CHAPTER-II

A COMPARATIVE STUDY OF

SPECTRAL BAND ABSORPTION
2.1 Introduction

The elementary models that are used to calculate the frequency-dependent absorptive power — i.e. absorptance — of a gas may be classed into four types: the Elsasser model, the statistical model, the random Elsasser model and the quasi-random model [1]. The absorption for a band composed of identical spectral lines that are uniformly spaced has been studied by Elsasser [2]. On the other hand, the statistical model of a band assumes that the positions of the individual spectral lines occur at random and that the intensities can be represented by some probability distribution. The statistical model was suggested by Teller [3] and worked out by Mayer [4] and independently by Goody [5]. The random Elsasser model is a combination of the statistical and Elsasser models: it describes absorption by a system of randomly distributed bands, each of which has a uniform, though perhaps different, spacing. The quasi-random model can handle the most general type of band arrangements: for example CO₂ and H₂O.

2.2 Absorption by a Single Line

In this section, equations are given for the absorption by a single spectral line with Lorentz, Doppler and combination of line shapes. It will be shown in section 2-4 that the absorption for the statistical model of a band is a function of the absorption of a single line only. Since it is necessary to understand these single-line equations before studying the statistical model, some well-known results for single line absorption have been briefly presented here.

(a) Single line with Lorentz shape: The total absorption by a single line over a frequency interval Δv is given by
\[ W' = \int_{\lambda} (1 - e^{-\lambda}) d\nu \]  

where \( W' \) represents the equivalent width of a single line, \( u \) is the mass of the absorbing gas per unit area, \( S \) is the total intensity of the line, and \( b \) is the line shape factor normalized to unity, so that:

\[ \int_0^\infty b(\nu) d\nu = 1 \]  

(2)

For the Lorentz line shape,

\[ b(\nu) = \frac{\alpha / \pi}{(\nu - \nu_0)^2 + \alpha^2} \]  

(3)

where \( \alpha \) is the half-width of the line whose centre is at \( \nu_0 \). The frequency interval \( \Delta \nu \) in Eq. (1) is usually allowed to approach to infinity. Physically this requires that there is no spectral line outside of this interval. When Eq. (3) is substituted in Eq. (1), the resulting integral can be solved exactly [6] for the equivalent width to obtain

\[ W_{\text{al}} = 2\pi \alpha x e^{-x} [I_0(x) + I_1(x)] \]  

(4)

where

\[ x = Su / 2\pi \alpha \]  

(5)

\( I_0 \) and \( I_1 \) are the Bessel functions of imaginary argument. Here it is assumed that there is no absorption outside of the frequency interval \( \Delta \nu \). Mathematically, this is equivalent to integrating Eq. (1) over all frequencies from zero to infinity. There are well-known approximations [7] to Eq. (4) for small and large values of \( x \). For small values of \( x \), Eq. (4) becomes
\[ W_{sl} = 2\pi \alpha x = d\beta x \leq 0.02q \quad (6) \]

\[ \beta = 2\pi \alpha / d \quad (7) \]

where \( d \) represents the average spacing between the spectral lines. Here it is assumed that the spectral lines are spaced sufficiently far apart so that there are no effects due to overlapping. In this case the absorption from several spectral lines is merely the sum of the absorption from each individual line. The preceding approximation is valid within \( q \) percent, if \( x \leq 0.02q \) as has been shown by Plass [8]. Eq. (6) is accurate within 10% if \( x \leq 0.2 \). For large values of \( x \), Eq. (4) becomes

\[ W_{sl} = 2\alpha \sqrt{2\pi x} = d \sqrt{\frac{2\beta^2 x}{\pi}}, \quad x \geq 12.5q^{-1} \quad (8) \]

If \( x \geq 12.5q^{-1} \), Eq. (8) is accurate within \( q \) percent. For a single line there is no upper limit to the region of validity of Eq. (8). Expressions for \( W_{sl} \) are required when the frequency interval \( \Delta v \) is finite. If there are \( n \) nonoverlapping lines in the frequency interval \( D \) with an average spacing between the lines of \( d \), then \( \Delta v = D = nd \). The equivalent width \( W_{sl} \) for a single line over the frequency interval \( D \) can be obtained from Eq. (1) in the weak and strong line approximation [9]. For small values of \( x \) the result is

\[ W_{sl,D} = \frac{2}{\pi} d\beta \tan^{-1}(n\alpha / \pi) \quad (8a) \]

while for large values of \( x \) it is found that

\[ W_{sl,D} = \sqrt{\frac{2\beta^2 x}{\pi n^2}} [1 - \phi(z)] + 1 = \exp(-z^2) \quad (8b) \]

where
and $\beta$ is defined by Eq. (7). In the limit as $n$ approaches to infinity, $\Delta v$ also approaches infinity and the above equations become identical with Eqs. (6) and (8).

(b) Single line with Doppler shape

For the Doppler line shape,

$$\begin{align*}
S_b(v) &= k_0 \exp\left\{-\frac{(v-v_0)^2}{2\Delta v^2}\right\} \\
\text{where} \\
k_0 &= \left(\frac{S}{\Delta v_0}\right)^{\frac{\ln 2}{\pi}}
\end{align*}$$

and $\Delta v_0$ is half the Doppler width at half-maximum. From Eqs. (1) and (9) the following expressions can be obtained [10] for the equivalent width:

$$W_d = \frac{2}{\pi} d\beta \sum_{n=0}^{\infty} \frac{(-1)^n x_{d^n}}{(n+1)! (n+1)^{1/2}}$$

and

$$W_d = \frac{2}{\pi} d\beta \sqrt{\ln x_{d^n} \left[1 + \left(\frac{C}{2} \ln x_{d^n}\right)^{-1} \right]}$$

where
and

$$\beta_1 = \sqrt{\frac{2}{\pi}} \left( \frac{\Delta V_{t,1}}{d} \right)$$

when $x \leq 2^{1/2} \gamma / 50$, then the first term of the Eq. (11) represents $W_s$ within $\gamma$ percent. Of course, the equivalent width always varies linearly with $S$ and $u$ for small values of $x_0$. For large values of $x_0$, Eq. (12) shows that the equivalent width increases very slowly as $(\ln x_0)^{1/2}$.

**c) Single line with both Doppler and Lorentz broadening**

When both the Doppler effect and Lorentz collision damping contribute to the shape, the general terms for both the Taylor series and the asymptotic expansions of the line shape have been derived [11]. The equivalent width is obtained by substituting these series into Eq. (1). For small values of $S$ and $u$, the usual result is obtained that $W_s = Su$. For large values of $x$, the equivalent width is

$$W_{sl} = d \left( \frac{2 \beta^2 x}{\pi} \right) \left\{ - (1 - \ldots) \chi(8x)^{-1} + \ldots \right\} \quad (13)$$

where

$$a = \left( \frac{\alpha}{\Delta V_{t,1}} \right) \sqrt{\ln 2}$$

Both $x$ and $\beta$ are defined by Eqs. (5) and (7). If the Doppler half-width is considerably larger than the Lorentz half-width, the square root approximation is still valid if the second term in the series is small, i.e., for sufficiently large values of $x$. The physical reason for this is that the absorption coefficient falls off exponentially in the wings of the
Doppler line shape and only as $v^2$ in the wings of the Lorentz line shape. Thus at sufficiently large optical depths to absorb completely the centre of the line, the variation in the absorption with path is determined by the wings of the lines where the $v^2$ term dominates.

2.3 Absorption by an Elsasser Band

If a band is assumed to consist of an infinite number of spectral lines, each with the same intensity $S$ and half-width $a$ and with all the lines equally spaced by $d$ units from their neighbours is shown in Fig. 2(a). Elsasser found that an expression can be obtained for the absorption coefficient summed over the band [2,6]. When the expression is substituted in Eq. (1), it is found that the fractional absorption $A$, integrated over a frequency range $d$ for an Elsasser band, can be written in the form

$$A = 1 - \frac{1}{2\pi} \int \exp[-\beta x \sinh \beta \text{cosh } \beta - \cos z)] \, dz$$ (14)

where $x$ and $\beta$ are defined by Eqs. (5) and (7) respectively. Although the integral in Eq. (14) cannot be evaluated in terms of elementary function, a number of approximate expressions have been formulated.

$$A = \phi\left[\sqrt{\frac{1}{2}} \beta^3 x\right]$$ (15)

and

$$A = 1 - e^{-\beta h}$$ (16)

where $\phi$ is the error function integral defined below Eq. (8b). Eq. (15) is correct within 10% when $x > 1.25$ and $\beta < 0.3$. Eq. (16) is valid when $\beta > \frac{1}{3}$ for all $x$. Further, when $A$ is set equal to $W_d/d$ and expression given by Eq. (4) is substituted for $W_{sh}$, this expression is...
correct within 10% provided $x < 0.06 \beta^{-2}$, if $\beta < 0.3$ and provided $x < 0.2 \beta^{-1}$, if $\beta > 0.3$

The limits given here have been derived by the method given by Plaxs [6]. They differ somewhat from the approximate results given by Elsasser [6] and Goody [9].

Results of numerical calculations of the absorption from Eq (14) (or from the approximate equations in their range of validity) are shown in Fig. 2(b) for $\beta = 0.01, 0.1, 1.0, \text{ and } 10$. The fractional absorption $A$ is plotted against $\beta^2 x = 2\pi Su_0 d^2$. The weak and strong line approximations given by Eqs. (6) and (8) respectively are shown in Fig. 2(b) for comparison.

2.4 Statistical Model of a Band

This model assumes that the position and intensity of a spectral line can be specified by probability distribution functions. Previous derivations have all assumed that the positions of spectral lines occur at random within a given frequency interval and there is no correlation between position of the various lines. In this section the equation for the absorption by a statistical band has been derived in somewhat different manner than was used by Mayer [2] and Goody [3]. For any given number of spectral lines in the frequency interval of interest, it is shown that an exact and useful expression can be given for the absorption. In its general form, this equation from the absorption is a function only of the equivalent width for a single line. Since this latter quantity is known for a number of different line shapes, the exact expressions for the absorption by a statistical band can then be obtained directly.

Let $N(v_1, \ldots, v_n) dv_1 \ldots dv_n$ be the probability that the centre of the first line is in the frequency interval $dv_1$, when the centre of the second line is in the interval $dv_2$, etc., up to line $n$. Take the origin for frequency at the centre of a band of width $D$. Let $P(S_i) dS_i$ be the probability of the $i^{th}$ line having an intensity in the range $dS_i$. The average absorption over the frequency interval $D$ is [2, 3]
\[ A = 1 - \frac{\frac{\mathcal{J}_p}{\mathcal{J}_c} \int \cdots \int N(v_1 \ldots v_n)dv_1 \ldots dv_n \prod P(S_i) \exp[-S,u_0(b(v))]}{\mathcal{J}_c^{2n}} } \]

since the transmission of the \( i \)th line when its centre at \( v \), is \( \exp[-S,u_0(b(v))] \). Now assume that it is equally probable that each line has its centre at a given frequency interval within frequency interval \( D \), regardless of the position of the other lines. Then \( N(v_1 \ldots v_n) \) is a constant. Further, always normalize \( P(S) \) so that

\[ \int_0^{\infty} P(S_i) dS = 1 \] (18)

Then

\[ A = 1 - D^{-n} \frac{\mathcal{J}_p}{\mathcal{J}_c} \int \cdots \int N(v_1 \ldots v_n)dv_1 \ldots dv_n \prod P(S_i) \exp[-S,u_0(b(v))]dS. \] (19)

However, the integral

\[ \overline{W}_v = \int_{-\mathcal{J}_0}^{0} d\nu \int_0^\infty \{ P(S) e^{-S,u_0(v)} dS \} \] (20)

This may be written as

\[ A = 1 - \left\{ 1 - \frac{\mathcal{J}_p}{\mathcal{J}_c} \int \cdots \int P(S) \left[ 1 - e^{-S,u_0(v)} \right] dS \right\} \] (21)

After an interchange of the order of integration, this can be expressed as
A = 1 - \left\{ 1 - \frac{1}{D} \int P(S) dS \left[ \left( 1 - e^{-\alpha S} \right) d\nu \right] \right\} ^n \tag{22}

The integral over frequency in Eq. (22) is equal to the equivalent width of a single line evaluated over the frequency interval D. W_{\alpha \nu} as defined by Eq. (1) with \Delta \nu = D. The absorption can be written as

\[ A = 1 - \left\{ 1 - \frac{1}{D} \int W_{\alpha \nu} (S, \alpha) P(S) dS \right\} ^n \tag{23} \]

Now the average value of the equivalent width over the distribution of line strength is

\[ \overline{W}_{\alpha \nu} (S_0, \alpha) = \left[ \int W_{\alpha \nu} (S, \alpha) P(S) dS \right] \tag{24} \]

where \( S_0 \) indicates some mean line intensity that occurs in the distribution function \( P(S) \). Since \( D = nd \), Eq. (23) becomes

\[ A = 1 - \left\{ 1 - \overline{W}_{\alpha \nu} / nd \right\} \tag{25} \]

If the number of lines in the band \( n \), is allowed to approach infinity while the mean spacing, \( d \) is held constant, the frequency interval, \( D = nd \) also approaches infinity. From the definition of the exponential, it follows that in the limit as \( n \) approaches infinity, \( A = 1 - \exp \left( - \frac{\overline{W}}{d} \right) \) \tag{26} \]

where \( W_{\alpha \nu} \), without the subscript \( D \) represents the equivalent width of a single line for an infinite frequency interval. The general expression for the absorption from a finite
number $n$, and from an infinite number of randomly spaced spectral lines is given by Eqs. (25) and (26) respectively. These equations show that the absorption of a band including all effects caused by the overlapping of the spectral lines can be obtained from the single quantity $s_i$, the equivalent width for a single spectral line for the appropriate frequency interval averaged over the distribution of intensities of the spectral lines in the band. Since the expressions for the equivalent width of a single spectral line reduce to tractable mathematical expressions in many cases mentioned in section 2.2, it follows from the corresponding expression for the absorption from the statistical model of the band can be obtained directly from Eqs. (25) and (26).

(a) **Statistical model with finite number of spectral lines**

An experimental determination of the absorption of a band over a given frequency interval necessarily measures the effect of a finite number of spectral lines. In this section the change in the predicted absorption as the number of spectral lines in the interval increases is studied. For simplicity it is assumed that the spectral lines have the Lorentz line shape and that all of the spectral lines are equally intense, i.e. $P(S) = \delta(S-S_0)$. In this section, the influence of the line shape and of the form $P(S)$ on the absorption have been studied. Eq. (25) for the absorption from $n$ spectral lines with an average spacing $d$ can be written in the form,

$$A = \frac{W_{d,D}}{d} \left[ \frac{1}{2} \left( \frac{W_{d,D}}{d} \right)^2 + \frac{1}{3} \left( \frac{1-1/n}{2} \right) \left( \frac{W_{d,D}}{d} \right)^3 + \cdots \right]$$  \hspace{1cm} (27)

This series has exactly $n$ term, since small values of $W_{d,D}/d$ the absorption is equal to $W_{d,D}$. Thus the absorption is just $n$ times the absorption of each of the individual spectral lines over the frequency interval $D$. Of course, this result must hold for sufficiently small path lengths, since there is no overlapping of the spectral lines. For larger values of $W_{d,D}/D$, the remaining terms in Eq. (27) gives the effect of the overlapping of the
spectral line according to the statistical model. The form of the first few coefficients in Eq. (27) shows that their numerical value is essentially independent of \( n \), provided that \( n \) is reasonably large \((n \geq 10)\). Thus in the region where overlapping first becomes important, the absorption curve does not depend on \( n \), if \( n \geq 10 \). For very large path lengths, the absorption depends on \( n \) through the higher term in Eq. (27). However, the absorption is so close to unity that this is of no practical interest. As \( n \) approaches infinity the coefficients of Eq. (27) approach those of the exponential, so that they agree with the series expansion of Eq. (26). Since it is assumed in this section \( P(S) = \delta(S-S_0) \), it follows that

\[
\overline{W}_{n,D} = W_{n,D} \left(S_0, \alpha \right)
\]  

(28)

the equivalent width for a single line, which can be obtained from Eq. (1). For some intermediate values of \( x \), Eq. (1) is evaluated directly from a computer. The result of these calculations is shown in Fig. 2(c) for the cases \( n = 2, 10 \) and infinity and \( \beta = 0.1 \) and 1.0. Hence, the transmission (equal to \( 1-A \)) is plotted against \( \beta^2x \). The curves differ appreciably only when the effect of overlapping is important. When the absorption is large, the curve for \( n = 2 \) gives more absorption for a given value of \( x \) then the curves for \( n = 10 \) and infinity. The physical reason for this is that more of the lines overlap appreciably as the number of lines in the interval is increased; thus they lose part of their absorbing power. However, it has been observed that the curves for \( n = 10 \) and infinity are already vary close together over the entire range shown here. Thus, if there are at least 10 spectral lines in the frequency interval \( D \), the more convenient equations may be used and that can be applied in the limit as \( n \) approaches infinity. However, when there are only a small number of spectral lines in a given frequency interval, the appropriate absorption curve for the statistical model can be obtained from either Eq. (25) or (27).
(b) **Statistical model in the limit of infinite number of spectral lines**

Provided that there are at least 10 spectral lines in the frequency interval of interest, the results of section (a) shows that the absorption curves for a finite and an infinite number of spectral lines are nearly the same for any value of the absorption of practical interest. In this Section, the absorption relation Eq. (26), valid in the limit of an infinite number of spectral lines, has been used. The influence of the absorption of various intensity distributions for the spectral lines in a band and of various line shapes is studied in this section.

(i) **Lorentz Line Shape**

(a) All spectral lines equally intense

If the Lorentz line shape Eq. (3) is valid and if all the spectral lines are equally intense so that \( P(S) = \delta(S-S_0) \), the exact expression for the absorption from the statistical model of a band can be written immediately. Because of the assumed distribution of intensities, the average value of the equivalent width \( \overline{W_{SI}} \) over \( P(S) \) is obtained merely by replacing \( S \) by \( S_0 \) in \( \overline{W_{SI}} \) as indicated in Eq. (28). Thus the exact expression for the absorption is given by Eq. (26) with Eq. (4) substituted in \( \overline{W_{SI}} \). Of course, the approximate Eqs.(6) and (8) can be used for \( \overline{W_{SI}} \) with in their range of validity. In Fig. 2(c) the absorption is shown as a function of \( \beta^2 \alpha = 2\pi S \sigma a/d^2 \) for \( \beta = 0.01, 0.1, 1.0 \) and 10. The absorption calculated from the weak and strong line approximations, Eqs. (6) and (8) respectively is also indicated. In Fig. 2(e) the absorption predicted by the Elsasser and statistical models are compared. When the absorption is small, all the absorption calculated from the two models are the same, since there is no overlapping of the spectral lines. However, at larger values of \( \alpha \) the absorption calculated from the Elsasser model is always greater than that from the statistical model. This is because there is always more overlapping of the spectral lines in a random distribution of the positions of the line that
in the regular distribution used in the Elsasser model. Thus the line strength is not used as effectively for absorption in the statistical model as in the Elsasser model. In section 2.4, \( S_1 = S_2 \), the regions where these two models predict different values for the absorption has been studied in detail together with the regions of validity of the customary approximations to both theories.

[b] - Exponential Distribution of Line Intensities

There are many bands the probability of finding a spectral lines of intensity \( S \) in a given frequency range decreases approximately and exponentially,

\[
P(S) = \frac{1}{S_0} \times \exp\left(\frac{S}{S_0}\right)
\]

(29)

If \( W_{d,d} \) in Eq. (4) is substituted into Eq. (24) the average value of the equivalent width \( \bar{W}_{sl} \) can be written as

\[
\bar{W}_{sl} = \frac{4\pi}{S_0 u} \int_0^\infty [I_0(x) + I_1(x)] \exp\left\{ -\left[ 1 + \left( \frac{2\pi \alpha}{S_0 u} \right)^2 \right] x \right\} dx
\]

(30)

After evaluation of the integral [10] and substitution of the result into Eq. (26) the absorption by this statistical band is found to be given exactly by [2,3]

\[
A = 1 - \exp\left[ -\frac{\beta x_0}{\sqrt{1 + 2x_0}} \right]
\]

(31)

where \( x_0 = S_0 u / 2\pi \alpha \). In order to compare the absorption predicted for the model in this section with that of Section 2.2.(a), the absorption for these as a function of \( \beta x_0 = S_0 u / d \) is shown in Fig. 2(f) after this correlation has been made. For \( \beta = 0.01 \) and 0.1, there is no difference between this absorption curves that can be shown on the scale of the figure.
For $P = 1.0$ the two curves differ appreciably only when $x_0 < 1.25$. For comparison, the absorption for an Elsasser band is also shown in Fig. 2(f).

(ii) Square line shape

The shape of the spectral line is approximately given by a square shape factor

$$b(v) = \frac{1}{\delta} \nu_0 \langle \nu (\nu_0 + \delta) \rangle = 0 \nu \langle \nu_0 ; \nu \rangle (\nu_0 + \delta)$$

The equivalent width can be from Eq. (1) to be

$$W_u = \delta \left[ 1 - \exp \left( -\frac{Su}{d} \right) \right]$$

If it is assumed that all the spectral lines are equally intense, $P(S) = \delta(S-S_0)$, the absorption from the statistical model of the band is

$$A = 1 - \exp \left\{ -\frac{\delta}{d} \left[ 1 - \exp \left( -\frac{Su}{d} \right) \right] \right\}$$

On the other hand, if the line intensity of the band decreases exponentially from Eqs. (24), (29) and (33) the average equivalent width is

$$\overline{W}_{u,D} = \delta \left[ \frac{1}{1+\left( \frac{\delta}{Su} \right)} \right]$$

Thus the absorption of the statistical band in this case is

$$A = 1 - \exp \left\{ -\frac{\delta}{d} \left[ 1 + \frac{\delta}{Su} \right]^{-1} \right\}$$

The square line shape can sometimes provide a useful approximation to the Doppler line shape. The absorption from a spectral line with the Doppler line shape is limited at large path lengths by the very rapid decrease in the absorption coefficient beyond the half-width. This property is approximated by the square line shape whose absorption is exactly
zero beyond the half-width. In addition, the constants for the square line shape can be chosen so that it predicts the same absorption as the Doppler line shape for small path lengths. Since the equations for the absorption from the statistical model are considerably simpler for the square line shape than for the Doppler line shape, they can provide a useful first approximation. The square line shape is of no value in approximating the Lorentz line shape and the equations for the absorption are not appreciably simpler. The absorption from the Doppler and square line shapes is compared further in the following section.

(iii) Doppler Line Shape

The line shape is determined by the Doppler effect under certain conditions of temperature and pressure. But in the case of equivalent width can be represented \([7, 8]\) either by Taylor series

\[
W_d = d \beta_d x_d \sum_{n=0}^{\infty} \frac{(-1)^n x_d^n}{(n+1)! \sqrt{(n+1)}}
\]  

(37)

or by the asymptotic expansion

\[
W_d = \frac{2}{\pi} \beta_d \sqrt{\ln x_d} \left( 1 + \frac{C}{\ln x_d^2} + \ldots \right)
\]  

(38)

where \(\beta_d\) and \(x_d\) are defined for the Doppler line shape as

\[
\beta_d = \sqrt{\frac{\pi}{\ln 2}} \times \frac{\Delta V_{hi}}{d}
\]  

(39)

and

\[
x_d = \sqrt{\frac{\ln 2}{\pi}} \left( \frac{S_{hi}}{\Delta V_d} \right)
\]  

(40)
A \nu_0 is half the total Doppler width at half-maximum and C = 0.5772 is Euler's constant.

When \( P(S) = \delta(S-S_0) \), then \( W_{sl} \) is immediately obtained by writing \( S_0 \) for \( S \) in Eqs. (37) and (38). From the leading term of each of these expansions it follows that the absorption from the statistical model is

\[
A = 1 - \exp(-\beta_0 x_{sl}) \quad x_{sl} < 0.28
\]  

and

\[
A = 1 - \exp\left[-\frac{2}{\sqrt{\pi}} \beta_0 \sqrt{\ln x_{sl}}\right] \quad x_{sl} > 18
\]  

where \( x_{sl} \) is \( x_0 \) with \( S \) replaced by \( S_0 \). The limits for \( x_{sl} \) given for these equations show the range of \( x_0 \) where the expressions for \( W_{sl} \) is accurate to 10%. The absorption for a statistical band with the Doppler line shape as a function of \( x_0 \) is shown in Fig. 2(g). In addition, the absorption for a statistical band with the square line shape, given in Eq. (34), is shown in Fig. 2(g). Hence in this case, \( \beta_0 \) and \( x_0 \) have been set equal to \( (\pi/2)(d/5) \) and \( (2/\pi)(uS_0/6) \), respectively. It is seen that the square line shape accurately represents the absorption for a Doppler line shape when \( x_0 \) is small and that it saturating at the correct value of \( x_0 \). However, the square line shape completely saturates as \( x_0 \) increases so that the absorption cannot, beyond a certain value, be determined by the probability of the spectral lines overlapping, whereas the absorption from the Doppler line shape slowly increases towards complete absorption as \( x_0 \). When \( x_0 < 1 \) the difference in the absorption between these two lines shapes is small. The value of absorption at which the square line shapes saturates can be changed by choosing a different correlation of its constants with \( \beta_0 \) and \( x_0 \) than the one used above.

(iv) Combination of Lorentz and Doppler Line Shapes

When the Doppler effect and damping contribute to the line shape the general term of both a Taylor series and an asymptotic expansion for the line absorption

\[
A = 1 - \exp\left[-\frac{2}{\sqrt{\pi}} \beta_0 \sqrt{\ln x_{sl}}\right] \quad x_{sl} > 18
\]
coefficient can be given [8]. These series can be integrated to obtain the equivalent width $W_{st}$ in both the weak and strong line approximations [8]. In the weak line approximation $W_{st}$ for any line shape depends only on the line strength and path length so that

$$W_{st} = S u = d \beta,$$  \hspace{1cm} (43)$$

In the strong line approximation, the equivalent width is

$$W_{st} = d \sqrt{\frac{2\beta^2 x}{\pi}} \left[1 - \frac{1}{8x} \left(1 - \frac{3}{2\alpha^2}\right) + \ldots\right]$$  \hspace{1cm} (44)$$

where

$$A = \frac{\alpha}{\Delta v_D} \times \sqrt{\ln 2}$$  \hspace{1cm} (45)$$

The absorption from the statistical model of a band can be obtained immediately by the substitution of Eqs. (43) and (44) into Eqs. (24) and (26). It should be noted that even though the Doppler width is considerably larger than the Lorentz width, the absorption for large values of $x$ is determined essentially only by the Lorentz width. The physical reason for this is that the Doppler line shape falls off exponentially in the wings of the line whereas the Lorentz line shape falls as $(v - v_0)^2$. Since the absorption is independent of line shape in the weak line approximation, it follows that the Doppler width can influence the absorption only for the immediate values of $x$.

### 2.5 Random Superposition of Elsasser Bands

As $x$ increases the results calculated from these two models begin to diverge; the Elsasser model always gives more absorption than the statistical model for a given value of $x$. The reason for this is that, there is always more overlapping of the spectral line in the statistical model than with the regular arrangement of the lines of the Elsasser model; thus for a given path length and pressure the total line strength is not used as effectively for absorption in the statistical model. Moreover, the actual band has its spectral lines
arranged neither completely at random nor at regular intervals. The actual pattern is formed by the superposition of many systems of lines. For a molecule such as CO₂, one such system of lines may dominate the absorption in some frequency intervals. But other weak system of lines due to other transitions and to isotopic species are almost always present. On the other hand, the absorption of a molecule of H₂O is due to the superposition of many systems of lines with comparable strength. Clearly intermediate cases of superposition of systems of lines also occur. The absorption can be represented in all of these cases more accurately by the random Elsasser band model than by either the statistical or Elsasser model alone. The random Elsasser band model is a natural generalization of the original models. It assumes that the absorption can be represented by the random superposition of Elsasser bands. The individual bands may have different line spacings, half-widths and intensities. As the number of superposed Elsasser bands becomes large, the predicted absorption approaches that of the usual statistical model.

In order to distinguish the frequency intervals that occurs in this section, as Δᵢ be the separation of the spectral lines in the ith Elsasser band and N be the number of superposed Elsasser bands. The average spacing δ between the spectral lines is

\[ δ = \left( \sum \frac{1}{Δᵢ} \right)^{-1} \]  \hspace{1cm} (46)

an equivalent width \( W_{E,i} \) for ith Elsasser band by

\[ W_{E,i}(x_i, \beta_{Δi}) = Δᵢ A_{E,i}(x_i, \beta_{Δi}) \]  \hspace{1cm} (47)

where \( β_{Δi} = 2πα/Δi \) and \( A_{E,i} \) is the absorption from an Elsasser band given by Eq. (14). The line intensity, spacing and half-width may be different for each Elsasser band. Then by same argument by which Eq. (19) has been derived the absorption from a random superposition of \( N \) Elsasser band is
where $b_i(v_i)$ is the line shape factor summed over all of the spectral lines in the $i$th Elsasser band and $P_i(S_i)\,dS_i$ is the probability of the $i$th Elsasser band having an intensity in the range $dS_i$. After an interchange of the order of integration and the using Eqs. (1), (14) and (47) the absorption can be written as

$$A = 1 - \prod_{i=1}^{n} \left[ 1 - \frac{1}{\Delta_i} \int_{\Delta_i}^{\infty} P_i(S_i) \exp[-S_i \cdot b_i(v_i)] \,dS_i \right]$$

(49)

If the average value of the equivalent width $W_{E,i}$ of an Elsasser band over the intensity distribution for the band is defined as in Eq. (24) then

$$A = 1 - \prod_{i=1}^{n} \left[ 1 - \frac{W_{E,i}}{\Delta_i} \right]$$

(50)

This is the general result for the absorption from a random superposition of $N$ Elsasser bands of which may have different line spacing, half-width and intensity. The absorption for the random superposition of various Elsasser bands is shown in Fig. 2(g). The cases given include the superposition of two Elsasser bands of equal intensity; two Elsasser bands with one having ten times the intensity of the other; five Elsasser bands of equal intensity. Now for comparison, a single Elsasser band and the statistical model of a band with all lines of equal intensity are also given.

In this context, it is observed that the random Elsasser band model provides a smooth transition between the absorption curves for the limiting cases of the Elsasser and statistical models of a band. As the number of random superposed Elsasser bands increases and the average line spacing $\delta$ is constant, the absorption curves approaches that of the statistical model. This can be proved most readily if it is assumed for simplicity that $\Delta_i$ are the same for each Elsasser band. Eq. (50) can be written as
A = W, since A = Nδ as N approaches infinity for fixed δ, A also approaches infinity and β approaches zero. However, for any finite value of β, as β approaches zero the absorption from an Elsasser band approaches that from a single line, i.e. \(\overline{W}_a\) approaches \(\overline{W}_d\). Thus, in the limit of Eqs. (51) and (25) are identical and the absorption is given by Eq. (26). The usual statistical model can be considered as the random superposition of a large number of Elsasser bands.

2.6 Regions of Validity of various models and approximations

Table 2 consists of Regions of validity of various models and approximations for Lorentz line shape. The range of values of \(x = Su/2\pi\alpha\) is indicated where the absorption model or approximation named is valid with an error of less than 10%. For all column except the last i.e P(S) = \(\delta(S-S_0)\) is, \(x > 1.25\) has been made for the line intensity distribution of the statistical model. For the last column it is assumed that the constant in the different line intensity distribution functions are correlated in the square root region. This section of regions of validity of the linear and square root approximation have been stated. In addition, a derivation is given for the range over which overlapping can be neglected, where the absorption is a function only of the single variable \(\beta^2x \sim pu\), and where the absorption is less than 10% more than 90%. Further, it is shown in what conditions the Elsasser and the statistical models predict the same absorption as well as when the shape of the absorption curve is changed by different intensity distribution functions, P(S) for the spectral lines in the band. In many cases in this section the complete proof of the region of validity of a given model or approximation is not given, since all of these proofs are similar. The technique is to expand the mathematical expression for the absorption into an appropriate power series. In all cases, which are
treated in this section, these turns out to be convergent, alternating series. The error made in retaining only a finite number of terms in such a series is always less than the magnitude of the first omitted term. Thus it is possible the regions of validity of various approximations, which usually retain only the first term of the series, if the second term of the series is known [6].

(a) Non-overlapping of spectral lines

For the statistical model, the condition for the neglect of overlapping of the spectral lines can easily be derived by expansion of Eq. (26). But the error in the absorption A, made by neglecting the overlap is always less than \( q\% \) when

\[
\frac{W_o}{d} < 0.02q \tag{52}
\]

The corresponding condition for \( x \) can be obtained by substituting Eqs. (6) and (8) in Eq. (52). Thus the error made in neglecting the overlapping is less than \( q\% \) when

\[
x < 0.00628q^2 \beta^{-2}, \quad \text{if } \beta < 0.0314q
\]

\[
x < 0.02q \beta^{-1}, \quad \text{if } \beta > 0.0314q \tag{53}
\]

otherwise, the statistical model with all the spectral lines having equal intensity is assumed in this section. Other assumptions of \( P(S) \) change the numerical factors of the inequalities only slightly. For convenience, the regions in which the neglect of overlapping causes less than 10% error (\( q = 10 \)) in A are given in Table 2.

(i) Linear and Square-root approximation

The linear approximation Eq. (6) is valid when this is a suitable approximation for a single line and when in addition there is no appreciable overlapping Eq. (53). Hence the
linear approximation is accurate within q% for the statistical model with all spectral lines of equal intensity when

\[ x < 0.02q, \quad \text{if } \beta < 1 \]

\[ x < 0.02q\beta^{-1}, \quad \text{if } \beta > 1 \]  \hspace{1cm} (54)

Similarly the square root approximation Eq. (8) is valid when this is a suitable approximation for a single line and when there is no overlapping. The square root approximation is accurate within q% for the statistical model with all lines equally intense when

\[ 12.5 \times \frac{1}{q} \left< \frac{q^2}{\beta^2} \right> \beta (7.09q^{3/2}10^{-3} \hspace{1cm} (55) \]

When

\[ \beta > 7.0910^{-3} q^{3/2} \]

there is no region of validity of the square root approximation. For the Elsasser band, the square root approximation is accurate within q% when [6]

\[ 12.5q^{-1} < x < 0.06q^{-2} \]  \hspace{1cm} (56)

The Elsasser band can be represented by the square-root approximation at considerably larger values of x. This occurs because the overlapping of the spectral lines becomes important at smaller values of x. When their distribution is random than if it is regular as required by the Elsasser band. This difference, in this absorption can be seen by comparing Fig. 2 (a) and Fig. 2(c). The range of values for x where the absorption is represented within 10% by the linear and square root approximation is given in Table 2.
[c] Product $pu$ determines absorption

The absorption is a function only the product of pressure and path length $pu$. In the regions of validity the theoretical analysis of problems where the pressure varies along the path is considerably simplified. Over the appropriate range of variables this technique has been used by Plass [11,13] to calculate the infrared radiation in the atmosphere in the region of the CO$_2$ and O$_3$ bands. In the statistical model, the absorption as given by Eq. (26) is determined by the equivalent width for a single line $W_{\lambda i}$. Now from Eq. (8) $W_{\lambda i}$ is represented within $q\%$ by the square-root approximation whenever $x > 12.5q^{-1}$. Now the square-root approximation is a function only of $\beta^2x$ which is proportional to $pu$. Hence a simple analysis shows that an error of $q\%$ in $W_{\lambda i}$ is substituted in Eq. (26) results in an error less than $q\%$ in $A$. Thus, for a statistical model the absorption is a function of pressure and path length $pu$ within $q\%$ when

$$x > 12.5q^{-1}$$

The region in which this is valid within $10\%$ is given in Table 2. Eq. (57) is also the condition for the Elsasser model to be function of $pu$ only when $\beta < 0.3$. For larger values of $\beta$, this approximation rapidly breaks down as Eq. (16) begins to represent the absorption. The difference in the region of validity between the Elsasser and the statistical bands for large $\beta$ results from entirely different manner in which the spectral lines overlap.

(d) Regions of small and large absorption

For many applications it is important to know when the absorption is small and when it is nearly unity. In these regions detailed calculations of the spectrum are not
necessary since a good approximation is obtained by setting the absorption equal to zero or unity respectively. The absorption is less than 10% when

\[ x < 0.0157\beta^{-2}, \quad \beta < 0.157 \]

\[ x < 0.1\beta^{-1}, \quad \beta > 0.157 \]  \hspace{1cm} (58)

The conditions for small and large \( \beta \) are obtained from Eqs. (8) and (6) respectively, since the overlapping can be neglected and these conditions are valid for any model of a band. The absorption is more than 90% for the statistical band when

\[ x > 8.30\beta^{-2}, \quad \beta < 3.6 \]

\[ x > 2.30\beta^{-1}, \quad \beta > 3.6 \]  \hspace{1cm} (59)

These conditions are derived from Eqs. (6), (8) and (26) respectively. Again from Eqs. (14), (15) and (16) it gives that the absorption is greater than 90% for the Elsasser band when,

\[ x > 2.70\beta^{-2} \quad \beta < 1.17 \]

\[ x > 2.30\beta^{-1} \quad \beta > 1.17 \]  \hspace{1cm} (60)

Hence the Elsasser model saturates at smaller values of \( x \) than the statistical model.

(e) Same absorption for Elsasser and Statistical models

For \( x \) less than a certain limit the Elsasser and statistical models predict the same value for the absorption. For small values of \( \beta \), these models predict same absorption for those values of \( x \) where the overlapping of the spectral lines can be neglected and the limit for \( x \) in this case is obtained from Eq. (53). Now, for \( \beta > 3 \) the absorption from the
Elsasser band Eq. (16) is identical with the absorption from the statistical band, Eq. (26) when \( W_d \) is given by the linear approximation Eq. (6). The appropriate limit for \( x \) is thus given by Eq. (54) which for \( \beta > 1 \) is identical with the limit given by Eq. (53). Hence from these two Equations it is found that the absorption predicted by the Elsasser and statistical models is the same within \( q \% \) when

\[
x < 0.000628q^2 \beta^{-2}, \quad \text{if } \beta < 0.134q
\]

\[
x < 0.02q \beta^{-1}, \quad \text{if } \beta > 0.0314q (61)
\]

Thus the limits on \( x \) for 10% accuracy are given in Table 2. It is instructive to compare these results with Fig. 2 (d).

(f) **Influence of \( P(S) \) on absorption in the statistical model**

In this section, we have shown that the absorption by the statistical model is independent of the form of the intensity distribution \( P(S) \) over a large range of \( x \). The average value of the equivalent width \( \bar{W}_d \) over the intensity distribution \( P(S) \) is defined by Eq. (24) and \( \bar{W}_d \) depends on some mean intensity \( S_0 \), which occurs in the distribution function \( P(S) \). Now by the Mean value theorem a value of the line intensity \( \bar{S} \), can always be found so that the equivalent width evaluated for \( \bar{S} \) is equal to its average value.

\[
W_d(\bar{S}, \alpha) = \bar{W}_d(S_0, \alpha)
\]

Now, the absorption predicted by the statistical model Eq. (26) is a function only of \( \bar{W}_d \) and if the parameters \( S_0^{(1)} = S_0^{(2)} \) that occurs in two different distribution functions \( P^{(1)}(S) \) and \( P^{(2)}(S) \) are correlated, so that \( \bar{S}^{(1)} = \bar{S}^{(2)} \) and for the absorptions are
equal for these two distributions. However, the functional relationship between
\( \tilde{S}^{(1)} \) and \( \tilde{S}^{(2)} \) may change in different regions. Moreover, the absorption from the two
distributions is then equal over most of the range of practical interest. An example will
clarify this point consider the absorption from the following distributions:

\[
\begin{align*}
(1) & & P^{(1)}(S) = \delta(S - S_0^{(1)}) \\
(2) & & P^{(2)}(S) = \frac{1}{S_0^{(1)}} \exp \left( -\frac{S}{S_0^{(1)}} \right)
\end{align*}
\]

For the first distribution \( \tilde{S}^{(1)} = S_0^{(1)} \) and for the second distribution from Eqs. (24), (30),
(31) and (62) it gives us that

\[
\tilde{S}^{(2)} \exp \left( -\frac{\tilde{S}^{(2)} u}{2\pi \alpha} \right) \left[ I_0 \times \frac{\tilde{S}^{(2)} u}{2\pi \alpha} + I_1 \times \frac{\tilde{S}^{(2)} u}{2\pi \alpha} \right] = S_0^{(2)} \left[ 1 + \frac{1}{\frac{S_0^{(2)} u}{\pi \alpha}} \right]
\]

(63)

If \( \frac{\tilde{S}^{(2)}}{2\pi \alpha} > 1.25 \) the square-root approximation may be used to evaluate the left hand side
of Eq. (63) with the result that

\[
\tilde{S}^{(2)} = \frac{1}{4} \pi S_0^{(2)}
\]

(64)

Hence if we set \( S_0^{(2)} = \left( \frac{4}{\pi} \right) S_0^{(1)} \) then the absorption from the statistical model for each of
this distribution is the same within 10\% whenever \( x > 1.25 \). For \( \beta \geq 1 \), the lower limit for
\( x \) is even considerably less than this, since there is appreciably overlapping for these
values of \( x \). The general proof for any two-distribution functions
\( P^{(1)}(S) \) and \( P^{(2)}(S) \) follows the same arguments as given for the above example. Thus,
if the constants $S_{\alpha}^{i+j}$ and $S_{\beta}^{i+j}$ associated with two distribution functions are correlated so that $\tilde{S}_{\alpha}^{i+j} = \tilde{S}_{\beta}^{i+j}$ in the square-root region, the absorption calculated by the statistical model for these two distributions is the same within at least $q$% when

$$x > 12.5q^{-1}$$

where $q > 0$. From the above, we conclude that all the absorption curves for the statistical model have the same line shape when $x > 12.5q^{-1}$, regardless of the distribution of the line intensities in the band. For $\beta < 1$ this includes the entire range in which the absorption has an appreciable value.

2.6 Quasi-random model

It has already been indicated that the lines in a given interval are likely to contribute to the absorptance in an adjacent interval. These wing contributions are already taken into account in Elsasser bands, because of the assumed infinite extent of the band. Hence, the absorption becomes a constant for each band. In the case of the statistical model, however, there has been no mention of the effect of lines in an adjacent interval. The quasi-random model treats these contributions in essentially the same manner as those in the interval under consideration, since one cannot, in general, neglect their effect. This implies that the calculation of the transmittance at the center of the interval $\Omega_j$, containing $n_1$ randomly distributed lines, involves the contributions of the $n_2$ randomly distributed lines in the right adjacent interval $\Omega_k$, the contributions of the $n_3$ randomly distributed lines in the left adjacent interval $\Omega_l$ etc. in equal measures. Fig. 2(h) the quasi-random model provides a means of accurately representing molecular absorptance over finite frequency intervals. It includes absorption by the wings of spectral lines in as many neighbouring intervals as necessary. This model has several other
distinguishing features. Each interval Ω, over which the average transmittance is required, is further divided into smaller intervals δ. Each line may be said to be localized within an error defined by the interval size δ. These elementary intervals are chosen small enough to ensure an accurate description of the important band characteristics, yet large enough to simplify the calculations. All the lines falling within a given order of magnitude are subsequently averaged, and each line in each intensity sector is then associated with the appropriate average value. Since the top five intensity sectors in each frequency interval are found to be sufficient to describe the absorptive properties associated with the lines therein, the transmittance at a frequency ν as affected by the np lines in the frequency interval δp is given by [44]

\[
T(\nu) = \left[ \frac{1}{n} \int_{-\delta_p}^{\delta_p} \left( \frac{1}{d} \right) \exp \left\{ -S,ub(\nu, v_i)dv_i \right\} \right]^{n} \tag{66}
\]

where n, represents the number of lines within the intensity range i, which itself is characterized by an average intensity S, u is the absorber thickness in atm-cm (atmosphere centimeter), and b(ν, v_i) is the Lorentz shape factor defined by

\[
b(\nu, v_i) = \frac{\alpha/\pi}{\left( (\nu - v_i)^2 + \alpha^2 \right)} \tag{67}
\]

(α is the half-width, i.e. half the frequency difference between the half-maximum points, v_i refers to the center of the line).

The quasi-random model does away with artificial effects associated with the imposition of a frequency mesh. As can be expected, the calculation based on this model is quite time-consuming and complex. However, it gives a good representation of spectral characteristics of most molecules at low temperatures (< 400 K).
Table No 2: Regions of validity of various absorption models and approximations for Lorentz line shape.

<table>
<thead>
<tr>
<th>Sl. No</th>
<th>β</th>
<th>All models linear approx. Eq.(6) valid</th>
<th>All models square root approx. Eq.(8) valid</th>
<th>Statistical model square root approx.</th>
<th>Elsasser model square root approx.</th>
<th>Statistical model pu determine Absorption</th>
<th>All models A&lt;10%</th>
<th>Elsasser model A&gt;90%</th>
<th>Statistical model identical</th>
<th>Elsasser and statistical model independent of choice P(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.001</td>
<td>x&lt;62800</td>
<td>x&lt;0.2</td>
<td>1.25&lt;</td>
<td>1.25&lt;x&lt;6</td>
<td>x&lt;62800</td>
<td>x&lt;0.2</td>
<td>1.25&lt;</td>
<td>1.25&lt;x&lt;6</td>
<td>x&lt;15700</td>
</tr>
<tr>
<td>2</td>
<td>0.01</td>
<td>x&lt;628</td>
<td>x&lt;0.2</td>
<td>1.25&lt;</td>
<td>1.25&lt;</td>
<td>x&lt;628</td>
<td>x&lt;0.2</td>
<td>1.25&lt;</td>
<td>1.25&lt;</td>
<td>x&lt;157</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>x&lt;6.28</td>
<td>x&lt;0.2</td>
<td>1.25&lt;</td>
<td>1.25&lt;</td>
<td>x&lt;6.28</td>
<td>x&lt;0.2</td>
<td>1.25&lt;</td>
<td>1.25&lt;</td>
<td>x&lt;157</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>x&lt;0.2</td>
<td>x&lt;0.2</td>
<td>None</td>
<td>None</td>
<td>x&gt;1.25</td>
<td>x&lt;0.1</td>
<td>x&gt;8.30</td>
<td>X&gt;2.70</td>
<td>x&lt;0.2</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>x&lt;0.02</td>
<td>x&lt;0.2</td>
<td>None</td>
<td>None</td>
<td>x&gt;1.25</td>
<td>x&lt;0.01</td>
<td>x&gt;0.23</td>
<td>X&gt;0.23</td>
<td>x&lt;0.02</td>
</tr>
<tr>
<td>6</td>
<td>100</td>
<td>x&lt;0.002</td>
<td>x&lt;0.2</td>
<td>None</td>
<td>None</td>
<td>x&gt;1.25</td>
<td>x&lt;0.001</td>
<td>x&gt;0.23</td>
<td>X&gt;0.023</td>
<td>x&lt;0.002</td>
</tr>
</tbody>
</table>
Fig. 2(b): Absorption for an Elsasser band as a function of $\beta'x = 2\pi Su_\alpha/d$. The linear and square root approximation is also indicated to show their regions of validity. Curves are shown for $\beta = 0.01, 0.1, 1.0, 10$ where $\beta = 2\pi \alpha/d$. 
Fig. 2(c): Transmission for the statistical model of a function of $\beta^2 x = 2\pi S_\omega \omega / d^2$. It has been considered that all spectral lines are equally intense and that the Lorentz line shape factor is valid. Curves are shown for two, ten and infinite number of lines with random spacing but with same mean spacing $d$ in each case.
Fig. 2: Absorption for the statistical model of a band as a function of $\beta^tv$. It has been considered that all spectral lines are equally intense and Lorentz line shape factor is valid, and the absorption is from a large number of spectral lines with an average spacing $d$. The absorption is calculated from the linear and square root approximation.
Fig. 2(e): Absorption for the statistical and Elsasser model of a band. In each case it has been considered that all the spectral lines are equally intense and the Lorentz line shape factor is valid.
Fig. 2(f): Absorption as a function of $\beta x = S_0 u/d$ for the statistical model with an exponential distribution of line intensities $P(S) = S_a^{-1} \exp(-S/S_a)$ and with line equally intense [$P(S) = \delta(S-S_a)$]. Now for comparison purposes, $S_a$ being the exponential distribution function has been multiplied by $1/4\pi$ in order to compare with the case where all the lines are equally intense. For comparison the absorption curves for an Plasner band have been shown.
Fig. 2: Absorption as a function of $\beta_0 x_0 = \pi^{1/2} (\ln 2)^{1/2} \Delta y_{\nu} S y d$ for the Doppler, Lorentz and square line shape. In each case the statistical model of the band has been used with all spectral lines considered to have the same intensity. Now for comparison $\beta_0$ and $x_0$ were set equal to $\beta$ and $x$ for Lorentz line shape factor and to $(\sqrt{\pi}/2\delta \nu d)$ and $(2\sqrt{\pi}S y d)$ for the square line shape respectively.
Fig. 2(Γ): Absorption is a function of $\beta x = Su/d$ for $\beta = 0.1$. The absorption is shown for single Elsasser band ($N = 1$), the random superposition of two Elsasser bands ($N = 2$) where the intensities are equal ($S_1 = S_2$) and one band is ten times as intense as the other ($S_1 = 10S_2$), the random superposition of five Elsasser bands ($N = 5$) where intensities in each band are equal ($S_1 = S_2 = \ldots = S_5$), and the statistical model where all the intensities are equal.
Fig. 2(i): Three typical adjacent intervals showing the position and relative intensities of the lines contained in...
References


[7] Plass G N, *J Meteor* **11** (1954) 163. Only values of \( \beta \) less than unity were considered in this reference. Some of the limits are generalized in this work to include all values of \( \beta \).


[11] Equations essentially identical with Equations (26) and (50) have been derived by Kaplan L D *Proc Toronto Meteor Conf* (1953) 43.


