -G\cos a\theta_{00} - G\cos a\phi_0 - \left(M + \frac{1}{K}\right) \]  
(9.3.24)

\[ u_{100}'' + u_{100}' - \left(M + \frac{1}{K} + i\omega\right)u_{100} = -u_{000}' - G\cos a\theta_{10} - G\cos a\phi_0 \]  
(9.3.25)

\[ u_{010}'' + u_{010}' - \left(M + \frac{1}{K}\right)u_{010} = -G\cos a\theta_{010} \]  
(9.3.26)

\[ u_{110}'' + u_{110}' - \left(M + \frac{1}{K} + i\omega\right)u_{110} = -u_{010}' - G\cos a\theta_{110} \]  
(9.3.27)

\[ \theta_{010}'' + Pr\theta_{010}' - FPr\theta_{010} = -Pr u_{000}'^2 - PrMu_{000}^2 \]  
(9.3.28)

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\[ \theta_{110}'' + Pr \theta_{110}' - (F + i\omega)Pr \theta_{110} \]
\[ = -Pr \theta_{010}' - 2Pr u_{000}'u_{100}' - 2Pr M u_{000}u_{100} \]  \hspace{1cm} (9.3.29)

\[ u_{001}'' + u_{001}' - \left( M + \frac{1}{K} \right)u_{001} = -u_{000}''' \]  \hspace{1cm} (9.3.30)

\[ u_{101}'' + u_{101}' - \left( M + \frac{1}{K} + i\omega \right)u_{101} = -u_{010}' - u_{100}''' + i\omega u_{100}'' - u_{000}''' \]  \hspace{1cm} (9.3.31)

\[ u_{011}'' + u_{011}' - \left( M + \frac{1}{K} \right)u_{011} = -u_{010}''' - grcos\theta_{011} \]  \hspace{1cm} (9.3.32)

\[ u_{111}'' + u_{111}' - \left( M + \frac{1}{K} + i\omega \right)u_{111} = -u_{110}''' + i\omega u_{110}'' + u_{010}''' - u_{011}' - grcos\theta_{111} \]  \hspace{1cm} (9.3.33)

The transformed boundary conditions are:
\[ y = 0; \ u_{000} = 0, u_{001} = 0, u_{010} = 0, u_{011} = 0, u_{100} = 0, u_{101} = 0, u_{110} = 0, \]
\[ u_{111} = 0, \theta_{100} = 1, \theta_{101} = 0, \theta_{110} = 0, \theta_{111} = 0, \theta_{010} = 0, \theta_{011} = 0 \]
\[ y \to \infty; u_{000} = 1, u_{001} = 0, u_{010} = 0, u_{011} = 0, u_{100} = 0, u_{101} = 0, u_{110} = 0, \]
\[ u_{111} = 0, \theta_{100} = 0, \theta_{101} = 0, \theta_{110} = 0, \theta_{111} = 0, \theta_{010} = 0, \theta_{011} = 0 \]  \hspace{1cm} (9.3.34)

The equations (9.3.24)-(9.3.33) are ordinary differential equations and solutions of these equations under the boundary conditions (9.3.34) are:

\[ u_{000} = 1 + B_5 e^{-a_0 y} + B_6 e^{-S_0 y} + B_7 e^{-a_1 y} \]
\[ u_{001} = B_1 e^{-a_0 y} + B_2 e^{-S_0 y} + B_3 e^{-a_1 y} \]

\[ u_{010} = B_{32} e^{-2a_0 y} + B_{33} e^{-2S_0 y} + B_{34} e^{-2a_1 y} + B_{35} e^{-(a_0 + S_0) y} + B_{36} e^{-(S_0 + a_1) y} \]
\[ + B_{37} e^{-(a_0 + a_1) y} + B_{38} e^{-S_0 y} + B_{39} e^{-a_0 y} + B_{40} e^{-a_1 y} \]

\[ u_{011} = B_{92} e^{-2a_0 y} + B_{93} e^{-2S_0 y} + B_{94} e^{-2a_1 y} + B_{95} e^{-(a_0 + S_0) y} + B_{96} e^{-(a_0 + a_1) y} \]
\[ + B_{97} e^{-(S_0 + a_1) y} + B_{98} e^{-a_0 y} + B_{99} e^{-S_0 y} + B_{100} e^{-a_1 y} \]

\[ u_{100} = B_8 e^{-a_0 y} + B_9 e^{-S_0 y} + B_{10} e^{-a_1 y} + B_{11} e^{-a_0 y} + B_{12} e^{-a_1 y} + B_{13} e^{-a_0 y} \]
\[ u_{101} = B_{14} e^{-a_0 y} + B_{15} e^{-S_0 y} + B_{16} e^{-a_1 y} + B_{17} e^{-a_0 y} + B_{18} e^{-a_1 y} + B_{19} e^{-a_0 y} + B_{20} e^{-a_1 y} + B_{21} e^{-a_0 y} + B_{22} e^{-a_1 y} \]
\[ u_{110} = B_{63} e^{-2a_0 y} + B_{64} e^{-2S_0 y} + B_{65} e^{-2a_1 y} + B_{66} e^{-(a_0 + S_0) y} + B_{67} e^{-(a_0 + a_1) y} \]
\[ + B_{68} e^{-(a_0 + a_2) y} + B_{69} e^{-(a_0 + a_1) y} + B_{70} e^{-(a_0 + a_1) y} + B_{71} e^{-(S_0 + a_1) y} \]
\[ + B_{72} e^{-(S_0 + a_2) y} + B_{73} e^{-(S_0 + a_1) y} + B_{74} e^{-(S_0 + a_1) y} + B_{75} e^{-(a_0 + a_2) y} \]
\[ + B_{76} e^{-(a_0 + a_2) y} + B_{77} e^{-(a_0 + a_1) y} + B_{78} e^{-(a_0 + a_1) y} + B_{79} e^{-S_0 y} + B_{80} e^{-a_1 y} \]
\[ + B_{81} e^{-a_1 y} + B_{82} e^{-a_1 y} + B_{83} e^{-a_1 y} + B_{84} e^{-a_1 y} \]

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\[ u_{111} = B_{123} e^{-2a_0 y} + B_{124} e^{-2Scy} + B_{125} e^{-2a_1 y} + B_{126} e^{-(a_0 + Sc)y} + B_{127} e^{-(a_0 + a_1)y} + B_{128} e^{-(a_0 + 2a_1)y} + B_{129} e^{-(a_0 + a_3)y} + B_{130} e^{-(a_0 + a_4)y} + B_{131} e^{-(Sc + a_1)y} + B_{132} e^{-(Sc + a_2)y} + B_{133} e^{-(Sc + a_3)y} + B_{134} e^{-(Sc + a_4)y} + B_{135} e^{-(a_3 + a_1)y} + B_{136} e^{-(a_3 + a_2)y} + B_{137} e^{-(a_3 + a_4)y} + B_{138} e^{-(a_4)y} + B_{139} e^{-Scy} + B_{140} e^{-a_1y} + B_{141} e^{-a_2y} + B_{142} e^{-a_3y} + B_{143} e^{-a_4y} + B_{144} e^{-a_3y} + B_{23} e^{-2a_0 y} + B_{24} e^{-2Scy} + B_{25} e^{-2a_1 y} + B_{26} e^{-(a_0 + Sc)y} + B_{27} e^{-(Sc + a_3)y} + B_{28} e^{-(a_3 + a_0)y} + B_{29} e^{-a_3y} + B_{30} e^{-Scy} + B_{31} e^{-a_0y} + B_{85} e^{-2a_0 y} + B_{86} e^{-2Scy} + B_{87} e^{-2a_1 y} + B_{88} e^{-(a_0 + Sc)y} + B_{89} e^{-(a_0 + a_3)y} + B_{90} e^{-(Sc + a_3)y} + B_{91} e^{-a_0y} + B_{41} e^{-2a_0 y} + B_{42} e^{-2Scy} + B_{43} e^{-2a_1 y} + B_{44} e^{-(a_0 + Sc)y} + B_{45} e^{-(a_0 + a_1)y} + B_{46} e^{-(a_0 + a_2)y} + B_{47} e^{-(a_0 + a_3)y} + B_{48} e^{-(a_0 + a_4)y} + B_{49} e^{-(Sc + a_1)y} + B_{50} e^{-(Sc + a_2)y} + B_{51} e^{-(Sc + a_3)y} + B_{52} e^{-(Sc + a_4)y} + B_{53} e^{-(a_3 + a_1)y} + B_{54} e^{-(a_3 + a_2)y} + B_{55} e^{-(a_3 + a_4)y} + B_{56} e^{-(a_0)y} + B_{57} e^{-Scy} + B_{58} e^{-a_3y} + B_{59} e^{-a_0y} + B_{60} e^{-a_3y} + B_{61} e^{-a_0y} + B_{62} e^{-a_3y} + B_{101} e^{-2a_0 y} + B_{102} e^{-2Scy} + B_{103} e^{-2a_1 y} + B_{104} e^{-(a_0 + Sc)y} + B_{105} e^{-(a_0 + a_3)y} + B_{106} e^{-(a_0 + a_2)y} + B_{107} e^{-(a_0 + a_3)y} + B_{108} e^{-(a_0 + a_4)y} + B_{109} e^{-(Sc + a_1)y} + B_{110} e^{-(Sc + a_2)y} + B_{111} e^{-(Sc + a_3)y} + B_{112} e^{-(Sc + a_4)y} + B_{113} e^{-(a_3 + a_1)y} + B_{114} e^{-(a_3 + a_2)y} + B_{115} e^{-(a_3 + a_4)y} + B_{116} e^{-a_0y} + B_{117} e^{-Scy} + B_{118} e^{-a_1y} + B_{119} e^{-a_2y} + B_{120} e^{-a_3y} + B_{121} e^{-a_4y} + B_{122} e^{-a_3y} \]

where the constants of the problem are:

\[
a_0 = \frac{Pr + \sqrt{Pr^2 + 4Pr}}{2}, \quad a_1 = \frac{Sc + \sqrt{Sc^2 + 4i\omega Sc}}{2},
\]

\[
a_2 = \frac{Pr + \sqrt{Pr^2 + 4Pr(F + i\omega)}}{2}, \quad a_3 = \frac{1 + \sqrt{1 + 4 \left( M + \frac{1}{K} \right)}}{2},
\]

\[
a_4 = \frac{1 + \sqrt{1 + 4 \left( M + \frac{1}{K} + i\omega \right)}}{2}, \quad a_5 = \frac{Pr + \sqrt{Pr^2 + 4Pr(F + i\omega)}}{2},
\]

\[
B_1 = 1 + \frac{Sc}{i\omega}, \quad B_2 = -\frac{Sc}{i\omega}, \quad B_3 = \frac{Pra_0}{a_0^2 - Pra_0 - (i\omega + F)Pr'},
\]

\[
B_4 = \frac{Pra_0}{a_0^2 - Pra_0 - (i\omega + F)Pr'}.
\]

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\[
\begin{align*}
B_5 &= \frac{-Gc\cos a}{a_3^2 - a_0 - \left( M + \frac{1}{K} \right)}, \\
B_7 &= -(1 + B_5 + B_6), \\
B_9 &= \frac{ScB_6 - Gm\cos aB_2}{Sc^2 - Sc - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{11} &= \frac{-Gm\cos aB_3}{a_2^2 - a_2 - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{13} &= -(B_8 + B_9 + B_{10} + B_{11} + B_{12}), \\
B_{15} &= \frac{-Sc^3B_6}{Sc^2 - Sc - \left( M + \frac{1}{K} \right)}, \\
B_{17} &= \frac{a_0B_{14} + a_3^2B_9 - i\omega a_3^2B_6 + a_3^3B_5}{a_3^2 - a_0 - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{19} &= \frac{a_3^2B_{10} - i\omega a_3^2B_{10}}{a_2^2 - a_2 - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{21} &= \frac{a_3B_{16} + a_3^2B_{12} - i\omega a_3^2B_{12} + a_3^3B_7}{a_3^2 - a_3 - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{23} &= \frac{-a_3^2PrB_2^2 + PrMB_2^2}{4a_0^2 - 2Pr\alpha_0 - FP\alpha}, \\
B_{25} &= \frac{-2B_2^2PrM}{4a_3^2 - 2Pr\alpha_3 - FP\alpha}, \\
B_{27} &= \frac{-2Pr\alpha_3B_6B_7Sc + 2PrMB_6B_7}{(a_3 + Sc)^2 - Pr(a_3 + Sc) - FP\alpha}, \\
B_{28} &= \frac{-2Pr\alpha_6B_2a_2 + 2PrMB_6B_2}{(a_3 + a_0)^2 - Pr(a_3 + a_0) - FP\alpha}, \\
B_{29} &= \frac{-Pr\alpha_3^2B_2^2 + 2PrMB_7}{a_3^2 - Pr\alpha_3 - FP\alpha}, \\
B_{31} &= -(B_{23} + B_{24} + B_{25} + B_{26} + B_{27} + B_{28} + B_{29} + B_{30}), \\
B_6 &= \frac{-Gm\cos a}{Sc^2 - Sc - \left( M + \frac{1}{K} \right)}, \\
B_8 &= \frac{a_0^2B_5 - Gm\cos aB_4}{a_0^2 - a_0 - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{10} &= \frac{-Gm\cos aB_1}{a_1^2 - a_1 - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{12} &= \frac{a_3B_7}{a_3^2 - a_3 - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{14} &= \frac{-a_3B_5}{a_3^2 - a_0 - \left( M + \frac{1}{K} \right)}, \\
B_{16} &= -(B_{14} + B_{15}), \\
B_{18} &= \frac{ScB_{15} + Sc^3B_6 - i\omega Sc^2B_9 + Sc^3B_6}{Sc^2 - Sc - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{20} &= \frac{a_3^2B_{11} - i\omega a_3^2B_{11}}{a_3^2 - a_3 - \left( M + \frac{1}{K} + i\omega \right)}, \\
B_{22} &= -(B_{17} + B_{18} + B_{19} + B_{20} + B_{21}), \\
B_{24} &= \frac{-(PrSc^2B_6^2 + PrMB_6^2)}{4Sc^2 - 2PrSc - FP\alpha}, \\
B_{26} &= \frac{-2Pr\alpha_3B_6B_7Sc + 2PrMB_5B_6}{(a_0 + Sc)^2 - Pr(a_0 + Sc) - FP\alpha}, \\
B_{30} &= \frac{-2B_6PrM}{Sc^2 - PrSc - FP\alpha}.
\end{align*}
\]
\[ B_{32} = \frac{-Gc\cos aB_{23}}{4a_0^2 - 2a_0 - \left(M + \frac{1}{K}\right)} \]

\[ B_{33} = \frac{-Gc\cos aB_{24}}{4Sc^2 - 2Sc - \left(M + \frac{1}{K}\right)} \]

\[ B_{34} = \frac{-Gc\cos aB_{25}}{4a_3^2 - 2a_3 - \left(M + \frac{1}{K}\right)} \]

\[ B_{35} = \frac{-Gc\cos aB_{26}}{(a_0 + Sc)^2 - (a_0 + Sc) - \left(M + \frac{1}{K}\right)} \]

\[ B_{36} = \frac{-Gc\cos aB_{27}}{(a_3 + Sc)^2 - (a_3 + Sc) - \left(M + \frac{1}{K}\right)} \]

\[ B_{37} = \frac{-Gc\cos aB_{28}}{(a_3 + a_0)^2 - (a_3 + a_0) - \left(M + \frac{1}{K}\right)} \]

\[ B_{38} = \frac{-Gc\cos aB_{29}}{Sc^2 - Sc - \left(M + \frac{1}{K}\right)} \]

\[ B_{39} = \frac{-Gc\cos aB_{31}}{a_0^2 - a_0 - \left(M + \frac{1}{K}\right)} \]

\[ B_{40} = -(B_{32} + B_{33} + B_{34} + B_{35} + B_{36} + B_{37} + B_{38} + B_{39}) \]

\[ \alpha_1 = 2Pra_0B_{23} - 2Pra_0^2B_5B_8 - 2PrMB_5B_8, \]

\[ \alpha_2 = 2PrScB_{24} - 2PrSc^2B_5B_9 - 2PrMB_5B_9, \]

\[ \alpha_3 = 2Pra_3B_{25} - 2Pra_3^2B_7B_12 - 2PrMB_7B_{12}, \]

\[ \alpha_4 = Pr(a_0 + Sc)B_{26} - 2Pra_0B_5B_9Sc - 2Pra_0B_6B_8Sc - 2PrM(B_5B_9 + B_6B_8), \]

\[ \alpha_5 = Pr(a_3 + Sc)B_{27} - 2Pr(B_5B_12 + B_7B_9)Sc - 2PrM(B_5B_{12} + B_7B_9), \]

\[ \alpha_6 = Pr(a_3 + a_0)B_{28} - 2Pr(B_5B_{12} + B_7B_9)a_0a_3 - 2PrM(B_5B_{12} + B_7B_9), \]

\[ \alpha_7 = Pr_3B_{29} - 2PrMB_{12}, \quad \alpha_8 = PrScB_{30} - 2PrMB_9, \quad \alpha_9 = Pra_0B_{31} - 2PrMB_8, \]

\[ \alpha_{10} = -2Pra_0B_5B_{10}a_1 - 2PrMB_5B_{10}, \quad \alpha_{11} = -2Pra_0B_5B_{11}a_2 - 2PrMB_5B_{11}, \]

\[ \alpha_{12} = -2Pra_0B_5B_{12}a_3 - 2PrMB_5B_{13}, \quad \alpha_{13} = -2PrB_5B_{10}Sc - 2PrMB_5B_{10}, \]

\[ \alpha_{14} = -2PrB_6B_{11}Sc - 2PrMB_6B_{11}, \quad \alpha_{15} = -2PrB_6B_{13}Sc - 2PrMB_6B_{13}, \]

\[ \alpha_{16} = -2PrB_7B_{10}a_3 - 2PrMB_7B_{10}, \quad \alpha_{17} = -2PrB_7B_{11}a_2 - 2PrMB_7B_{11}, \]

\[ \alpha_{18} = -2PrB_7B_{13}a_4 - 2PrMB_7B_{13}, \quad \alpha_{19} = -2PrMB_7B_8, \]

\[ \alpha_{20} = -2PrMB_{11}, \quad \alpha_{21} = -2PrMB_{13}, \]

\[ B_{41} = \frac{a_1}{4a_0^2 - 2Pra_0 - (F + i\omega)Pr} \quad B_{42} = \frac{a_2}{4Sc^2 - 2PrSc - (F + i\omega)Pr} \]

\[ B_{43} = \frac{a_3}{4a_3^2 - 2Pra_3 - (F + i\omega)Pr} \quad B_{44} = \frac{a_4}{(a_0 + Sc)^2 - Pr(a_0 + Sc) - (F + i\omega)Pr} \]

\[ B_{45} = \frac{a_5}{(a_0 + a_1)^2 - Pr(a_0 + a_1) - (F + i\omega)Pr} \quad B_{46} = \frac{a_6}{(a_0 + a_2)^2 - Pr(a_0 + a_2) - (F + i\omega)Pr} \]
\[ B_{47} = \frac{\alpha_6}{(a_3 + a_3)^2 - Pr(a_0 + a_3) - (F + i\omega)Pr'} \]
\[ B_{48} = \frac{\alpha_{12}}{(a_0 + a_4)^2 - Pr(a_0 + a_4) - (F + i\omega)Pr'} \]
\[ B_{49} = \frac{\alpha_{13}}{(Sc + a_1)^2 - Pr(Sc + a_1) - (F + i\omega)Pr'} \]
\[ B_{50} = \frac{\alpha_{14}}{(Sc + a_2)^2 - Pr(Sc + a_2) - (F + i\omega)Pr'} \]
\[ B_{51} = \frac{\alpha_5}{(Sc + a_3)^2 - Pr(Sc + a_3) - (F + i\omega)Pr'} \]
\[ B_{52} = \frac{\alpha_{15}}{(Sc + a_4)^2 - Pr(Sc + a_4) - (F + i\omega)Pr'} \]
\[ B_{53} = \frac{\alpha_{16}}{(a_3 + a_1)^2 - Pr(a_3 + a_1) - (F + i\omega)Pr'} \]
\[ B_{54} = \frac{\alpha_{17}}{(a_3 + a_2)^2 - Pr(a_3 + a_2) - (F + i\omega)Pr'} \]
\[ B_{55} = \frac{\alpha_{18}}{(a_3 + a_4)^2 - Pr(a_3 + a_4) - (F + i\omega)Pr'} \]
\[ B_{56} = \frac{\alpha_9}{a_0^2 - Pr a_0 - (F + i\omega)Pr'} \]
\[ B_{57} = \frac{\alpha_8}{Sc^2 - PrSc - (F + i\omega)Pr'} \]
\[ B_{58} = \frac{\alpha_{10}}{a_1^2 - Pr a_1 - (F + i\omega)Pr'} \]
\[ B_{59} = \frac{\alpha_{20}}{a_2^2 - Pr a_2 - (F + i\omega)Pr'} \]
\[ B_{60} = \frac{\alpha_7}{a_3^2 - Pr a_3 - (F + i\omega)Pr'} \]
\[ B_{61} = \frac{\alpha_{21}}{a_4^2 - Pr a_4 - (F + i\omega)Pr'} \]
\[ B_{62} = -(B_{41} + B_{42} + B_{43} + B_{44} + B_{45} + B_{46} + B_{47} + B_{48} + B_{49} + B_{50} + B_{51} + B_{52} + B_{53} + B_{54} + B_{55} + B_{56} + B_{57} + B_{58} + B_{59} + B_{60} + B_{61}) \]
\[ B_{63} = \frac{2a_0 B_{32} - GrcosaB_{41}}{4a_0^2 - 2a_0 - \left(M + \frac{1}{K} + i\omega\right)} \]
\[ B_{64} = \frac{2ScB_{33} - GrcosaB_{42}}{4Sc^2 - 2Sc - \left(M + \frac{1}{K} + i\omega\right)} \]
\[ B_{65} = \frac{2a_3 B_{34} - GrcosaB_{43}}{4a_3^2 - 2a_3 - \left(M + \frac{1}{K} + i\omega\right)} \]
\[ B_{66} = \frac{-GrcosaB_{45}}{(a_0 + a_1)^2 - (a_0 + a_1) - \left(M + \frac{1}{K} + i\omega\right)} \]
\[ B_{67} = \frac{-GrcosaB_{46}}{(a_0 + a_2)^2 - (a_0 + a_2) - \left(M + \frac{1}{K} + i\omega\right)} \]
\[
B_{69} = \frac{(a_0 + a_3)B_{37} - G\cos aB_{47}}{(a_0 + a_3)^2 - (a_0 + a_3) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{70} = -\frac{G\cos aB_{48}}{(a_0 + a_4)^2 - (a_0 + a_4) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{71} = -\frac{G\cos aB_{49}}{(Sc + a_1)^2 - (Sc + a_2) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{72} = \frac{-G\cos aB_{50}}{(Sc + a_2)^2 - (Sc + a_2) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{73} = \frac{(Sc + a_3)B_{36} - G\cos aB_{51}}{(Sc + a_3)^2 - (Sc + a_3) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{74} = -\frac{G\cos aB_{52}}{(Sc + a_4)^2 - (Sc + a_4) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{75} = -\frac{G\cos aB_{53}}{(a_3 + a_1)^2 - (a_3 + a_4) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{76} = -\frac{G\cos aB_{54}}{(a_3 + a_2)^2 - (a_3 + a_2) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{77} = -\frac{G\cos aB_{55}}{(a_3 + a_4)^2 - (a_3 + a_4) - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{78} = \frac{a_0B_{39} - G\cos aB_{56}}{a_0^2 - a_0 - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{79} = \frac{ScB_{38} - G\cos aB_{57}}{Sc^2 - Sc - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{80} = -\frac{G\cos aB_{58}}{a_1^2 - a_1 - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{81} = -\frac{G\cos aB_{59}}{a_2^2 - a_2 - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{82} = -\frac{G\cos aB_{60}}{a_3^2 - a_3 - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{83} = -\frac{G\cos aB_{62}}{a_5^2 - a_5 - \left(M + \frac{1}{R} + i\omega\right)},
\]
\[
B_{84} = -(B_{63} + B_{64} + B_{65} + B_{66} + B_{67} + B_{68} + B_{69} + B_{70} + B_{71} + B_{72} + B_{73} + B_{74} + B_{75} + B_{76} + B_{77} + B_{78} + B_{79} + B_{80} + B_{81} + B_{82} + B_{83})
\]
\[
B_{85} = -\frac{2PrB_{14}a_0^2 + PrB_{6}a_0^3 - 2PrMB_{14}}{a_0^2 - Pra_0 - FPr},
\]
\[
B_{86} = -\frac{2PrB_{15}Sc^2 + PrB_{6}Sc^3 - 2PrMB_{15}}{4Sc^2 - PrSc - FPr},
\]
\[ B_{87} = \frac{-2PrB_7B_{14}a_3^2 + PrB_7^2a_3^3 - 2PrMB_7B_{16}}{4a_3^2 - Pra_3 - FPr} , \]

\[ B_{88} = \frac{-2Pra_0Sc(B_5B_{15} + B_6B_{14}) + PrB_5B_6(a_0Sc^2 + a_0^2Sc)}{-2PrM(B_5B_{14} + B_6B_{12})}, \]

\[ B_{89} = \frac{-2Pra_0a_3(B_5B_{16} + B_7B_{14}) + PrB_5B_7(a_0a_3^2 + a_0^2a_3)}{(a_0 + a_3)^2 - Pr(a_0 + a_3)}, \]

\[ B_{90} = \frac{-2PrSc^2a_3(B_5B_{16} + B_7B_{15}) + PrB_5B_7(Sca_3^2 + Sc^2a_3)}{(Sc + a_3)^2 - Pr(Sc + a_3)}, \]

\[ B_{91} = -(B_{85} + B_{86} + B_{87} + B_{88} + B_{89} + B_{90}), \]

\[ a_{22} = 2Pra_0B_{85} - 2Pra_0^2(B_5B_{17} + B_6B_{14}), \]

\[ a_{23} = 2PrScB_{86} - 2PrSc^2(B_5B_{18} + B_9B_{13}), \]

\[ a_{24} = 2Pra_0B_{87} - 2Pra_0^2(B_7B_{21} + B_8B_{12}), \]

\[ a_{25} = Pr(a_0 + Sc)B_{88} - 2Pra_0Sc(B_5B_{18} + B_6B_{17} + B_{14}B_9 + B_{13}B_8), \]

\[ a_{26} = -2Pra_0a_1(B_5B_{19} + B_{14}B_{10}), \]

\[ a_{27} = -2Pra_0a_2(B_5B_{20} + B_{14}B_{11}), \]

\[ a_{28} = Pr(a_0 + a_3)B_{89} - 2Pra_0a_3(B_5B_{21} + B_7B_{17} + B_{14}B_{12} + B_{14}B_9), \]

\[ a_{29} = -2Pra_0a_4(B_5B_{22} + B_{14}B_{13}), \]

\[ a_{30} = -2PrSc^2a_1(B_5B_{19} + B_{15}B_{10}), \]

\[ a_{31} = -2PrSc^2a_2(B_6B_{20} + B_{15}B_{11}), \]

\[ a_{32} = Pr(Sc + a_3)B_{90} - 2PrSc^2a_3(B_6B_{21} + B_7B_{18} + B_{15}B_{12} + B_{14}B_9), \]

\[ a_{33} = -2PrSc^2a_4(B_6B_{22} + B_{15}B_{13}), \]

\[ a_{34} = -2Pra_3a_1(B_7B_{19} + B_{16}B_{10}), \]

\[ a_{35} = -2Pra_3a_2(B_7B_{20} + B_{16}B_{11}), \]

\[ a_{36} = -2Pra_3a_4(B_7B_{22} + B_{16}B_{13}), \]

\[ a_{37} = PrB_{91}, \]

\[ a_{38} = i\omega B_5B_6a_0^2 + 2B_5B_8a_3^3, \]

\[ a_{39} = i\omega B_6B_9Sc^2 + 2B_6B_9Sc^3, \]

\[ a_{40} = i\omega B_7B_{12}a_3^2 + 2B_7B_{12}a_3^3, \]

\[ a_{41} = i\omega a_0Sc(B_5B_9 + B_6B_8) + (B_5B_9 + B_6B_8)a_0Sc^2 + (B_5B_9a_0 + B_6B_8Sc)a_0Sc, \]

\[ a_{42} = i\omega a_0a_1B_5B_{10} + B_5B_{10}a_0a_1^2 + B_5B_{10}a_0a_1^2a_1, \]

\[ a_{43} = i\omega a_0a_2B_5B_{11} + B_5B_{11}a_0a_2^2 + B_5B_{11}a_0a_2^2a_2, \]

\[ a_{44} = i\omega a_0a_3(B_5B_{12} + B_7B_8) + B_5B_{12}a_0a_3^2 + B_7B_{8}a_0a_2a_3 + B_5B_{12}a_0a_2^2a_3, \]

\[ + B_7B_8a_3^2a_0, \]

\[ a_{45} = i\omega a_0a_4B_5B_{13} + B_5B_{13}a_0a_4^2 + B_5B_{13}a_0a_4^2a_4, \]

\[ a_{46} = i\omega Sc^2a_1B_6B_{10} + B_6B_{10}Sc^2a_1^2 + B_6B_{10}Sc^2a_1^2a_1, \]

\[ a_{47} = i\omega Sc^2a_2B_6B_{11} + B_6B_{11}Sc^2a_2^2 + B_6B_{11}Sc^2a_2^2a_2. \]
\[ \alpha_{48} = i\omega S\alpha_3(B_6B_{12} + B_7B_n) + B_6B_{12}S\alpha_3^2 + B_7B_9S\alpha_3^2 \alpha_3 + B_6B_{12}S\alpha_3 \]
\[ + B_7B_6S\alpha_3^2 S, \]
\[ \alpha_{49} = i\omega S\alpha_4 B_6B_{13} + B_6B_{13}S\alpha_4^2 + B_6B_{13}S\alpha_3^2 \alpha_4, \]
\[ \alpha_{50} = i\omega S\alpha_3 B_7B_{10} + B_7B_{10}S\alpha_3^2 + B_7B_{10}S\alpha_3^2 \alpha_3, \]
\[ \alpha_{51} = i\omega S\alpha_3 B_7B_{11} + B_7B_{11}S\alpha_3^2 + B_7B_{11}S\alpha_3^2 \alpha_2, \]
\[ \alpha_{52} = i\omega S\alpha_4 B_7B_{13} + B_7B_{13}S\alpha_4^2 + B_7B_{13}S\alpha_3^2 \alpha_4, \]
\[ \alpha_{53} = B_5B_{14} + B_1B_8, \quad \alpha_{54} = B_6B_{15} + B_{15}B_9, \quad \alpha_{55} = B_7B_{16} + B_{16}B_{12}, \]
\[ \alpha_{56} = B_6B_{15} + B_2B_8, \quad \alpha_{57} = B_5B_{16} + B_{14}B_9, \quad \alpha_{58} = B_5B_{17} + B_{14}B_{11}, \]
\[ \alpha_{59} = B_5B_{18} + B_{14}B_{12} + B_7B_{14} + B_{16}B_8, \quad \alpha_{60} = B_5B_{19} + B_{14}B_{13}, \]
\[ \alpha_{61} = B_6B_{16} + B_{15}B_{10}, \quad \alpha_{62} = B_6B_{17} + B_{15}B_{11}, \quad \alpha_{63} = B_6B_{18} + B_{15}B_{12} + B_7B_{15} + B_{16}B_9, \quad \alpha_{64} = B_6B_{19} + B_{15}B_{13}, \]
\[ \alpha_{65} = B_7B_{16} + B_{16}B_{11}, \quad \alpha_{66} = B_7B_{17} + B_{16}B_{11}, \quad \alpha_{67} = B_7B_{19} + B_{16}B_{13}, \]
\[ \alpha_{68} = B_{14}, \quad \alpha_{69} = B_{15}, \quad \alpha_{70} = B_{16}, \quad \alpha_{71} = B_{17}, \quad \alpha_{72} = B_{18}, \quad \alpha_{73} = B_{19}, \]
\[ B_{101} = \frac{\alpha_{22} + Pr\alpha_{38} - 2PrM\alpha_{53}}{a_0^2 - Pr\alpha_0 - (F + i\omega)Pr'}, \quad B_{102} = \frac{\alpha_{23} + Pr\alpha_{39} - 2PrM\alpha_{54}}{4S\alpha_3^2 - 2PrSc - (F + i\omega)Pr'}, \]
\[ B_{103} = \frac{\alpha_{24} + Pr\alpha_{40} - 2PrM\alpha_{55}}{4a_3^2 - 2Pr\alpha_3 - (F + i\omega)Pr'}, \]
\[ B_{104} = \frac{\alpha_{25} + Pr\alpha_{41} - 2PrM\alpha_{56}}{(a_0 + S\alpha_3)^2 - Pr(a_0 + S\alpha_3) - (F + i\omega)Pr'}, \]
\[ B_{105} = \frac{\alpha_{26} + Pr\alpha_{42} - 2PrM\alpha_{57}}{(a_0 + a_1)^2 - Pr(a_0 + a_1) - (F + i\omega)Pr'}, \]
\[ B_{106} = \frac{\alpha_{27} + Pr\alpha_{43} - 2PrM\alpha_{58}}{(a_0 + a_2)^2 - Pr(a_0 + a_2) - (F + i\omega)Pr'}, \]
\[ B_{107} = \frac{\alpha_{28} + Pr\alpha_{44} - 2PrM\alpha_{59}}{(a_0 + a_3)^2 - Pr(a_0 + a_3) - (F + i\omega)Pr'}, \]
\[ B_{108} = \frac{\alpha_{29} + Pr\alpha_{45} - 2PrM\alpha_{60}}{(a_0 + a_4)^2 - Pr(a_0 + a_4) - (F + i\omega)Pr'}, \]
\[ B_{109} = \frac{\alpha_{30} + Pr\alpha_{46} - 2PrM\alpha_{61}}{(a_0 + a_3)^2 - Pr(a_0 + a_3) - (F + i\omega)Pr'}, \]
\[ B_{110} = \frac{\alpha_{31} + Pr\alpha_{47} - 2PrM\alpha_{62}}{(a_0 + a_2)^2 - Pr(a_0 + a_2) - (F + i\omega)Pr'}, \]
\[ B_{111} = \frac{\alpha_{32} + Pr\alpha_{48} - 2PrM\alpha_{63}}{(a_0 + a_3)^2 - Pr(a_0 + a_3) - (F + i\omega)Pr'}, \]
\[ B_{112} = \frac{\alpha_{33} + Pr\alpha_{49} - 2PrM\alpha_{64}}{(a_0 + a_4)^2 - Pr(a_0 + a_4) - (F + i\omega)Pr'}. \]
\[
B_{113} = \frac{\alpha_{34} + Pr\alpha_{50} - 2PrMa_{65}}{(a_3 + a_1)^2 - Pr(a_3 + a_1) - (F + i\omega)Pr'}
\]
\[
B_{114} = \frac{\alpha_{35} + Pr\alpha_{51} - 2PrMa_{66}}{(a_3 + a_2)^2 - Pr(a_3 + a_2) - (F + i\omega)Pr'}
\]
\[
B_{115} = \frac{\alpha_{36} + Pr\alpha_{52} - 2PrMa_{67}}{(a_3 + a_4)^2 - Pr(a_3 + a_4) - (F + i\omega)Pr'}
\]
\[
B_{116} = \frac{\alpha_{37} - 2PrMa_{68}}{a_0^2 - Pra_0 - (F + i\omega)Pr'}
\]
\[
B_{117} = \frac{-2PrMa_{69}}{Sc^2 - PrSc - (F + i\omega)Pr'}
\]
\[
B_{118} = \frac{-2PrMa_{70}}{a_1^2 - Pra_1 - (F + i\omega)Pr'}
\]
\[
B_{119} = \frac{-2PrMa_{71}}{a_2^2 - Pra_2 - (F + i\omega)Pr'}
\]
\[
B_{120} = \frac{-2PrMa_{72}}{a_3^2 - Pra_3 - (F + i\omega)Pr'}
\]
\[
B_{121} = \frac{-2PrMa_{73}}{a_4^2 - Pra_4 - (F + i\omega)Pr'}
\]
\[
B_{122} = -(B_{101} + B_{102} + B_{103} + B_{104} + B_{105} + B_{106} + B_{107} + B_{108} + B_{109} + B_{110}
+ B_{111} + B_{112} + B_{113} + B_{114} + B_{115} + B_{116} + B_{117} + B_{118} + B_{119}
+ B_{120} + B_{121}),
\]
\[
B_{123} = \frac{(8a_0^3 + 4i\omega a_0^2)B_{63} + 8a_0^3 B_{32} + 2a_0 B_{92} - Gr\cos\alpha B_{101}}{4a_0^2 - 2a_0 - \left(M + \frac{1}{R} + i\omega\right)}
\]
\[
B_{124} = \frac{(8Sc^3 + 4i\omega Sc^2)B_{64} + 8Sc^3 B_{33} + 2Sc B_{93} - Gr\cos\alpha B_{102}}{4Sc^2 - 2Sc - \left(M + \frac{1}{R} + i\omega\right)}
\]
\[
B_{125} = \frac{(8a_3^3 + 4i\omega a_3^2)B_{65} + 8a_3^3 B_{34} + 2a_3 B_{94} - Gr\cos\alpha B_{103}}{4a_3^2 - 2a_3 - \left(M + \frac{1}{R} + i\omega\right)}
\]
\[
B_{126} = \frac{(a_0 + Sc)^2(a_0 + Sc + i\omega)B_{66} + (a_0 + Sc)B_{95} - Gr\cos\alpha B_{104} + (a_0 + Sc)^3 B_{35}}{(a_0 + Sc)^2 - (a_0 + Sc) - \left(M + \frac{1}{R} + i\omega\right)}
\]
\[
B_{127} = \frac{(a_0 + a_1)^2(a_0 + a_1 + i\omega)B_{67} - Gr\cos\alpha B_{105}}{(a_0 + a_1)^2 - (a_0 + a_1) - \left(M + \frac{1}{R} + i\omega\right)}
\]
\[
B_{128} = \frac{(a_0 + a_2)^2(a_0 + a_2 + i\omega)B_{68} - Gr\cos\alpha B_{106}}{(a_0 + a_2)^2 - (a_0 + a_2) - \left(M + \frac{1}{R} + i\omega\right)}
\]
\[
B_{129} = \frac{(a_0 + a_3)^2(a_0 + a_3 + i\omega)B_{69} + (a_0 + a_3)B_{96} - Gr\cos\alpha B_{107} + (a_0 + a_3)^3 B_{37}}{(a_0 + a_3)^2 - (a_0 + a_3) - \left(M + \frac{1}{R} + i\omega\right)}
\]
\[ B_{130} = \frac{(a_0 + a_4)^2(a_0 + a_4 + i\omega)B_{70} - \text{GrcosaB}_{108}}{(a_0 + a_4)^2 - (a_0 + a_4) - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{131} = \frac{(Sc + a_1)^2(Sc + a_1 + i\omega)B_{71} - \text{GrcosaB}_{109}}{(Sc + a_1)^2 - (Sc + a_1) - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{132} = \frac{(Sc + a_2)^2(Sc + a_2 + i\omega)B_{72} - \text{GrcosaB}_{110}}{(Sc + a_2)^2 - (Sc + a_2) - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{133} = \frac{(Sc + a_3)^2(Sc + a_3 + i\omega)B_{73} + (Sc + a_3)B_{97} - \text{GrcosaB}_{111} + (Sc + a_3)^3B_{36}}{(Sc + a_3)^2 - (Sc + a_3) - \left(M + \frac{1}{K} + i\omega\right)} \]

\[ B_{134} = \frac{(Sc + a_4)^2(Sc + a_4 + i\omega)B_{74} - \text{GrcosaB}_{112}}{(Sc + a_4)^2 - (Sc + a_4) - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{135} = \frac{(a_3 + a_1)^2(a_3 + a_1 + i\omega)B_{75} - \text{GrcosaB}_{113}}{(a_3 + a_1)^2 - (a_3 + a_1) - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{136} = \frac{(a_3 + a_2)^2(a_3 + a_2 + i\omega)B_{76} - \text{GrcosaB}_{114}}{(a_3 + a_2)^2 - (a_3 + a_2) - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{137} = \frac{(a_3 + a_4)^2(a_3 + a_4 + i\omega)B_{77} - \text{GrcosaB}_{115}}{(a_3 + a_4)^2 - (a_3 + a_4) - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{138} = \frac{a_0^2B_{78}(a_0 + i\omega) + a_0^3B_{39} + a_0B_{98} - \text{GrcosaB}_{116}}{a_0^2 - a_0 - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{139} = \frac{Sc^2B_{79}(Sc + i\omega) + Sc^3B_{38} + ScB_{99} - \text{GrcosaB}_{117}}{Sc^2 - Sc - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{140} = \frac{a_1^2B_{80}(a_1 + i\omega) - \text{GrcosaB}_{118}}{a_1^2 - a_1 - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{141} = \frac{a_2^2B_{81}(a_2 + i\omega) - \text{GrcosaB}_{119}}{a_2^2 - a_2 - \left(M + \frac{1}{K} + i\omega\right)}, \]

\[ B_{142} = \frac{a_3^2B_{82}(a_3 + i\omega) + a_3^3B_{40} + a_3B_{100} - \text{GrcosaB}_{120}}{a_3^2 - a_3 - \left(M + \frac{1}{K} + i\omega\right)}, \]

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\[ B_{143} = \frac{a_5^2 B_{132}(a_5 + i\omega) - Gr c o s a B_{122}}{a_5^2 - a_5 - \left( M + \frac{1}{K} + i\omega \right)} \]

\[ B_{144} = -(B_{123} + B_{124} + B_{125} + B_{126} + B_{127} + B_{128} + B_{129} + B_{130} + B_{131} + B_{132} + B_{133} + B_{134} + B_{135} + B_{136} + B_{137} + B_{138} + B_{139} + B_{140} + B_{141} + B_{142} + B_{143}) \]

### 9.4 RESULTS AND DISCUSSION

The fluid velocity is given by

\[ u = (u_{000} + k u_{001}) + Ec(u_{010} + k u_{011}) + \varepsilon e^{i\omega t}(u_{100} + k u_{101}) + Ec(u_{110} + k u_{111}) \]

The temperature is given by

\[ \theta = (\theta_{00} + Ec(\theta_{010} + k \theta_{011})) + \varepsilon e^{i\omega t}(\theta_{10} + Ec(\theta_{110} + k \theta_{111})) \]

The non-dimensional shearing stress \( \sigma \) at the plate \( y = 0 \) is given by

\[ \sigma = \frac{\partial u}{\partial y} - k \left( \frac{\partial^2 u}{\partial t \partial y} - (1 + \varepsilon e^{i\omega t}) \frac{\partial^2 u}{\partial y^2} \right) \]

The non-dimensional heat flux \( Nu \) at the plate \( y = 0 \) in terms of Nusselt number is given by

\[ Nu = \left( \frac{\partial \theta}{\partial y} \right)_{y=0} \]

The non-dimensional form of the rate mass transfer at the plate \( y = 0 \) in terms of Sherwood number \( Sh \) is given by

\[ Sh = \left( \frac{\partial \phi}{\partial y} \right)_{y=0} \]

The motivation of the present study is to investigate the effects of visco-elasticity on mixed convective MHD boundary layer flow past an inclined moving surface in presence of heat and mass transfer. The visco-elastic effect is exhibited through the non-dimensional parameter \( k \). The non-zero values of the parameter \( k \) characterize the visco-elastic fluid and \( k = 0 \) represents the Newtonian fluid flow phenomenon. The fluid velocity and the shearing stress at the plate are illustrated graphically for various values of flow parameters involved in the solution. Throughout the discussion \( Gr = 5, Gm = 4, Pr = 3, M = 3, Sc = 2, \omega = 5, K = 5, \varepsilon = 0.2, Ec = 0.1, \omega t = \frac{\pi}{2} \) are kept
fixed unless otherwise stated.

Figures 9.2 to 9.8 display the pattern of fluid velocity against the displacement \( y \) for various values of visco-elastic parameter. These figures illustrate that the flow behaviour is alike in both Newtonian and visco-elastic fluids. The maximum effects of visco-elastic parameters are detected in the neighbourhood of the inclined surface.

In figure 9.2, it is observed that the convection velocity follows a downward trend in the case of cooling of the boundary i.e. \( \text{Gr} > 0 \) in comparison with the heated boundary i.e. \( \text{Gr} < 0 \) in both Newtonian and non-Newtonian cases.

The effects of Prandtl number on both visco-elastic and Newtonian fluids, are analyzed in figure 9.3. The significance of Prandtl number cannot be ignored in heat transfer flow problems as it helps to study the simultaneous effects of momentum and thermal diffusion in fluid flow. It states that with the ascending value of Prandtl number fluid velocity also raises.

Figure 9.4 represents typical convection velocity in the boundary layer for various values of the solutal Grashof number \( G_m \), while all other parameters are kept at some fixed values. The ratio of the species buoyancy force to the viscous hydrodynamic force is solutal Grashof number. As expected, the fluid velocity increases due to increase in the species buoyancy force. The velocity distribution attains a distinctive maximum value in the vicinity of the plate for Newtonian fluid in comparison with visco-elastic fluid.

Figure 9.5 represents the flow behaviour against \( M \). It can be noticed from the figure that with the enhancement of Hartmann number \( M \), the fluid velocity also enhances for simple fluid as well as elastico-viscous fluid.

In mass transfer problems, the importance of Schmidt number cannot be neglected as it studies the combined effect of momentum and mass diffusion. Figure 9.6 notifies the effect of Schmidt number in this chapter. The rising nature of Schmidt number increases the viscosity of the fluid flow and hence it will slow down the speed of both Newtonian as well as visco-elastic fluid.

The effects of frequency parameter \( \omega \) and permeability parameter \( K \) are exhibited in the Figures 9.7 and 9.8 respectively. As expected, with the development of frequency parameter the fluid velocity drops off to a considerable amount. Also the speed of visco-elastic fluid is faster than that of the viscous fluid for various values of
frequency parameter. In Figure 9.8, we notice the behaviour of fluid velocity for different values of permeability parameter. In this case, the Newtonian fluid accelerates more in comparison with visco-elastic fluid and similar to frequency parameter $\omega$, for rising values of permeability parameter $K$, the fluid velocity diminishes.

Knowing the velocity field, it is important from practical point of view to know the effect of visco-elastic parameter on temperature field. Figure 9.9 depicts the transient temperature at the plate for the visco-elastic fluid ($k = 0.008$) for various values of flow parameters involved in the solution (Table 9.1). It is observed that the transient temperature increases with the increasing values of $Gr$ and $Gm$ but a decreasing trend can be observed for $Pr$, $Sc$, $\omega$ and $M$ for visco-elastic fluid. Also, the transient temperature weakens more with the enhancement of $Pr$ and $Sc$ than the other flow parameters.

Figures 9.10-9.14 illustrate the nature of viscous drag formed during the motion of Newtonian and visco-elastic fluids. The study of shearing stress experienced by the governing fluid flow gives the significance of the concerned problem. In all the Figures, it can be observed that shearing stress or viscous drag diminishes with the modified values of flow parameters.

Figures 9.10 and 9.11, illustrate the nature of shearing stress of both types of fluids against thermal Grashof number ($Gr$) and solutal Grashof number ($Gm$). In both the cases the shearing stress shows a decreasing trend for both Newtonian and non-Newtonian fluids.

The Schmidt number $Sc$ characterizes the behavior of simultaneous momentum and concentration diffusion. Figure 9.12, shows that the increasing value of $Sc$ declines the shearing stress at the plate for visco-elastic fluid in comparison with Newtonian fluid.

Figure 9.13 and 9.14 represent the effects of Lorentz force and radiation parameter on viscous drag. In non-Newtonian fluid flow mechanism, the visco-elastic fluid feels a reduction in shearing stress during the growth of Hartmann number ($M$) and radiation parameter ($F$).

The non-dimensional heat flux in terms of Nusselt number against various values of flow parameters are exhibited in Figures 9.15 to 9.18.
In these figures, it can be observed that Nu diminishes with the enhancement of solutal Grashof number Gr and Schmidt number Sc but a complete reverse trend is detected in case of thermal Grashof number Gm and permeability parameter K.

9.5 CONCLUSION:

The behaviour of mixed convective MHD boundary layer flow of a visco-elastic fluid past an inclined moving surface in presence of heat and mass transfer has been investigated in this study. Some of the important points are wrapped up as below:

- The effect of fluid velocity is pragmatic near the inclined surface of the plate.
- The temperature profile diminishes with the increasing values of flow parameters.
- The shearing stress formed at the plate is subdued with the growing trend of visco-elastic parameter.
- The Nusselt number boost up with the development of Gm and K.
- The rate of mass transfer is not significantly affected by visco-elastic parameter.

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Table 9.1: Various combinations of flow parameters.
Figure 9.1: Physical Model of the problem

Figure 9.2: Transient velocity $u$ versus $y$ for various values of Gr.

Figure 9.3: Transient velocity $u$ versus $y$ for various values of Pr.
Figure 9.4: Transient velocity $u$ versus $y$ for various values of Gm.

Figure 9.5: Transient velocity $u$ versus $y$ for various values of M.

Figure 9.6: Transient velocity $u$ versus $y$ for various values of Sc.
Figure 9.7: Transient velocity $u$ versus $y$ for various values of $\omega$.

Figure 9.8: Transient velocity $u$ versus $y$ for various values of $K$.

Figure 9.9: Transient temperature $\theta$ versus $y$. 
Figure 9.10: Variation of shearing stress $\sigma$ versus $Gr$.

Figure 9.11: Variation of shearing stress $\sigma$ versus $Gm$.

Figure 9.12: Variation of shearing stress $\sigma$ versus $Sc$. 
Figure 9.13: Variation of shearing stress $\sigma$ versus M

Figure 9.14: Variation of shearing stress $\sigma$ versus F.

Figure 9.15: Rate of heat transfer $\mathrm{Nu}$ versus Gr.
Figure 9.16: Rate of heat transfer Nu versus Gm.

Figure 9.17: Rate of heat transfer Nu versus Sc.

Figure 9.18: Rate of heat transfer Nu versus K.