CHAPTER II

Bifurcation points and Feigenbaum Universality on Two-Dimensional Nonlinear Mappings
CHAPTER-II
Bifurcation points and Feigenbaum Universality on Two-Dimensional Nonlinear Mappings

2.1 Introduction:

Our chief aim in this chapter is to demonstrate some numerical algorithms in order to find periodic points, and the limiting value of the sequence of period doubling bifurcation values of \( p \) for every \( b \) lying in \((-\infty, \infty)\) in the two parameter \((\mu, b)\) families of nonlinear maps. In particular we consider the Henon map, \( H_{\mu, b}(x, y) = (1 - \mu x^2 + y, \ bx) \). Our numerical methods are easily comprehensible and found to be effective in the sense that these give faster convergence.

Mapping of the plane into itself are often used to study the solutions of differential equations via the Poincare Map. In [44] Henon did this to reduce and simplify the Lorenz system of differential equations [67] to the two-dimensional map \( H_{\mu, b}(x, y) = (1 - \mu x^2 + y, \ bx) \). Henon was able to prove, among other things, that the transformation which he considered was the most general quadratic map which carries the plane into itself and has constant Jacobian determinant. Then in a remarkable sequence of computer graphics he gave strong numerical evidence that the transformation he studied has a strange attractor whose local structure is the product of a one-dimensional manifold by a Cantor set, at least in the neighbourhood of one of the stationary solutions.

Further, Henon was able to show for the specific parameter values which he considered, that there exists a compact set \( M \), called a 'trapping region', which is carried into itself by the action of the transformation. Subsequently, Feit in [36] has generalised the above result by giving a characterisation of the compact set of nondivergent points for Henon's transformation. A point in [36] is called nondivergent...
provided its forward orbit under the action of the transformation is bounded.

In [36] characteristic exponents were also computed for a substantial set of parameter values for the Henon map. If the characteristic exponent is less than zero then neighbouring trajectories approach each other at an exponential rate and if it is positive then we have exponential separation of nearby trajectories; hence, the characteristic exponent provides a measure of sensitivity to initial conditions. Feit found, for the parameter values studied by Henon, that the associated characteristic exponent was positive.

Another quantity which is an indicator of sensitivity to initial conditions is the decay of time correlations; the time correlation function is a normalised time covariance function and the decay of this function to zero provides some evidence that a given dynamical system is mixing [80]. By computing the Fourier transform of the covariance, i.e. the frequency spectrum, it is possible to determine which frequencies contribute most to the variance of a process. If the frequency spectrum consists of solitary narrow spikes then the underlying process is (multiply) periodic, while if there is a broad band of frequencies present then the process has continuous spectrum and is not periodic. Gollub and Swinney [40] have measured frequency spectra for the velocity field in a rotating fluid and have found a broad band of frequencies present.

The gently swirling, boomerang-like shape of the attractor that arises through the dynamics is very appealing aesthetically. In fact, it has become another icon of the chaos theory next to the Mandelbrot set, the Feigenbaum diagram, and the Lorenz attractor.

The Henon system leads from the one-dimensional dynamics of the quadratic transformation to higher-dimensional dynamics.
strange attractors. It is simple enough to allow analysis similar to the analysis of chaos in the logistic equation, yet it possesses features inherent in more complicated attractors such as the Lorenz attractor, about which we do not know nearly as much.

The Henon map is given by

$$H_{n,b}(x,y) = \left(1 - \mu x^2 + y, bx\right)$$

where $\mu$ and $b$ are adjustable parameters.

We can partition the application of the transformation $H$ into three steps.

(i) **Bend up**: The first step consists of a nonlinear bending in the $y$-coordinate

$$H_1(x,y) = (x, 1 - \mu x^2 + y)$$

For example, a horizontal line ($y = \text{constant}$) becomes a parabola with vertex at $(0, y + 1)$ and opening up at the bottom. In contrast, the remaining two steps are linear transformations.

(ii) **Contraction in $x$**: Next, a contraction in the $x$-direction is applied.

$$H_2(x,y) = (bx,y)$$

The contraction factor is given by the parameter $b$, which is 0.3 for the Henon attractor.

(iii) **Reflection**: Finally a reflection at the diagonal is applied

$$H_3(x,y) = (y,x)$$

Thus, we get

$$H(x,y) = H_3(H_2(H_1(x,y)))$$

An orbit of the system consists of a starting point $(x_0,y_0)$ and its iterated images

$$(x_{k+1},y_{k+1}) = \left(1 - \mu x_k^2 + y_k, bx_k\right), \quad k = 0,1,2,...$$

Similar to the Logistic equation, these dynamics depend dramatically on the choice...
of the constants $\mu$ and $b$ besides that of the starting point. For some parameters almost all orbits tend to a unique periodic cycle, while chaos seems to reign for other choices. Henon used the values $\mu = 1.4$ and $b = 0.3$

2.2 **Henon Map and the Feigenbaum Universality:**

The Henon map is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ depending on two real parameters $\mu$, $b$ and is given by

$$H_{\mu,b}(x, y) = \left(1 - \mu x^2 + y, bx\right).$$

If $b \neq 0$, $H_{\mu,b}$ is a diffeomorphism of $\mathbb{R}^2$ onto itself. Note that the Jacobian of $H_{\mu,b}$ is the constant $-b$, so $H_{\mu,b}$ is dissipative if $|b| < 1$, area-preserving if $b = \pm 1$, and the area-expanding if $|b| > 1$. Note also that the case $|b| > 1$ can be effectively reduced to the case $|b| < 1$, since $H_{b,\mu,b} = T^{-1}H_{-b,\mu}T$ where the mapping $T$ is given by $T(x, y) = (-y, -x)$.

The Henon map has two fixed points whose coordinates are given by

$$x = \frac{(b - 1) \pm \sqrt{(1 - b)^2 + 4\mu}}{2\mu}, \quad y = bx$$

From this, one finds that $H_{\mu,b}$ has no fixed point if $\mu < -\frac{1}{4}(1 - b)^2$. In this context, we also wish to point out that the stability theory is intimately connected with the **Jacobian matrix** of the map, and that the **trace** of the Jacobian matrix is the sum of its eigenvalues and the product of the eigenvalues equals the **Jacobian determinant**. For a particular value of $b$ in the closed interval $[-1, 1]$, the Henon map $H$ depends on the real parameter $\mu$, and so a fixed point $x_0$ (or a periodic point $x_0^p$) of this map depends on the parameter value $\mu$, i.e. $x_0 = x_0(\mu)$. Now, first consider the open interval $I_1 = \left(-\frac{1}{4}(1 - b)^2, \frac{3}{4}(1 - b)^2\right)$.

The fixed point $x_0$ remains stable for all values of $\mu$ lying in this interval and a stable periodic trajectory of period one appears around it. This means, the two eigenvalues of the Jacobian matrix

34
at \( x_0 \) remains less than one in modulus, and as a result all the neighbouring points (that is, points in the domain of attraction) are attracted towards \( x_0(\mu) \), \( \mu \) lying in \( I_1 \). Again, some negative values of \( b \) for which \( \mu \) lies in the region sandwiched between the boundary curves \( \mu = -b \pm (1 - b)\sqrt{-b} \) yield complex eigenvalues for the Jacobian \( J \). This region is exhibited in figure 2.3 by the striped lines between the curves \( \mu_0 \) and \( \mu_1 \). The significance of complex eigenvalues is that successive iterations of the map spiral into the stable fixed point, and that of real eigenvalues is that consecutive iterations approach the stable fixed point along the direction of the eigenvector corresponding to the higher eigenvalues in modulus. If we now begin to increase the value of \( \mu \), then it happens that one of the eigenvalues starts decreasing through \(-1\) and the other remains less than one in modulus, because their product is always equal to \((-b)\). When \( \mu \) equals \( \frac{3}{4}(1-b)^2 \), one of the eigenvalues becomes \(-1\) and then \( x_0 \) loses its stability, i.e. \( \mu_1 = \frac{3}{4}(1-b)^2 \) emerging as the first bifurcation value of \( \mu \). Again, if we keep on increasing the value \( \mu \), the point \( x_0 \) becomes unstable and there arises around it two points, say, \( x_{21}(\mu) \) and \( x_{22}(\mu) \) forming a stable periodic trajectory of period 2. All the neighbouring points except the stable manifold of \( x_0(\mu) \) are attracted towards these two points and this phenomenon continues for all \( \mu \) lying in the open interval \( I_2 = \left( \frac{3}{4}(1-b)^2, \frac{1}{4}(1+b)^2 + (1-b)^2 \right) \). Since the period, as emerged becomes double, the previous eigenvalue which was \(-1\) becomes \(+1\) and as we keep increasing \( \mu \), one of the eigenvalues starts decreasing from \(+1\) to \(-1\).
The values of $\mu$ in $I_2$ for which we obtain the inequality $[2(1-b)^2-2\mu+b]^2<0$ give complex eigenvalues for the Jacobian $J$. The shaded portion in fig. 2.3 between the curves $\mu_1$ and $\mu_2$ shows this region. Since the trace is always real, when eigenvalues are complex, they are conjugate to each other moving along the circle of radius $\sqrt{b_e}$, where $b_e = b^{2^n}$ is the effective Jacobian, in the opposite directions. When we reach $\mu = \left(\frac{1}{4}(1+b)^2+(1-b)^2\right)$, we find that one of the eigenvalues of the Jacobian of $H^2$ (because of the chain rule of differentiation, it does not matter at which periodic point
one evaluates the eigenvalues) becomes $-1$, indicating the loss of stability of the periodic trajectory of period two. Thus, the second bifurcation takes place at this value $\mu_2$ of $\mu$. We can then repeat the same arguments, and find that the periodic trajectory of period 2 becomes unstable and a periodic trajectory of period 4 appears in its neighbourhood. This phenomenon continues up to a particular value of $\mu$, say $\mu_3(b)$, at which the periodic trajectory of period 4 loses its stability in such a way that one of the eigenvalues at any of its periodic points become $-1$, and thus it gives the third bifurcation at $\mu_3(b)$.

Increasing the value $\mu$ further and further, and repeating the same arguments we obtain a sequence \{\mu_n(b)\} of bifurcation values for the parameter $\mu$ such that at $\mu=\mu_n(b)$ a periodic trajectory of period $2^n$ arises and all periodic trajectories of period $2^m (m<n)$ remain unstable. The sequence \{\mu_n(b)\} behaves in a universal manner such that $\mu_n(b) - \mu_{n-1}(b) \approx c(b)\delta^{-n}$, where $c(b)$ is independent of $n$ and $\delta$ is the Feigenbaum Universal constant. Since the Henon map has constant Jacobian $-b$, $|b|<1$ gives the dissipative case, that is, contraction of area and in this case $\delta$ equals $4.6692016091029\ldots$. For $|b|=1$ we have the conservative case, i.e. the preservation of area and in this case $\delta$ equals $8.721097200\ldots$. Furthermore, the Feigenbaum theory says that the Henon map $H$ at $\mu = \mu_n(b)$ has an invariant set $F$ of Cantor type encompassed by infinitely many unstable periodic orbits of period $2^n (n=0,1,2,\ldots)$, and that all the neighbouring points except those belonging to these unstable orbits and their stable manifolds are attracted to $F$ under the iterations of $H_{\mu_n,b}$.

2.3 Numerical Method For Obtaining Periodic Point:

To find a periodic point of the Henon map for the period $k$, we can apply the following two numerical methods.

(i) The Newton Recurrence formula

The Newton Recurrence formula is
\[
x_{n+1} = x_n - Df(x_n) \cdot f(x_n), \text{ where } n = 0, 1, 2, \ldots \text{ and } (Df)(x) \text{ is the Jacobian of the map } f \text{ at the vector } x.
\]

We see that this map \( f \) is equal to \( H^{k-1} \) in our case, where \( k \) is the appropriate period. The Newton formula actually gives the zero(es) of a map, and to apply this numerical tool in the Henon map one needs a number of recurrence formulae which are given below.

Let the initial point be \((x_0, y_0)\),

Then,

\[
H(x_0, y_0) = (1 - \mu x_0^2 + y_0, bx_0) = (x_1, y_1) \quad \text{(say)}
\]

\[
H^2(x_0, y_0) = H(x_1, y_1) = (x_2, y_2) \quad \text{(say)}
\]

where \( x_2 = 1 - \mu x_1^2 + y_1, \quad y_2 = bx_1 \).

Proceeding in this manner the following recurrence formula for the Henon map can be established.

\[
x_n = 1 - \mu x_{n-1}^2 + y_{n-1} \quad \text{and} \quad y_n = bx_{n-1} \quad \text{where } n = 1, 2, 3, \ldots
\]

Since the Jacobian of \( H^k \) (k times iteration of the Henon map) is the product of the Jacobian of each iteration of the map, we proceed as follows to describe our recurrence mechanism for the Jacobian Matrix.

The Jacobian \( J_1 \) for the transformation

\[
H(x_0, y_0) = (1 - \mu x_0^2 + y_0, bx_0)
\]

is

\[
J_1 = \begin{pmatrix} -2\mu x_0 & 1 \\ b & 0 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{(say)} \quad \text{where} \quad A_1 = -2\mu x_0, \quad B_1 = 1, \quad C_1 = b, \quad D_1 = 0.
\]

Next the Jacobian \( J_2 \) for the transformation \( H^2(x_0, y_0) = (x_2, y_2) \) where \( x_2 \) and \( y_2 \) are as mentioned above, is the product of the Jacobians for the transformations

\[
H(x_1, y_1) = (1 - \mu x_1^2 + y_1, bx_1) \quad \text{and} \quad H(x_0, y_0) = (1 - \mu x_0^2 + y_0, bx_0).
\]

So we obtain

\[
J_2 = \begin{pmatrix} -2\mu x_1 & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} -2\mu x_1 A_1 + C_1 & -2\mu x_1 B_1 + D_1 \\ bA_1 & bB_1 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \quad \text{(say)}
\]

38
where $A_2 = -2\mu x A_1 + C_1$, $B_2 = -2\mu x B_1 + D_1$, $C_2 = bA_1$, $D_2 = bB_1$.

Continuing this process in this way we have the Jacobian for $H^m$ as

$$J_m = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}$$

with a set of recursive formula as

$$A_m = -2\mu x_{m-1} A_{m-1} + C_{m-1}, \quad B_m = -2\mu x_{m-1} B_{m-1} + D_{m-1}, \quad C_m = bA_{m-1}, \quad D_m = bD_{m-1} \quad (m = 2, 3, 4, 5...).$$

Since the fixed point of this map $H$ is a zero of the map $H'(x,y) = H(x,y) - (x,y)$, the Jacobian of $H'(k)$ is given by

$$J_k-I = \begin{pmatrix} A_k - 1 & B_k \\ C_k & D_k - 1 \end{pmatrix}$$

Its inverse is

$$(J_k-I)^{-1} = \frac{1}{\Delta} \begin{pmatrix} D_k - 1 & -B_k \\ -C_k & A_k - 1 \end{pmatrix}$$

Where $\Delta = (A_k - 1)(D_k - 1) - B_k C_k$, the Jacobian determinant. Therefore, Newton's method gives the following recurrence formula in order to yield a periodic point of $H^k$

$$x_{n+1} = x_n - \frac{(D_k - 1)(\bar{x}_n - x_n) - B_k(\bar{y}_n - y_n)}{\Delta}$$

$$y_{n+1} = y_n - \frac{(-C_k)(\bar{x}_n - x_n) - (A_k - 1)(\bar{y}_n - y_n)}{\Delta}$$

where $H^k(x_0) = (\bar{x}_n, \bar{y}_n)$

For practical purposes this method is very useful and requires generally almost 20 iterations.

(ii) We now wish to describe some averaging iteration methods on the Henon map in order to obtain a periodic point of period $k$. Suppose an initial value $x_0 = (x_0, y_0)$ is given. By this method, after first iteration we take the average value $x_1 = \frac{1}{2}(x_0 + H^k x_0)$

39
or \( x_{n+1} = \frac{1}{2} (H^k x_n + H^2 x_n) \) as the initial value for the second iteration, instead of \( x_n \). We repeat this process by applying the average formula \( x_{n+1} = \frac{1}{2} (x_n + H^k x_n) \) or \( x_{n+1} = \frac{1}{2} (H^k x_n + H^2 x_n) \) and find that this process gives the fast convergence of the value of \( x \) owing to the following reasons:

Suppose our averaging formula is \( x_{n+1} = \frac{1}{2} (x_n + H^k x_n) \). Assume that \( x \) is a fixed point of \( H^k \) and that \( x_n = x + e_n \) for some vector \( e_n \). Let \( u \) and \( v \) be the basis eigenvectors for the Jacobian operator \( J_k \) at \( x \) with the corresponding eigenvalues \( \sigma \) and \( \rho \) such that for some scalars \( \alpha_n \) and \( \beta_n \), we have \( e_n = \alpha_n u + \beta_n v \). Then

\[
e_{n+1} = \frac{1}{2} (e_n + J_k e_n) + o(\| e_n \|^2)
\]

\[
= \frac{1}{2} (\alpha_n u + \beta_n v) + \frac{1}{2} (\alpha_n \sigma u + \beta_n \rho v) + o(\| e_n \|^2)
\]

\[
= \left(1 + \frac{\sigma}{2}\right) \alpha_n u + \left(1 + \frac{\rho}{2}\right) \beta_n v + o(\| e_n \|^2)
\]

If \( e_{n+1} = \alpha_{n+1} u + \beta_{n+1} v \), then we have

\[
\alpha_{n+1} = \left(1 + \frac{\sigma}{2}\right) \alpha_n + \text{some small error term}
\]

\[
\beta_{n+1} = \left(1 + \frac{\rho}{2}\right) \beta_n + \text{some small error term}
\]

At a stable periodic point, the absolute values for \( \sigma \) and \( \rho \) are less than 1 and their product is always equal to \((-b)^k\). It is noted that we are concerned with the period doubling bifurcations when one of the eigenvalues becomes \(-1\). As \( \mu \) approaches to a bifurcation value, one of the eigenvalues say \( \sigma \) approaches \(-1\) and \( \rho \) tends to \((-b)^k\).

Then, \( \frac{1 + \sigma}{2} \approx 0 \) and \( \frac{1 + \rho}{2} \approx \frac{1}{2} \) and so each iteration of the map reduces the error at least
by half of the previous error. Consequently, after a reasonably small number of iterations these two scalars tend to zero. This leads $e_{n+1}$ to zero approximately, and therefore, gives a fast convergence of $x_{n+1}$ to $x$.

2.4 Numerical Methods For Finding Bifurcation Values:

First of all, we recall our recurrence relations for the Jacobian Matrix of the map $H^k$ described in the Newton’s method and then the eigenvalue theory gives the relation $A_k + D_k = -1 - (-b)^k$ at the bifurcation value. Again the Feigenbaum theory says that

$$\mu_{n+2} = \mu_{n+1} + \frac{\mu_{n+1} - \mu_n}{\delta} \tag{2.1}$$

where $n = 1, 2, 3, \ldots$ and $\delta$ is the Feigenbaum Universal constant.

In the case of the Henon map, the first two bifurcation values $\mu_1$ and $\mu_2$ can be evaluated by their explicit formulae,

$viz.$ $\mu_1 = \frac{3}{4}(1 - b)^2$ and $\mu_2 = \frac{1}{4}(1 + b)^2 + (1 - b)^2$.

Furthermore, it is easy to find the periodic points for these $\mu_1$ and $\mu_2$ for any value of $b$. We note that if we put $I = A_k + D_k + 1 + (-b)^k$, then $I$ turns out to be a function of the parameter $\mu$. The bifurcation value of $\mu$ of the period $k$ occurs when $I(\mu)$ equals zero. This means, in order to find a bifurcation value of period $k$, one needs the zero of the function $I(\mu)$, which is given by the Secant method as

$$\mu_{n+1} = \mu_n - \frac{I(\mu_n)(\mu_n - \mu_n)}{I(\mu_n) - I(\mu_{n-1})}$$

Then using the relation (2.1), an approximate value $\mu'_3$ of $\mu_3$ is obtained. Since the Secant method needs two initial values, we use $\mu'_3$ and a slightly larger value, say, $\mu'_3 + 10^{-4}$ as the two initial values to apply this method and ultimately obtain $\mu_3$. In like manner, the same procedure is employed to obtain the successive bifurcation values $\mu_4, \mu_5, \ldots$ etc. to our requirement. For $b = -0.7$, we enlist in table 2.1 the bifurcation
values, the corresponding fixed points, the value of $\delta$ for the periods, $k = 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8 \ldots 2^{12}$. The corresponding computer program is given in the appendix.

Table 2.1

$(b = -0.7)$

<table>
<thead>
<tr>
<th>k</th>
<th>$b_d$</th>
<th>$b_{d_{10}}$</th>
<th>$b_{d_{10}10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>3.016127640524</td>
<td>0.571286672010</td>
<td>0.056446636624</td>
</tr>
<tr>
<td>8.0</td>
<td>3.025749850389</td>
<td>0.576344855997</td>
<td>0.054528780926</td>
</tr>
<tr>
<td>16.0</td>
<td>3.030368966077</td>
<td>0.574882852109</td>
<td>0.056352126455</td>
</tr>
<tr>
<td>32.0</td>
<td>3.031612052208</td>
<td>0.575330636100</td>
<td>0.056147344497</td>
</tr>
<tr>
<td>64.0</td>
<td>3.031804508577</td>
<td>0.575057111971</td>
<td>0.056393646128</td>
</tr>
</tbody>
</table>

$k = 2^{12}$
3.031835000491 0.575174103135 0.056302640902
3.031835001491 0.575174105116 0.056302693420
3.031834989243 0.575174080851 0.056302657570
4.66607485830 3.031843207382

k=256.0

3.031841450898 0.575126437373 0.056342116383
3.031841450998 0.575126436991 0.056342116705
3.031841448117 0.575126447988 0.056342107426
4.668616592658 3.031843208411

k=512.0

3.031842831584 0.575108702383 0.0563566667159
3.031842831594 0.575108702321 0.0563566667211
3.031842831450 0.575108703216 0.0563566666467
4.669068644347 3.031843208462

k=1024.0

3.031843127726 0.575111161929 0.056354710844
3.031843127727 0.575111161932 0.056354710842
3.031843127719 0.575111161912 0.056354710857
4.669173765120 3.031843208464

k=2048.0

3.031843191171 0.575112252843 0.056353834562
3.031843191171 0.575112252844 0.056353834561
4.669197491804 3.031843208464

k=4096.0

3.031843204760 0.575112492847 0.056353641728
3.031843204760 0.575112492847 0.056353641728
4.669195373917 3.031843208464

2.5 Conclusion:

By adopting double precision, we calculate the numerical values and so all sorts of errors are negligible. The numerical methods stated in this chapter give the fast convergence for obtaining periodic points, bifurcation values and the Feigenbaum constant. These methods, we hope, are applicable to study the similar properties of any dimensional nonlinear maps.