CHAPTER - 3

NUCLEAR MATTER CALCULATIONS
A. Nuclear Matter:

Nuclear matter is a hypothetical infinite system consisting of an equal number of neutrons and protons, the coulomb forces between the protons being switched off. Nuclear matter may be effectively used to compare the different methods in nuclear many body calculations in addition to its standard use to judge which of the many phase-equivalent nucleon nucleon potentials are likely to give reasonable results in actual nuclei. Nuclear matter calculations have also been successfully applied to derive an effective nucleon-nucleon interaction that can be used to obtain the properties of finite nuclei.

The influences of two-nucleon interaction on nuclear binding energies can be understood from a calculation of the binding energy of a many nucleon system. The Rayleigh-Schrödinger perturbation expansion, in powers of two-nucleon interaction $V$, for the ground state energy of many-nucleon system could be used for calculating the binding energy. However, because of the presence of the singular or nearly singular short range repulsion and strong tensor-forces in the two nucleon potential models, the expansion in $V$ does not converge. This convergence difficulty was handled by Brueckner-Goldstone\textsuperscript{9,12}. They obtained the perturbation expansion for the ground state energy in powers of the reaction matrix $G$. The $G$-matrix was obtained by summing over selected terms\textsuperscript{10-13} in the perturbation expansion. The $G$-matrix plays the role of an effective interaction of two nucleons in a nuclear medium and is finite.
Because of this finiteness, the Brueckner-Goldstone expansion converges more rapidly than the Rayleigh-Schrodinger expansion. However, the Brueckner theory becomes complicated for calculating binding energy of finite nuclei due to the presence of various effects such as the surface effect, the pairing effect, the shell effect etc. On the other hand, the hypothetical nuclear matter system are free from these effects as it possesses translational invariance which means that the single-particle wave-functions are plane waves. Thus the main difficulty in obtaining the single particle wave functions in finite nuclei is absent in nuclear matter. In this chapter we shall study the properties of nuclear matter in terms of the reaction matrix by solving the Brueckner-Bethe-Goldstone equation in the lowest order. The solution of the equation for G-matrix is done by the matrix inversion technique using the non-local Paris potential\(^1\).

The single particle wave functions in configuration space are given by

\[
\Phi_m(\vec{r}_i) = \langle \vec{r}_i | m \rangle = \frac{1}{\sqrt{\Omega}} e^{i \vec{k}_m \cdot \vec{r}_i} | s_m t_m \rangle \quad \text{......}(3.1)
\]

where \(\Omega\) is the volume that encloses the nucleon taken to normalise the single particle orbitals; \(s_m\) and \(t_m\) are respectively the spin and isospin state of the nucleon. For infinite nuclear matter both the number of particles A and the normalising volume \(\Omega\) become infinite but the particle density \(\rho \propto A/\Omega\) remains finite. In the ground state of nuclear matter the A particles fill the lowest available single particle orbitals consistent with the
Pauli exclusion principle and the periodic boundary conditions on the plane waves. The unperturbed many-body wave function for such a system is

$$\phi_0(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_A) = \mathcal{Q} \prod_{i=1}^{A} \phi_m(\vec{r}_m) \quad \ldots \ldots (3.2)$$

The antisymmetrisation operator $\mathcal{Q}$ assures that $\phi_0$ is normalised and antisymmetric upon interchange of any pair of particles. For an infinite system as $A$ and $\omega$ approaches infinity, the single particle orbitals have a continuous distribution in momentum upto a maximum momentum of $k_F$ --- the Fermi momentum. The summation $\sum_{i=1}^{A}$ implied in expression (3.2) for plane waves then becomes

$$\sum_{m} \frac{\omega}{s_m t_m (2\pi)^3} \int d^3 k_m k_m^3 \mathcal{Q}$$

As a consequence, the total kinetic energy of the system becomes

$$T = \frac{3}{5} \frac{\hbar^2 k_F^2}{2M A} \quad \ldots \ldots (3.3)$$

and the particle density becomes

$$\rho = \frac{2 k_F^3}{3 \pi^2} \quad \ldots \ldots (3.4)$$

where assumption has been made that the volume enclosed by the occupied states is spherical.

The main calculation in nuclear matter is the binding energy per particle and the saturation density. In a finite nucleus, the binding energy per particle is given by the semi-empirical mass formula.
Here the first term is the volume term, the second term is the surface term, the third term is coulomb term and the fourth is the symmetric term. For infinite symmetric nuclear matter all terms except the volume term disappears. Thus nuclear matter is related to the finite nuclei through the volume term only and exhibits the property of the interior region of a heavy nucleus. As the nucleons per fermi-cube in a heavy-nucleus saturate, the nuclear matter must posses saturation property. This means that, at a particular density the binding energy per particle should be a minimum. This saturation or equilibrium density is expected to be the interior density of a heavy nucleus which is 0.17 nucleons per fermi-cube and which corresponds to the Fermi-momentum \( k_F = 1.36 \text{ fm}^{-1} \). The expected binding energy per particle for the nuclear matter is the value given by the volume term of the mass formula which is \(-15.68\) MeV. An acceptable potential model must produce these values correctly.

B. Reaction Matrix \( G \) and Brueckner-Bethe-Goldstone Equation:

For a many particle system the Hamiltonian is given by

\[
H = \sum_{i=1}^{A} T_i + \sum_{i<j}^{A} V_{ij} = H_0 + H_1 \tag{3.6}
\]

where

\[
H_0 = \sum_{i=1}^{A} (T_i + U_i) \quad \text{and} \quad H_1 = \sum_{i<j}^{A} V_{ij} - \sum_{i=1}^{A} U_i \tag{3.7}
\]
Here \( A \) is the number of particles, \( T_i \) is the kinetic energy operator, \( U_i \) is the single particle potential operator, \( V_{ij} \) denotes the two nucleon interaction between particles \( i \) and \( j \), \( H_0 \) is the unperturbed Hamiltonian and \( H_1 \) is in some sense a small perturbation.

The unperturbed Hamiltonian satisfy the equation

\[
H_0 \Phi_0 = E_0 \Phi_0 \tag{3.8}
\]

where \( E_0 \) is the unperturbed ground state energy of the system and \( \Phi_0 \) is the unperturbed ground state wave-function represented by expression (3.2). The eigenvalue \( E_0 \) is the sum of the single particle energies

\[
E_0 = \sum_{n=1}^{A} E_n \tag{3.9}
\]

The exact Hamiltonian of the system satisfy the equation

\[
H \Psi = E \Psi \tag{3.10}
\]

where \( \Psi \) is the exact ground state wave function and \( E \) is the actual ground state energy.

Perturbation theory gives an expansion for \( E \) in terms of the perturbation \( H_1 \) as

\[
E = E_0 + \langle \Phi_0 | H_1 | \Phi_0 \rangle + \langle \Phi_0 | H_1 (E_0 - H_0)^{-1} P H_1 | \Phi_0 \rangle + \ldots \tag{3.11}
\]

The projection operator \( P = I - |\Phi_0 \rangle \langle \Phi_0| \) projects off \( \Phi_0 \) and ensures that \( \Phi_0 \) does not occur as an intermediate state in any of the matrix elements.
The above expansion (3.11) gives rise to different linked and unlinked diagrams. Goldstone\(^9\) shows that the unlinked diagrams cancel in a natural way and only the linked diagrams should be considered. However the Goldstone expansion is unsuitable for nuclear calculations because the short-range repulsion makes all the matrix elements very large and the series does not converge. To get rid of this difficulty, Brueckner arranged the diagrams of the Goldstone expansion in such a way that each matrix element of \(V\) is replaced by an infinite series which takes into account the two-body interaction to all orders of the potential. The quantity which replaces the potential is the reaction matrix \(G\). The \(G\)-matrix is well behaved even for a singular two-body force and hence the series now converges.

The Brueckner-Goldstone expansion in favour of \(G\)-matrix is given by

\[
G(W) = V - V \frac{Q}{e} G(W) \quad \text{..................} \quad (3.12)
\]

Here \(W\) is the starting energy, \(Q\) is the Pauli operator which blocks the already occupied states below the Fermi-sea, \(e\) is the energy denominator, and \(V\) is the two-body potential operator.

The two particle operators \(Q\) and \(e\) are defined by the equation

\[
Q|pq > = |pq > \quad \text{if } k_p, k_q > k_F
\]

\[
= 0 \quad \text{otherwise} \quad \text{............}(3.13)
\]

and

\[
e|pq > = (E_p + E_q - W)|pq > \quad \text{...........(3.14)}
\]

where \(E_p, E_q\) are single particle energies.
For a given starting energy $W$, $G$ is a well defined hermitian two-body operator because $Q,e,V$ are all hermitian.

Above integral equation (3.12) is the Brueckner-Bethe-Goldstone (BBG) equation which defines the reaction matrix $G$ and expresses the ground state energy of nuclear matter in powers of $G$. To lowest order in $G$, the ground state energy of nuclear matter is given by

$$E = T + \frac{1}{2} \sum_{m,n} <mn | G(W) | mn-nm > \quad \ldots \ldots \text{(3.15)}$$

Once the on-energy shell $G$-matrix is known, we can evaluate the ground state energy $E$. It is clear that $E$ is simply obtained from pair interaction energies summed over all pairs in the Fermi-sea. These interactions are represented by the diagonal $G$-matrix elements. However, the $G$-matrix approximately takes into account the presence of the other particles through the $Q$ operator and the single particle spectrum. We, therefore, aim to the solution for $G$. Since the on-energy-shell $G$-matrix satisfies an equation similar to the Lippmann-Schwinger equation for free scattering, the problem becomes similar to solving a two body problem.

C. Correlated Wave-function and Wound Integral:

The unperturbed two-particle wave-function is given by

$$|pq> = \phi_{pq}(\vec{r}_1,\vec{r}_2) = \phi_p(\vec{r}_1) \phi_q(\vec{r}_2) \quad \ldots \ldots \text{(3.16)}$$

where $\phi_p, \phi_q$ are the single particle wave functions defined in equation (3.1).

Let us define the relative and center of mass quantities as
\[ \vec{k} = \frac{1}{2}(\vec{k}_p - \vec{k}_q) \quad \vec{r} = \vec{r}_1 - \vec{r}_2 \]
\[ \vec{p} = \vec{k}_p + \vec{k}_q \quad \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) \]

where \( \vec{k}_p, \vec{k}_q \) label the momentum of the single particle orbitals and \( \vec{r}_1, \vec{r}_2 \) label the single particle coordinates for a pair 1,2.

Then the unperturbed two-particle wave function, in configuration space, can be expressed as
\[ \phi_{pq}(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{\mathcal{N}}} \ e^{i\vec{p} \cdot \vec{R}} e^{i\vec{k} \cdot \vec{r}} |s_p t_p> |s_q t_q> \]  \( \ldots \ldots \)(3.18)

The correlated two-particle wave function \( \Psi \) is related to the G-matrix by the equation
\[ G\phi = v\Psi \] \( \ldots \ldots \)(3.19)

where \( \phi = \phi_{pq} \) and \( \Psi = \Psi_{pq} \).

By equation (3.12) it follows that the correlated wave function satisfy the Bethe-Goldstone equation
\[ \Psi = \phi - \frac{Q}{e} v \Psi \] \( \ldots \ldots \)(3.20)

One may regard the correlated nuclear wave function as a wave function that describes the interaction of two-nucleons in the nuclear medium. There is an important difference between the nuclear wave function and the free-scattering wave function. The nuclear wave function approaches the free-scattering wave function i.e. the plane wave, in the asymptotic limit. This property called healing, arises due to the lack of singularity in equation (3.20) because of the presence of the Q operator. Healing is a consequence
of the Pauli exclusion principle. Because of the healing property correlations in the nuclear wave functions are limited to short distances. True scattering does not occur in a nuclear medium because the nuclear wave function does not contain a scattered wave in the asymptotic limit.

The correlations in the nuclear wave function is measured by the defect function defined by

$$\mathcal{I} = \Psi - \phi$$

Due to healing, the defect function vanishes in the asymptotic limit and thus measures the short range correlations only. It is found that strong correlations are mainly limited to distances less than 1 fm, which is considerably less than the average internucleon distance ($\sim 1.8$ fm) of nuclear matter. This means that the probability that three nucleons approach each other close enough to cause significant three-body correlations becomes fairly small.

One overall measure of the short-range correlations in nuclear matter is the wound integral $K$. The wound integral is defined by

$$K = \frac{\text{Correlated Volume}}{\text{Volume per particle}} = \rho \int \phi^2 - |\Psi|^2 \, d\mathcal{I} \quad \ldots(3.22)$$

Physically, the wound integral $K$ measures the probability of exciting two incident particles in the Fermi sea to momenta above the Fermi-sea. The value of $K$ is of great importance because it determines the convergence of the Brueckner-Goldstone expansion. Since the magnitude of $K$ indicates the degree of short-range correlations and how quickly the nuclear wave function heals, $K$ also indicates how important three body correlations may be. When two particles are
interacting the probability that a third particle be simultaneously correlated is of the order of K. It has a magnitude always less than unity.

D. BBG Equation --- Angle Average and Effective Mass Approximation:

In terms of the two-body matrix elements, the BBG equation (3.12) is given by

\[ \langle ab|G(W)|mn \rangle = \langle ab|V|mn \rangle - \sum_{p,q} \langle ab|V|pq \rangle \frac{Q}{E_p + E_q - W} \langle pq|G(W)|mn \rangle \] ....(3.23)

Using the relative and center of mass quantities defined through (3.17), keeping in mind infinite nuclear matter and replacing the summation over intermediate states with an integration over intermediate relative momentum, the above equation (3.23) reduces to an integral equation.

\[ \langle \vec{k}|G(\vec{P},W)|\vec{k}_0 \rangle = \langle \vec{k}|V|\vec{k}_0 \rangle - \int \frac{dk'}{e(k',\vec{P};W)} \frac{\langle \vec{k}'|V|\vec{k}' \rangle Q(k',\vec{P}) \langle \vec{k}|G(\vec{P},W)|\vec{k}_0 \rangle}{\langle \vec{k}'|G(\vec{P},W)|\vec{k}_0 \rangle} \] ....(3.24)

Here \( \vec{P} \) refers to the center-of-mass momentum which must be conserved; \( \vec{k}_0, \vec{k}' \) and \( \vec{k} \) refer respectively to the initial state intermediate state and final state relative momentum; \( Q(k',\vec{P}) \) is simply the eigenvalue of \( Q \) when it operates on a two-body intermediate state with center-of-mass momentum \( \vec{P} \) and relative momentum
\( e(k', \vec{P}; W) \) is simply the sum of the single particle energies for two-particle states having relative momentum \( \vec{k}' \) and center-of-mass momentum \( \vec{P} \) minus the starting energy.

The operator \( \langle \vec{k}' | G(\vec{P}, W) \rangle_{k_0} \) is defined by the equation

\[
\langle m'n'|G(W)|mn \rangle = \langle w'_m w'_n | \delta_{\vec{P}', \vec{P}} \frac{(2\pi)^3}{\Omega} \langle \vec{k}' | G(\vec{P}, W) \rangle_{\vec{k}} |w_m w_n \rangle
\]

\[\cdots \cdots (3.25)\]

where \( |0\rangle \) refers to the spin-isospin state of the respective particle. The Kronecker delta appears in above equation because the single particle orbitals in nuclear matter are normalised in a finite volume \( \Omega \) and the factor \( \frac{(2\pi)^3}{\Omega} \) appears because we want to evaluate \( G \) between relative states of the form \( \frac{1}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{r}} \) as has been done for the free scattering matrix \( R \).

In terms of relative and center-of-mass momentum, the operators \( Q \) and \( e \) are given by

\[
Q(k', \vec{P}) |pq \rangle = |pq \rangle \text{ if } \left| \frac{1}{2} \vec{P} + k' \right| > k_F
\]

\[\text{ = 0 otherwise} \qquad \cdots \cdots (3.26)\]

and

\[
e(k', \vec{P}; W) |pq \rangle = \left[ E\left(\frac{1}{2} \vec{P} + k'\right) + E\left(\frac{1}{2} \vec{P} - k'\right) - W\right] |pq \rangle \quad \cdots \cdots (3.27)
\]

We perform our calculation on the energy shell, that is, the starting energy \( W \) is always equal to the energy of the initial two particle states. Thus

\[
W = E\left(\frac{1}{2} \vec{P} + k_0\right) + E\left(\frac{1}{2} \vec{P} - k_0\right) \quad \cdots \cdots (3.28)
\]
As the eigenvalues of the operators $Q$ and $e$ when operating on a two-body state $|pq\rangle$, depend not only on the magnitudes of the relative and center-of-mass momentum $\mathbf{k}$ and $\mathbf{P}$ but also on the angle between them, $G$ depends on the magnitude and direction of the total momentum. The angular dependence of $G$ on the direction of the center-of-mass momentum $\mathbf{P}$ makes the partial wave expansion to the BBG equation complicated. This angular dependence also implies that $G$ cannot conserve total relative angular momentum $\mathbf{J}$.

However, Irwin\textsuperscript{59} showed that the effects of $J$ coupling on nuclear binding energies are less than .1 MeV. Neglecting this small effect one can replace the exact operator $Q$ with an angle averaged $Q$ operator. The angular dependence on $\mathbf{P}$ of the operator $e$ can be eliminated with a proper choice of single particle spectrum. Then it is possible to obtain a partial wave decomposition of the BBG equation.

The angle averaged Pauli operator $Q$ is given by (see Appendix- B)

$$\overline{Q}(\mathbf{k}', \mathbf{P}) = 0 \quad \text{for} \quad k'^2 < \sqrt{k_F^2 - \frac{P^2}{4}}$$

$$= 1 \quad \text{for} \quad k' > (k_F + \frac{P}{2}) \quad \ldots \quad (3.29)$$

$$= \frac{k'^2 + \frac{P^2}{4} - k_F^2}{k'P} \quad \text{otherwise}$$

The angular dependence on the direction of the center-of-mass momentum $\mathbf{P}$ in the energy denominator is avoided by the use of the effective mass approximation. In nuclear matter the single particle energy $U$ is a function of only the magnitude of the single particle momentum. The single particle energy, then, can be
written as

\[ E_a = E(k_a) = \frac{k_a^2}{2M^*} + U(k_a) \] 

\[ ........(3.30) \]

The energy denominator \( e \) in the BBG equation now becomes

\[ e = E(k', P) - E(k_0, P) \]

\[ = \frac{k_0^2}{M} - k_0^2 + U(\frac{1}{2} P + k') + U(\frac{1}{2} P - k') \]

\[ - U(\frac{1}{2} P + k_0) - U(\frac{1}{2} P - k_0) \] 

\[ ........(3.31) \]

which has the angular dependence because \( U \) terms depend on the angles between \( P \) and \( k' \), and \( P \) and \( k_0 \) as well as on the magnitudes of \( P \), \( k' \) and \( k_0 \).

In the effective mass approximation, the single particle energy is considered to have the quadratic form

\[ E(k_a) = \frac{k_a^2}{2M^*} + A \] 

\[ ........(3.32) \]

where \( M^* \) is "effective mass" and \( A \) is some parameter to be determined by the choice of \( U \).

E. Choice of Single Particle Spectrum:

The single particle potential \( U \) plays an important role in the convergence of the Brueckner-Bethe-Goldstone equation for \( G \)-matrix. There are available different choices for the single
particle spectrum. A proper choice of $U$ may produce good convergence of the Brueckner-Goldstone expansion cancelling out some cluster diagrams in higher orders. The best choice for such purpose is the standard choice of single particle spectrum. In this choice, the particles ($k_a > k_F$) are assumed to have no potential energy, that is their energy is only kinetic. However, the holes ($k_h < k_F$) have kinetic as well as potential energy. The standard choice of single particle spectrum is historically due to the belief that the three-body cluster diagrams and other higher order cluster diagrams contribute negligibly. Day in his recent calculations shows that this is not so. The third-order and other higher-order diagrams need to be considered. A possible way to take into account a large part of the three-body cluster diagrams is to choose the continuous choice of single particle spectrum. In this choice both the holes and the particles are assumed to have kinetic as well as potential energies. The particle and hole potentials are being treated on an equal footing. We consider both the choices --- the standard choice and the continuous choice --- in our calculation. This will provide a ground for justification of the choice of single particle spectrum.

The standard choice or gap choice of the single particle potential is the generally accepted Hartree-Fock potential defined by

$$U(m) = \sum_{n \leq k_F} \langle mn | G(W) | mn \rangle , \quad m \leq k_F$$

$$= 0 \quad , \quad m > k_F$$

......(3.33)

The continuous choice of single particle potential is
defined by

\[
U(m) = \sum_{n \leq k_F} \langle mn | G(W) | mn \rangle , \quad m \leq k_F
\]

\[U(a) = \text{Re} \sum_{n \leq k_F} \langle an | G(E_a + E_n) | an \rangle , \quad a \geq k_F\]

\section*{F. Partial Wave Expansions:}

With the angle-average and effective mass approximation, the BBG equation becomes

\[
\langle \mathbf{k}' | G(P,W) | k_o \rangle = \langle \mathbf{k}' | V | k_o \rangle - \int dk' \langle \mathbf{k}' | V | k' \rangle \frac{Q(k',P)}{E(k',P) - E(k_o,P)}
\]

\[
\langle \mathbf{k}' | G(P,W) | k_o \rangle \quad \ldots \ldots (3.35)
\]

Here \( G \) does not depend on the direction of \( \mathbf{P} \) but only on the magnitude. So, we can easily get the partial wave decomposition of the above equation.

The partial wave decomposition of the \( G \)-matrix is given by

\[
\langle \mathbf{k}' | G_{\alpha LL'}(P,W) | k_o \rangle = \frac{2}{\pi} \cdot \frac{\hbar^2}{M} \cdot \sum_{\alpha LL'} \frac{i^{L'-L}}{\mathcal{M}_{T \bar{T}}} \langle \mathbf{k}' | G_{\alpha LL'}(P,W) \rangle
\]

\[
\langle k_o | \gamma'_{L}(P) \gamma_{L'}^{\dagger}(k_o) \rangle \quad \ldots \ldots (3.36)
\]

It is to be noted here that the symbols to be used in partial wave decomposition have the same meaning as those used
in chapter-2.

The BBG equation (3.35) now reduces to

\[ < k | g_{Ll}^{\alpha}(P, W) | k_0 > = < k | v_{Ll}^{\alpha} | k_0 > - \frac{2}{\pi} \sum_{1}^{J} \int dk' k'^2 < k | v_{Ll}^{\alpha} | k > \]

\[ \frac{\frac{Q(k', P)}{E(k', P) - E(k_0, P)^2}}{E(k, P) - E(k_0, P)^2} < k' | g_{Ll}^{\alpha}(P, W) | k_0 > \quad (3.37) \]

which is the decomposed form of the Brueckner-Bethe-Goldstone equation.

The partial wave form of the correlated wave function, in configuration space, is given by

\[ \Psi_{Ll}^{\alpha}(r) = j_{L}(k_0 r) \delta_{Ll'} - \frac{2}{\pi} \int dk' k'^2 \frac{\frac{Q(k', P)}{E(k', P) - E(k_0, P)^2}}{E(k, P) - E(k_0, P)^2} \]

\[ < k' | g_{Ll}^{\alpha}(P, W) | k_0 > j_{l'}(k' r) \quad \ldots \ldots (3.38) \]

and, in momentum space, by

\[ \Psi_{Ll'}^{\alpha}(k) = \frac{\delta(k-k_0)}{k^2} \delta_{Ll'} - \frac{2}{\pi} \cdot \frac{\frac{Q(k, P)}{E(k, P) - E(k_0, P)^2}}{E(k, P) - E(k_0, P)^2} < k | g_{Ll}^{\alpha}(P, W) | k_0 > \]

\[ \quad \ldots \ldots (3.39) \]

The wound integral \( K \), in partial wave form is given by

\[ k_{Ll'}^{\alpha} = \frac{2k_F^3}{3\pi^2} \frac{2J+1)(2T+1)}{(2J+1)(2T+1)} \int dk' k'^2 \frac{b^2(k', P)}{[E(k', P) - E(k_0, P)]^2} \]

\[ \left| < k | g_{Ll}^{\alpha} | k_0 > \right|^2 \quad \ldots \ldots (3.40) \]
The single particle potential can be written as

\[
U(k_m) = \langle m | U | m \rangle = \frac{1}{4\pi^2} \frac{h^2}{2M} \sum_{\alpha L} (2J+1)(2T+1) \int_{k_n<k_F} \frac{d^3k_n}{k_n} \langle k_o | G_{LL}^\alpha (P,W) | k_o \rangle
\]

\[\ldots(3.41)\]

The integration over intermediate state particle momentum \(k_n\) is equivalent to two integrations --- one over the initial state relative momentum \(k_0\) and the other over the center of mass momentum \(P\). To reduce the dimension of the integral and the number of \(G\)-matrix elements, one assumption is made at this stage. The assumption is that

\[
\langle k_o | G_{LL}^\alpha (P,W) | k_o \rangle = G_{LL}^\alpha (P;k_o,k_o) \approx G_{LL}^\alpha (\bar{P};k_o,k_o) \ldots(3.42)
\]

where \(\bar{P}\) is the average value of \(P\). This value is taken to be the root mean square value of \(P\) for two particles in the Fermi sea with the constraint that one of the particles has momentum \(k_m\) and the relative momentum has magnitude \(k_o\).

Finally we can obtain

\[
U(k_m) = \frac{8}{\pi} \frac{k^2}{M} \sum_{\alpha L} (2J+1)(2T+1) \left[ \int_{0}^{\frac{1}{2}(k_F-k_m)} dk_o k_o^2 G_{LL}^\alpha(\bar{P};k_o,k_o) + \frac{1}{2k_m} \int_{0}^{\frac{1}{2}(k_F+k_m)} dk_o k_o \left[ \frac{1}{4}(k_F^2 - k_o^2) - k_o(k_o - k_m) \right] G_{LL}^\alpha(\bar{P};k_o, k_o) \right]
\]

\[\ldots(3.43)\]
This expression is applicable to both the choices of single particle spectrum. For the standard gap-choice \( k_m \leq k_F \) always. For continuous choice, \( k_m \) may be greater than \( k_F \). Obviously the first integral vanishes for \( k_m > k_F \) and the contribution will come from the second integral only.

The total binding energy per particle of nuclear matter is (for deduction see Appendix-C) given by

\[
\bar{E} = \frac{B}{A} = \bar{T} + \frac{1}{2} \sum_{m<k_F} \langle m | U | m \rangle
\]

\[
= \bar{T} + \frac{4 \kappa^2}{3 \pi^2 M} \sum_{\alpha \gamma} (2T+1)(2J+1) \int_0^{k_F} dk_0 k_0^2 \left( 1 - \frac{3}{2} \frac{k_0}{k_F} \right) \]

\[
+ \frac{1}{2} \frac{k_0^3}{k_F^3} \right) G_{\alpha \gamma}^L(\bar{T}; k_0, k_0) \quad \ldots \ldots \quad (3.44)
\]

where the kinetic energy per particle \( \bar{T} \) is obtained from the relation (3.3).

G. Solution to the BBG Equation—Numerical Procedure:

The Brueckner-Bethe-Goldstone integral equation was solved directly by Brueckner and Gammel. Later on Haftel gave an easier method --- the matrix inversion method for solving the BBG equation. We have used this method for our computation. Here we give a brief outline about the method.

With the help of equations (3.12) and (3.19) the G-matrix can be written as
\[ G = (1 - G \frac{Q}{e}) V = F^{-1} V \] \hspace{1cm} \text{(3.45)}

and the correlated two-body wave function as
\[ \Psi = (1 - \frac{Q}{e} G) \phi = \bar{F}^{-1} \phi \] \hspace{1cm} \text{(3.46)}

where \( \bar{F}^{-1} = 1 - G \frac{Q}{e} \) \hspace{1cm} \text{(3.47)}
defines the F-matrix and \( \bar{F}^{-1} \) is the adjoint of \( F^{-1} \).

The BBG equation can be solved for G-matrix in the same technique as has been done in solving the Lippmann-Schwinger equation. An N-point Gaussian integration formula is applied to equation (3.37) and the equation is evaluated at \( k = k_i, i = 1, 2, \ldots, N \) and at \( k = k_{N+1} = k_0 \) where \( k_i \)'s are the grid points of the integration formula.

For an uncoupled channel with a fixed \( \alpha, L, P \) and \( k_0 \), the equation (3.37) reduces to a set of \( (N+1) \) simultaneous, linear equations for the \( N+1 \) unknowns \( \langle k_i | G_{\alpha}^{\alpha,\alpha} | k_{N+1} \rangle \), \( i = 1, 2, \ldots, N+1 \).

For an uncoupled channel, we define the F-matrix as
\[ \langle k_i | F_{\alpha}^{\alpha,\alpha} | k_j \rangle = \delta_{ij} + w_j \langle k_i | V_{\alpha}^{\alpha,\alpha} | k_j \rangle \] \hspace{1cm} \text{(3.48)}

where
\[ w_j = \frac{2}{\pi} \frac{k_j^2 w_j \overline{Q}(E(k_j, P) - E(k_{N+1}, P))}{E(k_j, P) - E(k_{N+1}, P)} \] for \( j = 1, 2, \ldots, N \)
\hspace{1cm} \text{(3.49)}
\[ = 0 \] for \( j = N+1 \)
The \( w_j \)'s are the Gaussian weights corresponding to the grid points \( k_j \).
Then the G-matrix elements, with the help of equation (3.37) are given by

\[ \langle k_i \mid G_{\alpha LL} \mid k_{N+1} \rangle = \sum_{j=1}^{N+1} \langle k_i \mid F_{\alpha LL}^{-1} \mid k_j \rangle \langle k_j \mid V_{\alpha LL} \mid k_{N+1} \rangle \cdots (3.50) \]

Thus an inverse of the matrix F defined through equation (3.46) gives nuclear matter G-matrix elements.

For a coupled channel the procedure is the same as that for R-matrix discussed in chapter-2. In this case, the dimension of the matrices V, F and G becomes doubled and the equation (3.37) reduces to \((2N+2)\) linear equations.

For coupled channel, the F-matrix is defined by the equation

\[ \langle k_i \mid F_{\alpha LL'} \mid k_j \rangle = \delta_{ij} \delta_{LL'} + w_{ij} \langle k_i \mid V_{\alpha LL} \mid k_j \rangle \cdots (3.51) \]

and the G-matrix elements are given by

\[ \langle k_i \mid G_{\alpha LL'} \mid k_{N+1} \rangle = \sum_{j=1}^{N+1} \sum_{L'} \langle k_i \mid F_{\alpha LL'}^{-1} \mid k_j \rangle \langle k_j \mid V_{\alpha LL'} \mid k_{N+1} \rangle \cdots (3.52) \]

With the help of the F-matrix, the correlated two-body wave function (3.38) can be written as

\[ \Psi_{LL_o}(k_o r) = \sum_{j=1}^{N+1} \langle j_L(k_j r) \mid k_{N+1} \rangle \cdots (3.53) \]

where \(k_o = k_{N+1}\)

Once the correlated wave function is known, we can easily obtain the correlation function \(f\) as

\[ f_{LL_o}(k_o r) = \Psi_{LL_o}(k_o r) / j_{L_o}(k_o r) \cdots (3.54) \]
and the defect function as
\[ \mathcal{J}_{LL_0} = \mathcal{Y}_{L_0}^{(k_0)} - j_{L_0}^{(k_0)} r \] ...........(3.55)

In equations (3.53) to (3.55), \( L_0 \) refers to the entrance channel with relative momentum \( k_0 \) whereas \( L \) refers to the exit channel with relative momentum \( k \).

Calculating the G-matrix elements by the above procedure, the single particle potential energy, nuclear matter binding energy per particle and the healing integral are obtained by solving the equations (3.43), (3.44) and (3.40).

In the derivation of expressions (3.40), (3.43) and (3.44) an approximation, given by relation (3.42), of average center-of-mass momentum dependence in place of center-of-mass momentum dependence of \( G \) is made. The validity of this average center-of-mass approximation depends on \( G \) having only a weak dependence on center-of-mass momentum \( P \). For most realistic potentials \( G_{LL}^\alpha(P; k_0, k_0) \) varies by at most 10% with \( P \) for a given \( k_0 \). In single-particle spectrum, because one of the particle momentum \( (k_m) \) is fixed, the actual variation of \( G \) with \( P \) is somewhat less than 10%. It is to be noted that the average value of \( P \) to be considered for calculation of \( U(k_m) \) differs from that to be used for calculating the nuclear matter binding energy.

In the calculation of nuclear matter binding energy the average center-of-mass momentum \( \bar{P} \) is chosen to be the root mean square value of \( P \) given by (see Appendix-D)
As the G-matrix is a slowly varying function of $P$ according to Sprung, in the evaluation of single particle spectrum, we use the same $\bar{F}$ as the nuclear matter binding. However, the relative momentum $k_0$ can exceed $k_F$ in this case and equation (3.56) does not apply any more. For $k_0 > k_F$, we take the value

$$F^2 = 1.2 \frac{k_0^2}{k_F^2} \quad \quad \quad \ldots \ldots (3.57)$$

given by Day. This choice is expected to be valid at high values of $k_0$ but is not satisfactory for $k_0$ just above $k_F$. This implies that the single particle spectrum at $k_0 \approx k_F$ is not calculated very accurately in our formalism. Also the effective mass suffers an enhancement near $k_0 = k_F$ which we have not considered. However, binding energy should not depend particularly on this as it is insensitive to local variations in the single particle potential.

The angle averaged Pauli operator used is given by

$$\bar{Q}(k', P) = 0, \quad k^2 + \frac{1}{4} P^2 < k_F^2 \quad \quad \quad \quad \quad \quad \ldots \ldots (3.58)$$

$$= \min \left[ 1, \frac{k^2 + \frac{P^2}{4} - k_F^2}{k_F^2} \right], \text{ otherwise.}$$

The energy denominator used for the gap choice, with effective mass approximation, is given by (see Appendix-E)

$$e = \frac{\hbar^2}{M} (k^2 + \gamma^2) \quad \quad \quad \quad \quad \quad \ldots \ldots (3.59)$$
where
\[ \gamma^2 = -2U_0 \frac{M}{\hbar^2} + \frac{p^2}{4(1-M^*)} - \frac{k^2}{M^*} \] ........(3.60)

The single particle potential in this choice is given by
\[ U(k_a) = U_0 + \frac{k^2}{2M} \left( \frac{1}{M^*} - 1 \right) k_a^2 \] ........(3.61)

For the continuous choice the energy denominator is given by
\[ e = 2\sqrt{\hbar^2 c^2 (k_a^2 + \frac{p^2}{4}) + M^* c^4} - 2\sqrt{\hbar^2 c^2 (k_0^2 + \frac{p^2}{4}) + M^* c^4} \] ........(3.62)

and the single particle potential by
\[ U(k_a, k_F) = U_0(k_F) + M c^2 - M^* c^2 + \sqrt{\hbar^2 c^2 k_a^2 + M^* c^4} - \sqrt{\hbar^2 c^2 k_a^2 + M^* c^4} \] ........(3.63)

In continuous choice, the relativistic form of energy is used because we have found that the single particle energy fits this form in a natural way with a fewer number of parameters.

In relativistic form, the gap choice energy denominator is given by
\[ e = 2 \left[ \sqrt{\hbar^2 c^2 (k^2 + \frac{p^2}{4}) + M^2 c^4} - \sqrt{\hbar^2 c^2 (k_0^2 + \frac{p^2}{4}) + M^* c^4} - U_0 - M c^2 + M^* c^2 \right] \] ........(3.64)
In equations (3.59) to (3.64), $u_0$ and $M$ are respectively the potential depth and the effective mass to be determined self consistently. They are the characteristic parameters of the effective mass approximation and are density dependent. $M$ is the nucleon mass.

H. Results And Discussions:

We have already pointed out in chapter-2 that the three lap 16-point Gaussian integration upto 15 fm for potential matrix element in coordinate space and the four lap 4-point Gaussian integration upto 25 fm$^{-1}$ for potential matrix elements as well as R-matrix elements in momentum space give sufficiently accurate results. The laps are 0-1.5-4.5-15 fm for coordinate space and 0-2.5-5.5-13.5-25.0 for momentum space integration. We have evaluated the G-matrix elements with the same four lap 4-point Gaussian integration. These matrix elements of $G$ are then used to calculate the nuclear matter binding energy and the single particle potential by numerical evaluation of equations (3.43) and (3.44). The numerical integration is carried out by 4-point Gaussian integration. This gives reasonably accurate results. As a check to the result, we have reproduced the Reid-Soft-core binding energy in the $^1S_0$ state. It is found that the results agree within 0.1 MeV.

Before the calculation of the nuclear matter binding energy it is necessary to obtain a self consistent value of the single particle potential $U$. This is because $U$ is determined from $G$.
by equation (3.43) but the determination of G depends on the original choice of U. To obtain the self-consistency, an initial value of $M^*$ and $U_0$ are chosen to calculate G. With this G, U is calculated at different momenta by equation (3.43). These values of U corresponding to different $k_m$ are fitted to equation (3.61) or (3.63) to determine new values of $M^*$ and $U_0$. The value of U corresponding to $k_m=0$ gives $U_0$. These new $U_0$ and $M^*$ then determine a new G which, in turn, determines new values of U. Iteration of this process is continued until $M^*$ and $U_0$ change very little between successive iterations.

The self-consistent calculation is performed for the continuous choice of single particle spectrum only with both the Paris potential and the Reid soft-core potential. The self-consistency is done up to $k_m=3.0 \text{ fm}^{-1}$ for four densities corresponding to the Fermi momentum $k_F=1.10 \text{ fm}^{-1}$, $1.36 \text{ fm}^{-1}$, $1.60 \text{ fm}^{-1}$ and $1.75 \text{ fm}^{-1}$. The results for the Paris potential are displayed in Tables 10-13. The self-consistent single particle energies up to $6 \text{ fm}^{-1}$ for the four densities with detailed break down for each channel are shown in Tables 14-17. The potential energy becomes negligible in comparison with the kinetic energy around a high value of $k_m \sim 6 \text{ fm}^{-1}$ and the contribution to the binding energy also becomes very small. However, our motivation was to see whether at such a high value of momentum, the single particle energy turns out to be constant or it starts to decrease as it is supposed to do for a hard-core potential. But we have found that for Paris potential the single particle energy is still
an increasing function at this high value of momentum. The self-
consistency for the Reid soft core potential is shown in Tables
18-21. A comparison of the self consistent parameters for the RSC
and the Paris potential are made in Table-22. The potential energy
as well as the total binding energy per particle of nuclear matter
for the Paris potential is displayed in Table-23 showing the
individual channel contributions. It also shows a comparison with
the standard gap choice results at $k_F = 1.36 \text{ fm}^{-1}$. The similar
results for the RSC potential are shown in Table-24. It is to be
noted that for the gap choice calculation, the same energy deno-
minator as employed for the continuous choice calculation is used
with the exception that $M^*$ is replaced by $M$ and that potential
energy $U(k, k_F) = 0$ for particle. Thus relativistic calculation is
done for both the choices. We have found an increase in binding
energy of about 7 MeV in continuous choice for the Paris potential.
The gain is much larger than that obtained by Mahaux et al\textsuperscript{19} for
Reid soft core potential\textsuperscript{33}. For Paris potential, the main contri-
bution in increase in binding arises from $^3S_1 - ^3D_1$ channel
$\sim 5 \text{ MeV}$ whereas the difference in $^1S_0$ channel binding is small
($\sim 0.5 \text{ MeV}$). To understand this difference we have plotted the
$^3S_1 - ^3D_1$ defect function in Fig.-24, wave function in Fig.-25 and
correlation function in Fig.-26 at the same Fermi momentum
$k_F = 1.36 \text{ fm}^{-1}$ with the gap choice as well as the continuous choice.
The continuous choice $^3S_1 - ^3D_1$ defect function is significantly
larger than the gap choice result beyond 0.9 fm. The effect is
less pronounced for the $^1S_0$ channel shown in Fig.-27. The $^1S_0$
channel wave function and correlation function are shown in Figs. 28 and 29. The correlation function for the channels $^3S_1$, $^3D_1-^3S_1$ and $^3D_1$ are respectively shown in Figs. 30-32. It is clear from the figures that the correlation function is larger for the continuous choice but becomes smaller after 2 fm. The $^3D_1$-channel defect function shown in Fig. 33, is of remarkable nature being larger for the continuous choice after 0.9 fm. The continuous choice shifts the saturation point towards higher $k_F$. Fig. 34 shows the saturation curve for the continuous choice. The minimum binding energy per particle is about $-21$ MeV corresponding to $k_F = 1.55$ fm$^{-1}$ for the Paris potential. In the above results contributions from higher partial waves ($L \geq 5$) are not included. As pointed out by Lejeune, these higher order partial waves give a repulsive contribution of about $0.5$ MeV per particle. The wound integral is shown in Table-25 at a laboratory energy corresponding to $k_0 = 0.9112$ and $1.2656$ fm$^{-1}$. We have found that the wound is small which implies good convergence of the BBG equation.

Once the potential energy $U$ is obtained, one can parametrise it as a function of total energy $E$. Then $U$ can be interpreted as the potential felt by an incident particle in nuclear matter—this being the real part of the optical potential felt by it. The Liege group$^{62,63}$ analysed $U$ as a function of density $\rho$ and total energy $E$ in the form

$$U = \sum_{i,j=1}^{3} a_{ij} \rho^i E^j$$ ........................(3.65)
which again can be written in the form

\[ U = A + BE + CE^2 \]  \hspace{1cm} (3.66)

However, Liege group has neglected the second order contribution. We have parametrised the single particle potential energy for the Paris potential in both the form (3.65) and (3.66) and have compared the results with those obtained by Liege group for the Reid soft-core potential. Tables-26 and 27 represent these results. It is found that the potential depth \( A \) is larger and the energy dependence smaller for the Paris potential. The larger value of \( A \) can be understood by the fact that the saturation point for the Paris potential lies below the continuous Coester band though like the other modern potentials viz. Bonn\(^6\)\(^3\), Bethe-Johnson\(^6\)\(^4\), it also misses the empirical binding energy, being in a parallel second Coester band\(^6\)\(^5\). A reduction in binding energy with the inclusion of isobar degrees of freedom was observed by Green-Niskanen\(^3\)\(^9\), Day-Coester\(^4\)\(^0\), Dahlblom-Smulter\(^4\)\(^1\), Holinde-Machleidt\(^4\)\(^2\) and Samanta\(^4\)\(^3\). So, inclusion of isobar degrees of freedom with these modern potentials like Paris potential may yield a better binding energy and at the same time may bring the potential depth parameter \( A \) closer to the conventional value of 55 MeV or so as suggested by experiment of Becchetti and Greenlees\(^6\)\(^6\). Such calculations would also yield the isobar optical potential which can be derived in a way analogous to what we have outlined in the present paper. These are studied in the next chapter.