CHAPTER - IV

Non-Darcy convective heat transfer flow of a viscous dissipative fluid in a vertical channel with constant heat flux.
1. INTRODUCTION

Any substance with a temperature above zero transfers heat in the form of radiation. Thermal radiation always exits and can strongly interact with convection in many situations of engineering interest. The influence of radiation on natural or mixed convection is generally stronger than that on forced convection because of the inherent coupling between temperature and flow field [3]. Convection in a channel (or enclosed space) in the presence of thermal radiation continues to receive considerable attention because of its importance in many practical applications such as furnaces, combustion chambers, cooling towers, rocket engines and solar collectors. During the past several decades, a number of experiments and numerical computations have been presented for describing the phenomenon of natural (or missed) convection in channel or enclosures. These studies aimed at clarifying the effect of mixed convection of flow and temperature regimes arising from variations in the shape of the channel (or enclosure), in fluid properties, in the transition to turbulence, etc. Chawla and Chan [9] studied the effect of radiation heat transfer on thermally developing Poiseuille flow. The interaction of thermal radiation with conduction and convection in thermally developing, absorbing-emitting, non gray gas flow in a circular tube is investigated by Tabanfar and Modest [23]. Natural convection-radiation interaction is studied by Yucel et. al., [28] for a square cavity, Lauriat [18] for a vertical cavity, and Chang et. al., [7] for a complex enclosure. Akiyama and Chang [2] numerically analysed the influence of gray surface radiation on the convection of nonparticipating fluid in a rectangular enclosed space.

A combined free and forced convection flow of an electrically conducting fluid in a channel, in the presence of a magnetic field, is also of special technical significance because of its frequent occurrence on many industrial applications such as cooling of nuclear reactors, MHD marine propulsion, electronic packages, micro electric devices, etc., some other quite promising applications are in the field of metallurgy such as MHD stirring of molten metals and magnetic-levitation...

Non-Darcy effects on natural convection in porous media have received a great deal of attention in recent years because of the experiments conducted with several combinations of solids and fluids covering wide ranges of governing parameters which indicate that the experimental data of systems other than glass water at low Rayleigh numbers do not agree with theoretical predictions based on the Darcy flow model. This divergence in the heat transfer results has been reviewed in detail in Cheng [9] and Prasad et. al., [15] among others. Extensive efforts are thus being made to include the inertia and viscous diffusion terms in the flow equations and to examine their effects in order to develop a reasonable accurate mathematical model for convective transport in porous media. The work of Vafai and Tien [26] was one of the early attempts to account for the boundary and inertia effects in the momentum equations for a porous medium.

They found that the momentum boundary layer thickness is of order of \( \sqrt{\frac{k}{\varepsilon}} \).

Vafai and Thyagaraga [26] presented analytical solutions for the velocity and temperature fields for the interface region using Brinkmann – Forchhimer – Extended Darcy equation. Detailed accounts of accounts of recent efforts on non-Darcy convection have been recently reported in Tien and Hong [24], Chang [7], Prasad et. al., [17] and Kalidas and Prasad [12]. Poulikakos and Bejan [20]
investigated the inertia effects through the inclusion of Forchheimer's velocity squared terms and presented the boundary layer analysis for all cavities. They also obtained numerical results for a few cases in order to verify the accuracy of their boundary layer solutions. Later Prasad and Tuntomo [16] reported an extensive numerical work for a wide range of parameters and demonstrated that effects of Prandtl number remain almost unaltered while the dependence on the modified Grashof number $G$ changes significantly with an increase in the Forchheimer number. They also reported a criterion for the Darcy flow limit.

The Brinkmann-Extended Darcy model was considered in Tong and Subramamian [25] and Lauriat and Prasad [14] to examine the boundary effects on free convection in a vertical cavity. While Tong and Subramanian performed Weber-type boundary layer analysis. Lauriat and Prasad solved the problem numerically for $A = 1$ and $5$. It was shown that for a fixed Rayleigh number $Ra$, the Nusselt number decreases with an increase in the Darcy number, the reduction being larger at higher values of $Ra$. A scale analysis as well as the computational data also showed that the transport term $(v \cdot \nabla)v$, is of low order or magnitude compared to the diffusion plus buoyancy terms. A numerical study based on the Forchheimer-Brinkmann-Extended Darcy equation of motion has also been reported recently by Beckermann et. al., [4]. They demonstrated that the inclusion of both the inertia and boundary effects is important for convection in a rectangular packed-sphere cavity. Recently Sivasankar Reddy [21] has discussed the effects of radiation on a non-Darcy convective flow in a circular duct under different conditions.

All the above studies are based on the hypothesis that the effect of dissipation is neglected. This is possible in case of ordinary fluid flow like air and water under gravitational force. But this effect is expected to be relevant for fluids with high values of the dynamic viscosity force. Moreover Gebhart [10a], Gebhart and Mollendorf [10b] have shown that viscous dissipation heat in the natural convective flow is important when the fluid of extreme size or at
extremely low temperature or in high gravitational field. On the other hand Barletta [3a] and Zanchini [29,30] pointed out that relevant effects of viscous dissipations in the temperature profiles and in the Nusselt Number may occur in the fully developed convection in tubes, in view of this several authors notably, Soundalgekar and Pop [22a], Soundalgekar et al [22b], Barletta [3a,3b], Bulent Yasilats [4a] and Anwar Hossain [1] have studied the effect of viscous dissipation on the convective flows past an infinite vertical plates and through vertical channels and ducts. Ravindranath [18a] has analysed the dissipative effects on the convective heat transfer in channels under different conditions.

In the chapter we investigate the effect of radiation on the Non-Darcy convection heat transfer flow of a viscous dissipative electrically conducting fluid through a porous medium confined in a vertical channel. The non-linear, coupled equations governing the flow and heat transfer have been solved by employing a perturbation technique with $\delta$ the porous parameter as perturbation parameter. The velocity and temperature dissipation are analysed for different values of $G, D^1, N, \alpha, Ec$ and $N_t$. Also the shear stress and the rate of heat transfer on the walls are numerically evaluated for different sets of parameters.
Configuration of the Problem

\[ \frac{\partial T}{\partial y} = -q \]

\[ T = T_1 \]

\[ y = -L \]

\[ y = +L \]
2. FORMULATION OF THE PROBLEM

We consider the flow of a viscous, incompressible, electrically conducting fluid through a porous medium confined in a vertical channel in the presence of heat sources. A Cartesian coordinate system \( O(x,y,z) \) is used so that the boundaries are taken at \( y = \pm L \), where \( 2L \) is the distance between the walls. Boussinesq approximation is used so that the density variation is taken only in the buoyancy force term. A uniform magnetic field of strength \( H_0 \) is applied normal to the walls. Assuming the magnetic Reynolds number to be small we neglect the induced magnetic field in comparison to the applied field. The plate at \( y = -L \) is maintained at constant temperature \( T_1 \) and the plate at \( y = +L \) is maintained at a constant flux. The temperature gradient in the flow is sufficient to cause natural convection in the flow field. A constant axial pressure gradient is also imposed so that this resultant flow is a mixed convection flow. We also take dissipation into account in the energy equation. The porous matrix is assumed to be isotropic and homogeneous with constant porosity and the effective thermal diffusivity. The equations governing the flow and heat transfer are

Momentum equation:
\[
-\frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial y^2} \right) - \left( \frac{\mu}{k} + \frac{\sigma_m H_0^2}{\rho_0} \right) u - \frac{\rho \delta F}{\sqrt{k}} u^2 - \rho g = 0
\] (2.1)

Energy equation:
\[
\rho_C \rho^u \frac{\partial T}{\partial x} = \lambda \frac{\partial^2 T}{\partial y^2} + \frac{\partial (q_x)}{\partial y} + \mu \frac{\partial u}{\partial y}^2
\] (2.2)

Equation of State:
\[
\rho - \rho_c = -\beta (T - T_c)
\] (2.3)
Boundary conditions:

\[ u = 0 \quad \text{on} \quad y = \pm L \]

\[ T = T_1 \quad \text{on} \quad y = -L \]

\[ \frac{\partial T}{\partial y} = -q \quad \text{on} \quad y = +L \]

(2.4)

where \( T \) is the temperature of the fluid, \( \rho \) is the density of the fluid, \( C_p \) is the specific heat at constant pressure, \( k \) is the permeability of the porous medium, \( \mu \) is the coefficient of viscosity of the fluid, \( \delta \) is the porosity of the medium, \( \beta \) is the coefficient of thermal expansion, \( \lambda \) is the coefficient of thermal conductivity and \( F \) is the function that depends on the Reynolds number and microstructure of porous medium, \( \sigma \) is the electrical conductivity, \( \mu_e \) is the magnetic permeability, \( \lambda_0 \) is the strength of the magnetic field and \( Q \) is the strength of the heat source.

The axial gradient \( \frac{\partial T}{\partial x} \) is assumed to be a constant, say \( A \).

Invoking Rosseland approximation for radiation flux we get

\[ q_r = -4\sigma^* \frac{\partial \left( T^4 \right)}{\partial y} \]

and linearising \( T^4 \) about \( T_e \) by using Taylor's expansion and neglecting higher order terms we get

\[ T^4 \approx 4T_e^3T - 3T_e^4 \]

where \( \sigma^* \) is the Stefan-Boltzmann constant and \( \beta_R \) is the mean absorbing coefficient.
We introduce the following non-dimensional variables, as

\[(x', y') = (x, y)/L, u' = u/(v/L), p' = p/(\rho v^2/L^2)\]

\[\theta = \frac{T - T_s}{T_1 - T_2}\]  

(2.5)

Substituting (2.5) the equations in the non-dimensional form are

\[-\frac{dp}{dx} = \frac{d^2 u}{dy^2} - \delta (D^{-1} + M^2) u - \delta G \theta\]

(2.6)

\[P \eta u = \frac{d^2 \theta}{dy^2} + \alpha + PE_c \left(\frac{du}{dy}\right)^2 + \frac{4}{3N} \frac{d^2 \theta}{dy^2}\]

(2.7)

\[u(\pm 1) = 0,\]

\[\theta(-1) = 1,\]

\[\frac{d \theta}{dy} (+1) = -1\]

(2.8)

\[G = \beta g \Delta T L_1^3/v^2\]  

(Grashof Number)

\[D^{-1} = \frac{L^2}{k}\]  

(Darcy parameter)

\[M^2 = \frac{\sigma H^2 L_2^3}{v^2}\]  

(Hartmann Number)

\[\alpha = \frac{Q L_1^2}{k, \Delta T}\]  

(Heat Source parameter)
\[ N = \frac{\beta R k_f}{4\sigma' T_e^4} \]  
(Radiation parameter)

\[ N_T = \frac{A}{\Delta T} \]  
(Non-dimensional temperature gradient)

\[ P = \frac{\mu C_p}{k_f} \]  
(Prandtl Number)

\[ Ec = \frac{\nu^2}{C_p \Delta T} \]  
(Eckert number)

\[ P_1 = \frac{3NP}{3N + 4} \quad \alpha_1 = \frac{3N\alpha}{3N + 4} \]

**3. SOLUTION OF THE PROBLEM**

Assuming the parameter \( \delta \) to be small we assume the solutions as

\[ u(y) = u_0(y) + \delta u_1(y) + \delta^2 u_2(y) + \ldots \]

\[ \theta(y) = \theta_0(y) + \delta \theta_1(y) + \delta^2 \theta_2(y) + \ldots \]  
(3.1)

Substituting (3.1) in equations (2.5)-(2.8) and equating the like powers of \( \delta \) the equations to the zeroth order are

\[ \frac{d^2 u_0}{dy^2} = \pi \]  
(3.2)

\[ \frac{d^2 \theta_0}{dy^2} = (P_1 N_T) u_0 + P_1 E c \left( \frac{du_0}{dy} \right)^2 - \alpha \]  
(3.3)

to the first order are
\[
\frac{d^2 u_1}{dy^2} - \beta_1 u_1 = -G \theta_0
\] (3.4)

\[
\frac{d^2 \theta}{dy^2} = (P_N)f u_1 + 2P_iE_c \left( \frac{du_0}{dy} \right) \left( \frac{du_1}{dy} \right)
\] (3.5)

and to the second order are

\[
\frac{d^2 u_2}{dy^2} - \beta_1^2 u_2 = n(u_0)^2 - G \theta_1
\] (3.6)

\[
\frac{d^2 \theta_2}{dy^2} = (P_N)f u_2 + P_iE_c \left( \frac{du_1}{dy} \right)^2 + 2 \left( \frac{du_0}{dy} \right) \left( \frac{du_2}{dy} \right)
\] (3.7)

**Boundary Conditions**

\[
u_o (+1) = 0 ; \quad u_o (-1) = 0 ;
\]

\[
\frac{d \theta_0}{dy} (+1) = -1 ; \quad \theta_0 (-1) = 1
\] (3.8)

\[
u_i (± 1) = 0 ; \quad \theta_i (± 1) = 0
\]

\[
\frac{d \theta_i}{dy} (+1) = 0
\] (3.9)

\[
u_z (± 1) = 0 ; \quad \theta_z (± 1) = 0
\]

\[
\frac{d \theta_z}{dy} (+1) = 0
\] (3.10)

\[
\beta_1^2 = M^2 + D^{-1}
\]

Solving the equations (3.2)-(3.7) subject to the boundary conditions (3.8)-(3.10) we get.

\[
u_o = \frac{(\pi - \alpha)}{2} (y^2 - 1)
\]
\[ \theta_{1}(y) = 0.5a_{1}y(y + 1) + \frac{a_{2}}{12} y(y^3 - 1) - a_{5}(y + 1) + (y + 2) \]

\[ u_{1}(y) = a_{11}(1 - \frac{Ch(\beta_{1}y)}{Ch(\beta_{1})}) + a_{12}(y - \frac{Sh(\beta_{1}y)}{Sh(\beta_{1})}) + a_{13}(y^2 - \frac{Ch(\beta_{1}y)}{Ch(\beta_{1})}) + \]

\[ + a_{14}(y^4 - \frac{Ch(\beta_{1}y)}{Ch(\beta_{1})}) \]

\[ \theta_{1}(y) = B_{1}(ySh(\beta_{1}y) - ySh(\beta_{1}) - \beta_{1}yCh(\beta_{1}) - 2Sh(\beta_{1}) - \beta_{1}Ch(\beta_{1})) + \]

\[ + B_{2}(yCh(\beta_{1}y) + \beta_{1}ySh(\beta_{1}) - yCh(\beta_{1}) - \beta_{1}Sh(\beta_{1}) - 2Ch(\beta_{1})) + \]

\[ + B_{3}(Ch(\beta_{1}y) - \beta_{1}ySh(\beta_{1}) - \beta_{1}Sh(\beta_{1}) - Ch(\beta_{1})) + \]

\[ + B_{4}(Sh(\beta_{1}y) + \beta_{1}yCh(\beta_{1}) + Sh(\beta_{1})) + B_{5}(y^3 - 3y + 2) + \]

\[ + B_{6}(y^4 - 4y - 5) + B_{7}(y^7 - 6y - 7) + 3 \]

\[ u_{2}(y) = B_{17}(y^2 - 1)Sh(\beta_{1}y) + B_{18}(y^2 - 1)Ch(\beta_{1}y)) + \]

\[ + B_{19}(ySh(\beta_{1}y) - ySh(\beta_{1}) - Ch(\beta_{1})yCh(\beta_{1})) + \]

\[ + B_{20}(yCh(\beta_{1}y) - Ch(\beta_{1})ySh(\beta_{1})) + \]

\[ + B_{21}(\frac{Ch(\beta_{1}y)}{Ch(\beta_{1})} - y^6) + B_{22}(y^4 - \frac{Ch(\beta_{1}y)}{Ch(\beta_{1})}) + \]

\[ + B_{23}(\frac{Ch(\beta_{1}y)}{Ch(\beta_{1})} - Sh(\beta_{1}y) + B_{24}(y - \frac{Sh(\beta_{1}y)}{Sh(\beta_{1})}) + \]

\[ + B_{25}(1 - \frac{Ch(\beta_{1}y)}{Ch(\beta_{1})}) \]
4. SHEAR STRESS, NUSSELT NUMBER AND SHERWOOD NUMBER

The shear stress on the walls is given by

\[ \tau = \mu \left( \frac{du}{dy} \right)_{y=L} \]

which in the non-dimensional form is

\[ \tau = \frac{\tau}{\mu(T_1 - T_2)/L} = \left( \frac{du}{dy} \right)_{y=L} \]

and the corresponding expressions are

\[ (\tau)_{y=1} = \pi - \delta \left( a_{11} \beta_1 Th(\beta_1) + a_{12} (1 - \beta_1 Ch(\beta_1) + a_{13} (2 - \beta_1 th(\beta_1) + \right. \]
\[ + a_{14} (4 - \beta_1 th(\beta_1)) + \delta^2 (2B_{17}Sh(\beta_1) + 2B_{18}Ch(\beta_1) + B_{19} (Sh(\beta_1) + \]
\[ + \beta_1 Ch(\beta_1) - \beta_1 Th(\beta_1) Sh(\beta_1)) + B_{20} (Ch(\beta_1) + \beta_1 Sh(\beta_1) - \]
\[ - \beta_1 Ch(\beta_1) Ch(\beta_1) + B_{21} (6 - \beta_1 Th(\beta_1)) + B_{22} (4 - \beta_1 Th(\beta_1)) + \]
\[ + B_{23} (2 - \beta_1 Th(\beta_1)) + B_{24} (1 - \beta_1 Ch(\beta_1)) - B_{25} \beta_1 Th(\beta_1) \]

\[ (\tau)_{y=-1} = -\pi + \delta \left( a_{11} \beta_1 Th(\beta_1) + a_{12} (1 - \beta_1 Ch(\beta_1) - a_{13} (2 - \beta_1 Th(\beta_1) + \right. \]
\[ - a_{14} (4 - \beta_1 Th(\beta_1)) + \delta^2 (2B_{17} Sh(\beta_1) - 2B_{18} Ch(\beta_1) - B_{19} (Sh(\beta_1) + \]
\[ + \beta_1 Ch(\beta_1) + \beta_1 Th(\beta_1) Sh(\beta_1)) + B_{20} (Ch(\beta_1) + \beta_1 Sh(\beta_1) + \]
\[ + \beta_1 Ch(\beta_1) Ch(\beta_1) - B_{21} (6 - \beta_1 Th(\beta_1)) - B_{22} (4 - \beta_1 Th(\beta_1)) + \]
\[ - B_{23} (2 - \beta_1 Th(\beta_1)) + B_{24} (1 - \beta_1 Ch(\beta_1)) + B_{25} \beta_1 Th(\beta_1) \]

The rate heat transfer (Nusselt number) on the walls is given by

\[ q = -k \left( \frac{dT}{dy} \right)_{y=L} \]
which in the non-dimensional form is

\[ Nu = - \frac{qL}{(T_1 - T_2)k} = \left( \frac{d\Theta}{dy} \right)_{y=1} \]

and the corresponding expressions are

\[ (Nu)_{y=1} = -B_1 \delta \beta_1 Ch(\beta_1) + Sh(\beta_1)) + B_2 \delta (Ch(\beta_1) - \beta_1 Sh(\beta_1)) - \delta B_3 \beta_1 Sh(\beta_1) + \delta B_4 (Ch(\beta_1)) + 3\delta B_5 - 4\delta B_6 - 6\delta B_7 + a_{24} + a_1 - \frac{a_2}{3} + a_3 \]
5. DISCUSSIONS OF THE NUMERICAL RESULTS

In this analysis we discuss the effect of radiation on the non-Darcy convective heat transfer flow of viscous electrically conducting fluid through a porous medium in a vertical channel with quadratic density temperature variation by using a regular perturbation technique. The Governing equations are solve to obtain the expressions for velocity, temperature, shear stress and the rate of heat transfer.

Fig.8 1-7 represent variation of axial velocity with different values of G, D', M, α, N, N_T and Ec. Fig. 1 represents the variation u with Grashof number G. We find that the velocity is in the vertically downward direction. The magnitude of u depreciates everywhere in the flow region with increase in G. The variation of u with D' and M is shown in fig.2 & 3. It is found that lesser the permeability porous medium/higher the Lorentz force larger the magnitude of u in the entire flow region. The variation of u with heat source parameter α is exhibited fig. 4. An increase in the strength of heat source depreciates |u| in the flow region while it enhances with increase in the strength of heat sink (fig 4). From fig.5, we notice that an increase in the radiation parameter N leads to a depreciation in the magnitude of u. Also an increase in the temperature gradient N_T accelerates |u| (fig.6). The variation u with Eckert number Ec is shown in fig.7. It is found that |u| experiences a depreciation in magnitude with increase in Ec. Thus the viscous effects reduces the magnitude of u in the entire flow region.

The non-dimensional temperature (θ) is shown in figs 8-14 for different parametric values. The profile θ gradually reduces from prescribed value on the left boundaries y = -1 to attain the value ‘0’ on the right boundary y = 1, the variation of θ with Grashof number G is shown in fig. 8. It is found that the actual temperature experiences an enhancement with increase in G (fig.8). The variation of θ with heat source parameter α shows that a increase in the strength of the heat source reduces the actual temperature while it enhances with increase in the
Fig. 1: Variation of $u$ with $G$

- $G = 10^3$
- $G = 3 \times 10^3$
- $G = 5 \times 10^3$

Fig. 2: Variation of $u$ with $D$

- $D = 10^1$
- $D = 2 \times 10^2$
- $D = 3 \times 10^2$
Fig. 3: Variation of $u$ with $M$

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<tr>
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Fig. 4: Variation of $u$ with $\alpha$

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Fig. 5: Variation of $u$ with $N_T$

- I
- II
- III
- IV

$N_T = 0.25, 0.75, 1.2$

Fig. 6: Variation of $u$ with $N$

- I
- II
- III
- IV

$N = 1.5, 5, 10, 100$
Fig. 7: Variation of $u$ with $E_c$
I II III IV
$E_c$ 0.001 0.005 0.007 0.009

Fig. 8: Variation of $\theta$ with $G$
I II III
$E_c$ $10^3$ $3 \times 10^3$ $5 \times 10^3$
Fig. 9: Variation of $\theta$ with $D^{-1}$

- $D^{-1}$
  - $10^2$
  - $2 \times 10^2$
  - $3 \times 10^2$

Fig. 10: Variation of $\theta$ with $M$

- $M$
  - 2
  - 4
  - 6
### Table 1
Shear stress (τ) at y = 1

<table>
<thead>
<tr>
<th>G</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
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### Table 2
Shear stress (τ) at y = 1

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### Table 3
Shear stress (τ) at y = -1

<table>
<thead>
<tr>
<th>G</th>
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<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
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<th>VII</th>
<th>VIII</th>
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<tbody>
<tr>
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### Table 4
Shear stress (τ) at y = -1

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### Table - 5
Nusselt Number Nu at y = 1

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<th>II</th>
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<th>IX</th>
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<tbody>
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<td>0.5352</td>
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<td>0.9978</td>
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<td>0.9930</td>
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### Table - 6
Nusselt Number Nu at y = -1

<table>
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<tr>
<th>G</th>
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<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
<th>X</th>
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</thead>
<tbody>
<tr>
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<td>1.2609</td>
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</table>

| $N$   | 0.5    | 1.5    | 4      | 10     | 0.5    | 0.5    | 0.5    | 0.5    | 0.5    | 0.5    |
| $N_T$ | 0.5    | 0.5    | 0.5    | 0.5    | 0.5    | 2.5    | 0.5    | 0.5    | 0.5    | 0.5    |
| $Ec$  | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   |

### Table - 7
Nusselt Number Nu at y = 1

<table>
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<th>G</th>
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<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
<th>X</th>
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</tr>
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<tbody>
<tr>
<td>$M$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
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<td>2</td>
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</tr>
<tr>
<td>$\alpha$</td>
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</table>

### Table - 8
Nusselt Number Nu at y = -1

<table>
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<th>III</th>
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<th>V</th>
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<th>VII</th>
<th>VIII</th>
<th>IX</th>
<th>X</th>
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| $N$   | 0.5    | 1.5    | 4      | 10     | 0.5    | 0.5    | 0.5    | 0.5    | 0.5    | 0.5    |
| $N_T$ | 0.5    | 0.5    | 0.5    | 1.5    | 2.5    | 0.5    | 0.5    | 0.5    | 0.5    | 0.5    |
| $Ec$  | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   | 0.03   |
strength of the heat sink (fig.9). From fig. 10 we find that the actual temperature reduces with increase in the radiation parameter N. Also it enhances with temperature Gradient \( N_T \) with respect to the behaviour of \( \theta \). With \( E_c \), we find that the \( \theta \) exhibits a decreasing tendency with \( E_c \). Thus the dissipative heat reduces the actual temperature in the flow region.

The shear stress at \( y = \pm 1 \) is exhibited in tables 1-4 for different values of \( G, D^{-1}, M, \alpha, N, N_T \) and \( E_c \). We find that the magnitude of the stress enhances with increase in \( G > 0 \) and reduces with \( G < 0 \) at both the walls. The variations of \( \tau \) with \( D^{-1} \) and \( M \) shows that lesser the permeability porous medium/higher the Lorentz force larger the magnitude of \( \tau \) at \( y = \pm 1 \). \( |\tau| \) depreciates with increase in the strength of the heat source while it enhances with increase in the strength of the heat sink. With respect to radiation parameter \( N \) we find that \( |\tau| \) experiences a depreciation at both the walls. (tables 1-3). Also it reduces with increase in the temperature Gradient \( N_T \) at \( y = \pm 1 \) the variation of \( \tau \) with Eckert number \( E_c \) exhibits that at \( y = 1 \) \( |\tau| \) enhances with \( E_c \leq 0.15 \) and depreciate with higher \( E_c \geq 0.35 \) in the heating case while a reversed effect is observed in the cooling of channel walls at \( y = -1 \). The stress reduces with \( E_c \) in the heating case and enhances with \( E_c \) in the cooling case (Tables 2 and 4). The Nusselt number (Nu) which measures the rate of heat transfer at the boundaries \( y = \pm 1 \) is shown in tables in 5-11 for different values of \( G, D^{-1}, M, N, N_T, \alpha \) and \( E_c \). It is found the rate of heat transfer at \( y = +1 \) enhances with \( G > 0 \) and depreciates with \( G < 0 \) while a reversed effect is observed at \( y = -1 \). The variation of Nu with \( D^{-1} \) and \( M \) shows that lesser the permeability of the porous medium/higher the Lorentz force smaller the rate of heat transfer at \( y = 1 \) for all \( G \) while at \( y = -1 \) larger \( |Nu| \) in the heating case and smaller \( |Nu| \) in the cooling case. (tables 5 and 8). The variation of Nu with heat source parameter \( \alpha \) shows that the rate of heat transfer at \( y = 1 \) enhances with increase in \( \alpha > 0 \) and depreciates with \( \alpha < 0 \) in the heating case while reversed effect is observed in the cooling case. At \( y = -1 \) the rate of heat transfer depreciates with \( \alpha \leq 4 \) and enhances with higher \( \alpha \geq 6 \) and it enhances with
increase $|\alpha|$ (<0) in both heating and cooling cases (table 5 and 8). The variation of $\text{Nu}$ with radiation parameter $N$ shows that $|\text{Nu}|$ at $y = 1$ enhances with increase in $N$ in the heating case and depreciates in the cooling case while at $y = -1$ the rate of heat transfer experiences a depreciation with increase in $N$ for all $G$. An increase in the temperature gradient $N_T$ enhances $|\text{Nu}|$ in the heating case and depreciates in the cooling case at $y = 1$ while at $y = -1$ it depreciates with increase in $N_T \leq 1.5$ and enhances with $N_T \geq 2.5$ for all $G$ (table 6 and 10). The variation of $\text{Nu}$ with Eckert number $E_c$ shows that the rate of heat transfer enhances at $y = 1$ and depreciates at $y = -1$ with increase in $E_c$ in the heating case and a reversed behaviour is observed in the rate of heat transfer with $E_c$ in the cooling of the channel walls (tables 7 and 11).
6. REFERENCES


7. APPENDIX

\[
a_i = \frac{P_i N_i}{2\alpha_i}, \quad a_2 = \frac{2}{\alpha_1} + 1
\]

\[
a_8 = G\left(\frac{a_1 + Na_3}{\beta_1^2 Ch(\beta_1)} + a_4 a_5\right) \quad a_9 = G\left(a_4 + 0.5a_5\right)
\]

\[
a_{10} = G\left(a_1 a_2 + \frac{Na_3}{\beta_1^2 Ch(\beta_1)}\right) \quad a_{11} = G\left(\frac{Na_4}{\beta_1^2 Sh(\beta_1)} - 0.5\right)
\]

\[
a_{12} = G\left(\frac{a_4}{\beta_1^2 Sh(\beta_1)} + 0.5\right) \quad a_{13} = GN\left(1 + a_4\right)/2
\]

\[
a_{14} = \frac{a_8}{Ch(\beta_1) (\beta_1^2 - \beta_2^2)} \quad a_{15} = \frac{a_8}{\beta_2^2}
\]

\[
a_{16} = \frac{a_{11}}{Sh(\beta_1) (\beta_1^2 - \beta_2^2)} \quad a_{17} = \frac{a_{12}}{\beta_2^2}
\]

\[
D_3 = \frac{(a_{14} Ch(\beta_1) + a_{16} - a_{18})}{Ch(\beta_2)} \quad a_{19} = \frac{P_i N_i D_3}{\beta_1^2 - \beta_2^2}
\]

\[
a_{20} = \frac{P_i N_i D_4}{\beta_1^2 - \beta_2^2} \quad a_{21} = \frac{a_{16}}{\beta_2^2}
\]

\[
a_{22} = \frac{a_{12}}{\beta_2^2} \quad a_{23} = \frac{2a_{16} - a_{18}}{\beta_2^2}
\]
\[ a_{24} = \frac{a_{34}}{2\beta_1} \quad a_{23} = \frac{a_{35}}{2\beta_1} \]

\[ D_5 = \frac{a_{36}Sh(\beta_1) - a_{23} - a_{31} - a_{30}Ch(\beta_1)}{Ch(\beta_1)} \]

\[ D_6 = -\left( a_{36}Sh(\beta_2) + a_{22} - a_{25}Ch(\beta_1) \right) \frac{1}{Sh(\beta_1)} \]

\[ a_{38} = \frac{a_{39}}{\beta_1^2} \quad a_{39} = \frac{a_{38}}{\beta_1^2} \]

\[ a_{40} = \frac{a_{39}}{\beta_1^3} \quad a_{41} = \beta_1^2 a_{30} \]

\[ a_{42} = \beta_1^3 a_{31} \quad a_{43} = \frac{a_{32}}{12} \]

\[ a_{44} = \frac{a_{33}}{6} \quad a_{45} = \frac{a_{34}}{2} \]

\[ a_{46} = D_5 + N\alpha_{39} \]

\[ a_{47} = D_5 + N\alpha_{39} \quad a_{48} = a_{49} + N\alpha_{37} \]

\[ a_{49} = a_{30} + N\alpha_{38} \quad a_{50} = a_{25} - N\alpha_{41} \]

\[ a_{51} = a_{24} + N\alpha_{42} \quad a_{52} = G\alpha_{43} \]
\[ a_{53} = G N a_{44} , \quad a_{54} = a_{21} + N a_{45} \]

\[ a_{55} = a_{22} - N a_{35} , \quad a_{56} = a_{23} + N a_{36} \]

\[ a_{57} = a_{57} + \frac{\beta_1}{4} , \quad a_{58} = a_{54} + \frac{\beta_1^2}{4} \]

\[ a_{59} = \frac{\beta_1^2}{4} - a_{56} , \quad a_{60} = \frac{a_{46}}{\beta_1^2 - \beta_2^2} - \frac{2 P_i a_{50}}{(\beta_1^2 - \beta_2^2)^2} \]

\[ a_{61} = \frac{a_{47}}{\beta_1^2 - \beta_2^2} - \frac{2 P_i a_{41}}{(\beta_1^2 - \beta_2^2)^2} \]

\[ a_{62} = \frac{a_{48}}{2 \beta_2} , \quad a_{63} = \frac{a_{49}}{2 \beta_2} \]

\[ a_{64} = \frac{a_{50}}{\beta_1^2 - \beta_2^2} , \quad a_{65} = \frac{a_{51}}{\beta_1^2 - \beta_2^2} \]

\[ a_{66} = \frac{a_{47}}{\beta_1^2 - \beta_2^2} , \quad a_{67} = \frac{a_{53}}{\beta_2^2} \]

\[ a_{68} = -\frac{12 a_{51} - a_{58}}{\beta_1^2} - \frac{a_{58}}{\beta_2^2} , \quad a_{69} = \frac{6 a_{43} + a_{55}}{\beta_2^4} + \frac{a_{55}}{\beta_2^2} \]

\[ a_{70} = \frac{a_{60}}{\beta_2^2} , \quad a_{71} = \frac{(P_i N_j) D_8}{\beta_2^2 - \beta_1^2} \]

\[ a_{72} = \frac{(P_i N_j) D_8}{\beta_2^2 - \beta_1^2} , \quad a_{73} = \frac{2 \beta_2}{\beta_2^2 - \beta_1^2} \]
\[
a_{24} = \frac{a_{62} - P_1 N_T}{\beta_1^2 - \beta_1^2} , \quad a_{75} = \frac{a_{60} P_1 N_T}{2 \beta_1} + \frac{(P_1 N_T) D_3 + P_1 N_T a_{65}}{4 \beta_1^2} \\

a_{76} = -\frac{P_1 N_T a_{60}}{2 \beta_1} , \quad a_{77} = \frac{P_1 N_T a_{64}}{4 \beta_1^2} \\

a_{78} = \frac{P_1 N_T a_{65}}{4 \beta_1^2} , \quad a_{79} = \frac{P_1 N_T a_{66}}{4 \beta_1^2} \\

a_{80} = \frac{P_1 N_T a_{67}}{\beta_1^2} , \quad a_{81} = \frac{12 P_1 N_T a_{67}}{\beta_1^2} - \frac{P_1 N_T a_{68}}{\beta_1^2} \\

a_{82} = \frac{6 P_1 N_T a_{68}}{\beta_1^2} + \frac{P_1 N_T a_{69}}{\beta_1^2} , \quad a_{83} = \frac{P_1 N_T a_{70}}{\beta_1^2} \\

b_1 = \beta_1^2 D_0 + 2 \beta_1 a_{75} - 2 a_{77} , \quad b_2 = \beta_1^2 D_{10} + 2 \beta_1 a_{76} - 2 a_{78} \\

b_3 = \beta_1^2 a_{76} - 4 \beta_1 a_{78} , \quad b_4 = \beta_1^2 a_{73} \\

b_5 = \beta_2^2 a_{71} + 2 \beta_1 a_{74} , \quad b_6 = \beta_1^2 a_{72} + 2 \beta_3 a_{73} \\

b_7 = -\frac{ScSo}{N} \beta_1^3 a_{77} , \quad b_8 = -\frac{ScSo}{N} \beta_1^2 a_{74} \\

b_9 = -2 \frac{ScSo}{N} a_{79} - NcSa_{88} , \quad b_{10} = NcSa_{96} \\

b_{11} = NcSa_{67} , \quad b_{12} = -\frac{ScSo}{N} a_{80} + NcSa_{69} \\

b_{16} = \beta_1^2 b_1 + 2 \beta_1 b_4 , \quad b_{17} = \beta_2^2 b_2 + 2 \beta_1 b_3
\]
\begin{align*}
b_{18} &= \beta_1^2 b_1, \quad b_{19} = \beta_1^2 b_4 \\
b_{20} &= \beta_2^2 b_5 + 2 \beta_1 b_8, \quad b_{21} = \beta_1^2 b_7 \\
b_{23} &= \beta_2^3 b_8, \quad b_{24} = 112 b_1 \\
b_{27} &= -a_1 (\beta_2 \text{Th}(\beta_1) + 2) - a_1 a_2 \beta_1 \text{Th}(\beta_1) - \frac{\beta_1}{2} (\text{Th}(\beta_1) + \text{Cth}(\beta_1)) \\
b_{28} &= -a_1 (\beta_2 \text{Th}(\beta_1) + 2) + a_1 a_2 \beta_1 \text{Th}(\beta_1) + \frac{\beta_1}{2} (\text{Th}(\beta_1) - \text{Cth}(\beta_1)) \\
b_{29} &= -a_1 \beta_1 \text{Th}(\beta_1) + \frac{a_4}{\beta_1^3 \text{Sh}(\beta_1)} (\beta_1 \text{Ch}(\beta_1) - \text{Sh}(\beta_1)) - 0.2 - \frac{2a_5}{3} \\
b_{30} &= -\frac{a_3}{\beta_1^3} \text{Th}(\beta_1) + \frac{a_3}{\beta_1^3 \text{Sh}(\beta_1)} (\beta_1 \text{Ch}(\beta_1) - \text{Sh}(\beta_1)) - 0.2 + \frac{2a_5}{3} \\
b_{31} &= -a_{14} (\beta_2 \text{Th}(\beta_1) \text{Ch}(\beta_1) - \beta_1 \text{Sh}(\beta_1)) - a_5 (\beta_2 \text{Th}(\beta_1) + 2) \\
&\quad + a_{18} \beta_2 \text{Th}(\beta_2) + a_{15} (\beta_1 \text{Ch}(\beta_1) - \beta_2 \text{Cth}(\beta_2) - \text{Sh}(\beta_1)) + \\
&\quad + a_{17} (1 - \beta_2 \text{Cth}(\beta_2)) \\
b_{32} &= a_{14} (\beta_2 \text{Th}(\beta_2) \text{Ch}(\beta_1) - \beta_1 \text{Sh}(\beta_1)) + a_{16} (\beta_2 \text{Th}(\beta_2) + 2) - \\
&\quad - a_{15} \beta_1 \text{Th}(\beta_2) + a_{14} (\beta_1 \text{Ch}(\beta_1) - \beta_2 \text{Cth}(\beta_2) \text{Sh}(\beta_1)) + \\
&\quad + a_{17} (1 - \beta_2 \text{Cth}(\beta_2)) \\
b_{33} &= -a_{24} (\beta_2 \text{Th}(\beta_1) \text{Sh}(\beta_1) + \beta_1 \text{Ch}(\beta_1) + \text{Sh}(\beta_1)) + a_{23} \beta \text{Th}(\beta_1) + \\
&\quad + a_{21} (2 + \beta_2 \text{Th}(\beta_1)) - a_{15} (\beta_2 \text{Sh}(\beta_2) - \beta_2 \text{Th}(\beta_2) \text{Ch}(\beta_1)) + \\
&\quad + a_{22} (\beta_1 \text{Cth}(\beta_1) - 1) - a_{25} (\beta_1 \text{Sh}(\beta_1) - \text{ch}(\beta_1) + \beta_1 \text{Cth}(\beta_1) \text{Ch}(\beta_1))
\end{align*}
\[ b_{34} = a_{24} (\beta_1 Th(\beta_1) Sh(\beta_1) + \beta_1 Ch(\beta_1) + Sh(\beta_1)) - a_{23} \beta_1 Th(\beta_1) - a_{21} (2 + \beta_1 Th(\beta_1) + a_{19} (\beta_1 Sh(\beta_1) - \beta_1 Th(\beta_1) Ch(\beta_1)) + a_{22} (\beta_1 Cth(\beta_1) - 1) - a_{25} (\beta_1 Sh(\beta_1) - Ch(\beta_1) + \beta_1 Cth(\beta_1) Ch(\beta_1)) \]

\[ b_{35} = -a_{37} \beta_2 Th(\beta_1) + a_{38} (\beta_2 Ch(\beta_1) - Sh(\beta_1)) - a_{39} \beta_1 Sh(\beta_1) + a_{40} (\beta_1 Ch(\beta_1) - Sh(\beta_1)) + a_{41} \beta_1 Sh(\beta_1) + a_{42} (\beta_1 Ch(\beta_1) - Sh(\beta_1)) - \]

\[ - Sh(\beta_1) - 4a_{43} + 2a_{44} + 2a_{45} \]

\[ b_{36} = a_{36} \beta_2 Sh(\beta_1) + a_{34} (\beta_2 Ch(\beta_1) - Sh(\beta_1)) + a_{39} \beta_1 Sh(\beta_1) + a_{40} (\beta_1 Ch(\beta_1) - Sh(\beta_1)) + a_{41} \beta_1 Sh(\beta_1) + a_{42} (\beta_1 Ch(\beta_1) - Sh(\beta_1)) + \]

\[ 4a_{43} + 2a_{44} - 2a_{45} \]

\[ b_{37} = -a_{60} (\beta_2 Th(\beta_1) Ch(\beta_1) - \beta_1 Sh(\beta_1)) - a_{62} (\beta_2 Th(\beta_1) Sh(\beta_1) + \beta_1 Ch(\beta_1) + Sh(\beta_1)) - a_{64} (\beta_2 Th(\beta_1) + 4) + a_{65} (\beta_2 Th(\beta_1) + 2) - a_{67} \beta_2 Th(\beta_1) + a_{68} (\beta_2 Cth(\beta_1) Sh(\beta_1) - \beta_1 Ch(\beta_1)) - a_{69} (\beta_2 Cth(\beta_1) Ch(\beta_1) + \beta_1 Sh(\beta_1) - Ch(\beta_1)) - \]

\[ - a_{51} (\beta_2 Cth(\beta_1) - 3) + a_{52} (\beta_2 Ch(\beta_1) - 1) \]

\[ b_{38} = a_{60} (\beta_2 Th(\beta_1) Ch(\beta_1) - \beta_1 Sh(\beta_1)) + a_{62} (\beta_2 Th(\beta_1) Sh(\beta_1) + \beta_1 Ch(\beta_1) + Sh(\beta_1)) + a_{64} (\beta_2 Th(\beta_1) + 4) - a_{68} (\beta_2 Th(\beta_1) + 2) + a_{70} \beta_2 Th(\beta_1) + a_{63} (\beta_2 Cth(\beta_1) Sh(\beta_1) - \beta_1 Ch(\beta_1)) - a_{65} (\beta_2 Cth(\beta_1) Ch(\beta_1) + \beta_1 Sh(\beta_1) - Ch(\beta_1)) - \]

\[ - a_{67} (\beta_2 Ch(\beta_1) - 3) + a_{69} (\beta_2 Ch(\beta_1) - 1) \]
\[ b_{30} = -a_{71} (\beta_2 Sh(\beta_2) - \beta_1 Th(\beta_1)) + a_{74} (\beta_2 Ch(\beta_1) + Sh(\beta_2) + \beta_1 Cth(\beta_2) Sh(\beta_2)) + \\
+ a_{75} (\beta_1 Ch(\beta_1) + Sh(\beta_1) + \beta_2 Th(\beta_2) Sh(\beta_1)) - 2a_{77} Ch(\beta_1) + 4a_{78} + 2a_{61} + \\
+ a_{63} \beta_1 Th(\beta_1) + a_{72} (\beta_2 Ch(\beta_2) - \beta_1 Cth(\beta_1) Sh(\beta_2)) - a_{73} (\beta_2 Sh(\beta_2) + \\
+ \beta_1 Cth(\beta_1) Ch(\beta_2) - Ch(\beta_2)) - a_{76} (\beta_1 Sh(\beta_1) - Ch(\beta_1) + \beta_2 Cth(\beta_2) Ch(\beta_1)) + \\
+ 2a_{78} Sh(\beta_1) + a_{80} (3 - \beta_1 Cth(\beta_1) + a_{82} (1 - \beta_1 Cth(\beta_1)) \]

\[ b_{40} = a_{71} (\beta_2 Sh(\beta_2) - \beta_1 Th(\beta_1)) - a_{74} (\beta_2 Ch(\beta_1) + \beta_2 Th(\beta_2) Sh(\beta_2)) - \\
- a_{75} (\beta_1 Ch(\beta_1) + Sh(\beta_1) + \beta_2 Th(\beta_2) Sh(\beta_1)) + 2a_{77} Ch(\beta_1) \\
- 4a_{78} - 2a_{61} - a_{83} \beta_1 Th(\beta_1) + a_{72} (\beta_2 Ch(\beta_2) - \beta_1 Cth(\beta_1) Sh(\beta_2)) - \\
- a_{73} (\beta_2 Ch(\beta_2)) - a_{76} (\beta_1 Sh(\beta_1) - Ch(\beta_1) + \beta_2 Cth(\beta_1) Ch(\beta_1)) + 2a_{78} Sh(\beta_1) + \\
+ a_{80} (3 - \beta_1 Cth(\beta_1)) + a_{82} (1 - \beta_1 Cth(\beta_1)) \]

\[ b_{41} = -b_{16} \beta_1 Sh(\beta_1) + b_{16} (\beta_1 Ch(\beta_1) + Sh(\beta_1)) - b_{20} \beta_2 Sh(\beta_2) + b_{23} (\beta_2 Ch(\beta_2)) + \\
+ Sh(\beta_2) + 2b_{24} + b_{27} (\beta_1 Ch(\beta_1) - Sh(\beta_1)) - b_{18} \beta_1 Sh(\beta_1) + b_{21} (\beta_2 Ch(\beta_2) - \\
- Sh(\beta_2) - b_{22} \beta_2 Sh(\beta_2) \]

\[ b_{42} = b_{16} \beta_1 Sh(\beta_1) - b_{16} (\beta_1 Ch(\beta_1) + Sh(\beta_1)) + b_{20} \beta_2 Sh(\beta_2) - b_{23} (\beta_2 Ch(\beta_2)) + \\
+ Sh(\beta_2) - 2b_{24} + b_{17} (\beta_1 Ch(\beta_1) - Sh(\beta_1)) - b_{18} \beta_1 Sh(\beta_1) + b_{21} (\beta_2 Ch(\beta_2) - \\
- Sh(\beta_2) - b_{22} \beta_2 Sh(\beta_2) \]