CHAPTER V

Period Doubling
Bifurcation on Duffing's
Nonlinear differential
Equation
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5.1 Introduction:
In this chapter, we consider the Duffing equation having $2\pi$-periodic forcing term; and develop some useful numerical algorithm for the determination of Feigenbaum's fascinating sequence of period doubling bifurcation values of periods $2^0, 2^1, 2^2, 2^3, \ldots$ in the above equation.

The forced Duffing equation
\[
\ddot{x} + bx + ax + x^3 = F \cos \omega t, \quad F \geq 0
\] (5.1)
has been studied by a large number of authors since its introduction by Duffing [26]. Recently it has been used by Holmes [50], Guckenheimer & Holmes [39], and Ueda [98] as an example of a differential equation which has a chaotic response for a certain range of parameter values. Ueda, for example, studied the response with $a=0$ and $\omega=1$ for all values of $b$ and $F$ in the range $0 \leq b \leq 0.8$ and $0 \leq F \leq 25$. Byatt-Smith [96] also gives numerical results for $2\pi$-periodic solutions for a wide range of $F$ when $\omega=1$ and develops an asymptotic solution for large $F$. For small values of $F$, the relation between the numerical values of $b$, $a$, $F$, and $\omega$ are important. However, Byatt-Smith shows that the linear response term $ax$ does not appear in the first approximation and so the asymptotic result is valid for all finite values of $a$. If, when $a=0$, we make the transformation $(t, x) \rightarrow (t/\omega, \omega x)$ and write $b=wb^*$ and $F=w^3\gamma$, then equation (5.1) has the form
\[
\ddot{x} + b^* \dot{x} + x^3 = \gamma \cos t.
\] (5.2)
Now, for fixed values of $\gamma$ and $b$, it is clear that the effect of increasing $\omega$ is to decrease the effective damping.

Discussion of this Chapter is based on our paper entitled "Period-Doubling bifurcations of the Duffing equation" which was published in the Proceedings of the First national Conference on nonlinear systems and dynamics at I I T, Kharagpur.
Now, we want to study the structure of the solution of (5.2) for large \( \gamma \). This study is simplified by the introduction of the Poincare map \( P \). This is the transformation of the \((x, \dot{x})\) phase plane into itself according to

\[
P : (x_0, \dot{x}_0) \rightarrow (x(t_0 + 2\pi), \dot{x}(t_0 + 2\pi))
\]

(5.3)

where \( t_0 \) is given, and where \( x(*) \) is the solution of (5.2) determined by

\[
x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0
\]

Note that \( P \) is itself independent of time. Here we regard \( b^* \) as fixed, so that \( P \) depends only on \( \gamma \) and \( t_0 \), and we write \( P = P_{\gamma}(t_0) \). For large value of \( \gamma \) and for values of \( t_0 \) not close to \( \left( n + \frac{1}{2} \right) \pi \), for integral \( n \), we may also introduce the 'half' Poincare map denoted by \( P_{\gamma}(t_0) \). This is the transformation of the half plane into itself according to

\[
P_{\gamma}(t_0) : (x_0, \dot{x}_0) \rightarrow (-x(t_0 + \pi), -\dot{x}(t_0 + \pi))
\]

(5.4)

where \( x(t) \) is defined as for \( P_{\gamma}(t_0) \).

This reflects the symmetry of (5.2), since the transformation

\[
x \rightarrow -x, \quad \dot{x} \rightarrow -\dot{x}, \quad t \rightarrow t + \pi
\]

leaves the equation unchanged. This half map may, of course, be defined for any value of \( t_0 \), by identifying a point in the negative half plane with a suitable point in the positive half plane; clearly \( P_{\gamma}(t_0) = P_{\gamma}(t_0) \circ P_{\gamma}(t_0) = P_{2\gamma}(t_0) \).

Then we define symmetric \( 2\pi \)-periodic solution of (5.2) to be those which correspond to the fixed points of \( P_{\gamma} \) and asymmetric \( 2\pi \)-periodic solution of (5.2) to be those which correspond to the fixed points of \( P_{\gamma} \) which are not fixed points of \( P_{\gamma} \). A similar definition can be extended to cover the periodic points of \( P_{\gamma} \) and \( P_{\gamma} \). Then a pitchfork bifurcation of the \( P_{\gamma} \) map that produces two stable asymmetric solutions and one unstable symmetric solution from one stable symmetric solution corresponds to a period-doubling bifurcation of \( P_{\gamma} \). The importance of the half map is that, in general, pitchfork bifurcations are not necessarily structurally stable to perturbations of the map,
whereas period-doubling bifurcations are (39).

5.2 The asymptotic solution for large $\gamma$

Here, we study the form of the solutions of the equation

$$\ddot{x} + b\dot{x} + x^3 = \varepsilon^{-3}\cos t, \quad b > 0, \quad \varepsilon << 1,$$

(5.5)

where, for convenience, we have written $\gamma = \varepsilon^{-3}$

The main idea is to look for a solution of the form

$$x = \varepsilon^{-1}X(t, \tau)$$

(5.6)

where

$$\tau = \frac{f(t)}{\varepsilon}$$

(5.7)

For arbitrary initial conditions, the effect of damping is to reduce the magnitude of the solution, with increasing $t$, until there is an approximate match between the cubic restoring term and forcing term. After this time, and when $\cos t > 0$, the solution can be written as

$$\varepsilon x = \cos^3 t + \frac{1}{2} \varepsilon^6 A \frac{1}{t - \phi} \cos(\tau + \phi) + O(\varepsilon, \varepsilon^2 q)$$

(5.8)

with

$$\varepsilon \tau = \sqrt{3} \int_0^1 \cos^3 \xi d\xi - \frac{7 \sqrt{3}}{24} A^2 \varepsilon^2 q \int_0^1 e^{-b \xi} \cos^2 \frac{3}{2} \xi d\xi + O(\varepsilon^3 q)$$

(5.9)

When $\cos t < 0$, a similar form can be obtained by the use of the transformation $(t, x) \rightarrow (t + \pi, -x)$, which leaves (5.5) unaltered. The constants $A$ and $\phi$ are arbitrary and depend on the initial conditions. The exponent $q$ is also constant and is as yet undetermined. This solution is valid in any time interval for which $|t - (n + \frac{1}{2})\pi|$ is not small, where $n$ is any integer. In those time intervals where $|t - (n + \frac{1}{2})\pi|$ is small, which
in fact are of order $\frac{1}{\varepsilon^4}$, the equation (5.9) ceases to be valid, firstly because derivatives of $\cos^3 t$ becomes large and secondly because the ratio of successive terms in the perturbation series become large. To overcome this, we need to look for a different form of solution. Thus, for example, near $t = \left(2n - \frac{1}{2}\right)\pi$, we put

$$x = e^{\frac{3}{4}y(T)}$$

(5.10)

where

$$T = \left(t - 2n\pi + \frac{1}{2}\pi\right)e^{-\frac{3}{4}}$$

(5.11)

The first order approximation $y_0$ to the transformed equation satisfies

$$y_0'' + y_0^3 = T$$

(5.12)

The asymptotic form of this equation, as $T \to \infty$, is found to be

$$y_0 \approx T^\frac{3}{2} + \frac{A_+}{T^3}\cos\left(\frac{3\sqrt{3}}{4}T^\frac{3}{4} - \frac{7\sqrt{3}}{8}A_+^2T^\frac{1}{2} + \phi_+\right) + O\left(T^{-\frac{3}{2}}\right) \text{ as } T \to +\infty,$$

(5.13)

with a similar form

$$y_0 \approx -(-T)^\frac{3}{2} - \frac{A_-}{T^3}\cos\left(\frac{3\sqrt{3}}{4}(-T)^\frac{3}{4} - \frac{7\sqrt{3}}{8}A_-^2(-T)^\frac{1}{2} + \phi_-\right) + O\left(T^{-\frac{3}{2}}\right) \text{ as } T \to -\infty,$$

(5.14)

Similarly, in time intervals near $t = \left(2n + \frac{1}{2}\right)\pi$, the change of variable

$$x = -e^{\frac{3}{4}y(T)}$$

$$T = \left(t - 2n\pi - \frac{1}{2}\pi\right)e^{-\frac{3}{4}}$$

(5.14(a))

again gives (5.12) at the lowest approximation.

The requirement that the two solutions (5.8) and (5.13) match near (say) $t = \left(-\frac{1}{2}\right)\pi$
as \( T \to +\infty \) may be seen to require that \( q = \frac{3}{8} \) and, if \( \Lambda_+ \) and \( \phi_+ \) are given, that \( \Lambda \) and \( \phi \) satisfy

\[
\begin{align*}
\Lambda &= \Lambda_+ e^{1_{\text{b}}}, \\
\phi &= \phi_+ + \frac{\sqrt{3} \pi}{2} \int_0^1 \cos^3 \xi d\xi - \frac{7\sqrt{3}}{24\varepsilon^4} \int_0^1 e^{b\xi} \cos^2 \frac{1}{3} \xi d\xi
\end{align*}
\]

Equation (5.15) show how to relate \((\Lambda_+, \phi_+)\) of the inner solution of (5.13) to the outer solution (5.8) for \( t = -\frac{1}{2} \pi + \), corresponding to \( T \to +\infty \); similarly equation (5.16) give the corresponding relation for \( t = \frac{1}{2} \pi - \) \((T \to -\infty)\). To complete the description and obtain the next pair of values \((\Lambda_+, \phi_+)\) in terms of \((\Lambda_-, \phi_-)\) by integration of (5.12).
5.3 Properties of the connection problem and the Poincaré map

An important overall property of the connection problem is obtained by the symmetry that (5.12) inherits from (5.5), that is, that the transformation \((y_0, T) \rightarrow (-y_0, -T)\) leaves the equation unaltered. Thus, if the connection problem is regarded as a transformation of the \((A, \phi)\) plane into itself, then this transformation is its own inverse.

Equation (5.12) was integrated numerically for initial values \(A_\pm = 0(0.1)1.3\) and \(\phi_\pm = 0(0.04\pi)2\pi\). The line \(A_\pm = 0\) is mapped into the point \((0.9517 \ldots, 0.5959\ldots)\) for all values of \(\phi_\pm\). This point, denoted by \((A_s, \phi_s)\), represents a singular point of the transformation. For values of \(A_\pm\) less than \(A_s\), the images of the lines \(A_\pm = \text{constant}\) are closed loops surrounding the singular point. For values \(A_\pm\) exceeding \(A_s\), the corresponding images are not closed unless the line \(\phi = 2\pi\) is identified with \(\phi = 0\). The image of the line \(A_\pm = A_s\) (with \(\phi = \phi_s\)) divides the two sets of curves. The images of the lines \(\phi_\pm = \text{constant}\) all pass through the singular point, the images of \(\phi_\pm = \pi\) and \(\phi_\pm = \pi + \pi\) forming smooth curves through the singular point. This is to be expected, since the point \((-A_\pm, \phi_\pm)\) can be identified with \((A_s, \phi_s + \pi)\). This is shown in Fig. 1 plotted in the \((A, \phi)\) plane, and in Fig. 2 with \((A, \phi)\) plotted as polar coordinates. In the latter figure, the dividing curve is the one which passes through the origin and, in this plane, all curves corresponding to \(A_\pm = \text{constant}\) are automatically closed.

**Fig. 5.1.** Level Curves

\[
A_2(A_1, \phi_1) = \text{const tan } t
\]

and

\[
\phi_2(A_1, \phi_1) = \text{const tan } t,
\]

of the transition map. Also plotted are the locus of fixed points of the Poincaré map \(p^*\), for \(b = 0.25\) and \(b = 0\).
For larger values of \( A_\pm \), the computation becomes more difficult due to the large oscillations of the solution, and it is interesting to speculate on the form of the solution as \( A_\pm \to \infty \). Because of the symmetry of the equation, there is clearly an antisymmetric solution of (5.12) if \( y_0(0)=0 \), giving \( A_\pm = \Lambda \). The numerical results of Byatt-Smith indicate that \( \max (A_\pm /A_\pm) \) and \( \min (A_\pm /A_\pm) \) tend to one as \( A_\pm \to \infty \), indicating that the images of \( A_\pm =\text{constant} \) tend to circles (in Fig. 2) as \( A_\pm \to \infty \). Whether this or the stronger conjecture

\[
\frac{\max \left( \frac{A_+}{A_-} \right)}{\min \left( \frac{A_+}{A_-} \right)} = O\left( A_\pm^{-1} \right)
\]  

for some value \( r \), can be proved analytically is at the moment unknown.

The Poincaré maps \( P_y(t_0) \) and \( p_y(t_0) \) induce conjugate maps \( P^*_y \) and \( p^*_y \) on the \((A, \phi)\) plane. In particular, we look at the specific maps

\[
\begin{align*}
 p^*_y : (A_\pm, \phi_\pm) & \rightarrow (A_\pm, \phi_\pm) + n \pi, \\
p^*_y : (A_\pm, \phi_\pm) & \rightarrow (A_\pm, \phi_\pm) + 2n \pi = p^*_y^2
\end{align*}
\]

where \((A_\pm, \phi_\pm) + \frac{1}{2} \pi\) is the value of the constant \( A_i \) and \( \phi_i \) associated with the connection problem associated with \( t = \frac{1}{2} \pi \). From an initial value of \((A_\pm, \phi_\pm)\), one iteration of \( p^*_y \) is
obtained by determining \((A, \phi)\) via (5.15)-(5.16), and then using the solutions of the connection problem to determine the next value of \((A, \phi_i)\). Periodic solutions of (5.5) can then be determined by first calculating the fixed points of \(p^n\) with \(n\) integral. The odd values of \(n\) correspond to symmetric solutions, and the even ones to asymmetric solutions.

An alternative method of calculating the fixed points of \(p^*\) is given by Byatt-Smith [96]: \(b\) is fixed throughout. A value of \(A\) is chosen and then \(\phi\) is found so that the \(A+ (A, \phi)\) arising from the solution of the connection problem also satisfies (5.16a). For the calculated values \((A, \phi)\) to be a fixed point of \(p^*\), equation (5.16) must still be satisfied. Thus, equation (5.16b) now serves to determine the value (or values) of \(\varepsilon\) which make this calculated point \((A, \phi)\) a fixed point. While, to each point constructed in this process, there is associated a set of values of \(\varepsilon\), corresponding to \(n=1,2,...\) the whole locus of such points is independent of \(\varepsilon\). We stress this here, because it is not true of the set of fixed points of \(P^*\). The loci for \(b=0\) and for \(b=0.25\) are also shown in Fig. 5.1, the locus for \(b=\alpha\) being the point \((A, \phi)\). As \(b\) increases, the locus contracts uniformly to the singular point. Also if one complete circuit of the locus is made (\(b\) non-zero) in the clock-wise direction, the value of \(n\) is found to increase by one, showing that there is a quasi-periodic structure of the Poincare map when regarded as a function of \(\varepsilon\).

The above method calculates the fixed points of the conjugate map, but does not determine their stability. If, under the map \(p^*\), the point \(x=(A, \phi)\) is mapped into the point \(f(x)\), then the fixed points of the map are given by the solution

\[ x = f(x) \]  

Their stability can be determined from the eigenvalues \(\lambda\) of the 2x2 matrix \(Df\) given by

\[
(Df)_{i,j} = \frac{\partial f_i}{\partial x_j}
\]

If \(|\lambda| < 1\), the fixed points are stable, and if \(|\lambda| > 1\), they are unstable. Saddle-node bifurcations occur when one eigenvalue passes through +1, and period-doubling
occurs when one eigenvalue decreases through -1. Pitchfork bifurcations can also occur
at $\lambda=1$, but there are none for $p^*_{\gamma}$.

The period-two points of $p^*_{\gamma}$ are the fixed points of $p^{*2}_{\gamma}=p^{*}_{\gamma}$ and are solutions of
\[ x = f(f(x)) = f^2(x), \]
and their stability can be calculated in a similar fashion.

The other main feature of the $p^*_{\gamma}$ map is the occurrence of period-three points.
The number of all these points as a function of $\varepsilon$ is shown in Fig. 5.3.

\[ 0.105 \quad 0.103 \quad 0.101 \quad 0.099 \quad 0.097 \quad 0.095 \quad 0.093 \quad 0.091 \quad 0.089 \quad 0.087 \]

Fig.5.3 Bifurcation Diagram showing existence of fixed points, 2-cycles & 3-cycles

The interpretation of this diagram is facilitated by the observation that a fixed or
periodic point can only change stability as $|\lambda|$ passes through 1. Since there are no
pitchfork bifurcations of $p^*_{\gamma}$, an unstable point with $|\lambda|>1$ can only restabilized at a saddle
node, and a stable point which becomes unstable as a result of a period-doubling bifurcation
can only restabilize as a result of the reverse of a period-doubling bifurcation. Thus,
following the curve ABA', which represents the fixed point, starting at the saddle node A, the branch AB represents an unstable fixed point which changes to stable at the saddle node at B. The branch BA' is stable apart from the section between the period-doubling bifurcations (represented in Fig. 5.3 by the pitchfork bifurcation of P*) where the period-two points appear and disappear. After the saddle node at A', this whole process repeats itself indefinitely as ε → 0.

We now consider the following Duffing equation having 2π-periodic forcing term;
\[ \dot{x} + b\dot{x} - x + x^3 = F \cos t, \quad F \geq 0 \] (5.21)
and show our numerical mechanism for the determination of Feigenbaum's fascinating sequence of period doubling bifurcation values of periods \(2^0, 2^1, 2^2, 2^3, \ldots\) in the above equation.

5.4 Introduction of a Poincaré Map for the solution of the equation:

Let us first introduce a Poincaré map \(P\) of the above Duffing equation. This is the transformation of the \((x, \dot{x})\) phase plane into itself according to
\[ P:(x_0, \dot{x}_0) \rightarrow (x(t_0 + 2\pi), \dot{x}(t_0 + 2\pi)) \] (5.22)
where \(t_0\) is given, and where \(x(t)\) is the solution of (5.21) determined by
\[ x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0. \]
By the 2π-periodicity of the above equation, the following hold:
\[ P^2(x(0), y(0)) = (x(4\pi), y(4\pi)) \]
\[ P^3(x(0), y(0)) = (x(6\pi), y(6\pi)) \]
\[ \ldots \]
\[ P^n(x(0), y(0)) = (x(2n\pi), y(2n\pi)) \]
where \(P^n\) is the \(n\)-times functional composition of the map \(P\). Keeping \(b\) fixed we see that \(P\) depends on \(F\) i.e. \(P = P(F)\). Again, by the Floquet theory (see [39]) this map has
the constant Jacobian $e^{-2n}$. 

5.5 The Jacobian Matrix for the transformation P

Let the solution $x(t)$ of (5.21) be perturbed to $x(t) + u(t)$ with $u(t)$ small. Putting this perturbed value in (5.21), we have

$$\dot{x} + \ddot{u} + b(\dot{x} + \dot{u}) - (x + u) + (x + u)^3 = F \cos t \quad (5.23)$$

This gives,

$$\ddot{u} + b\dot{u} - u + 3x^2 u = 0 \quad (5.24) \quad \text{(retaining only first order terms in } u).$$

Now, (5.24) is a linear differential equation which can be written as a system of first order equation by means of

$$\begin{cases} \dot{u} = v \\ \dot{v} = -bv + u - 3x^2 u \end{cases} \quad (5.25)$$

If $(u(0), v(0))$ is an initial value of $(u(t), v(t))$ and if we define a map $G$ by

$$G(u(0), v(0)) = (u(2\pi), v(2\pi)),$$

there is a $2 \times 2$ matrix

$$J = \begin{pmatrix} Q(1) & Q(3) \\ Q(2) & Q(4) \end{pmatrix} \quad \text{such that}$$

$$\begin{pmatrix} u(2\pi) \\ v(2\pi) \end{pmatrix} = G \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} Q(1) & Q(3) \\ Q(2) & Q(4) \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}.$$

Now, by the Floquet-theory, the matrix $J$ is the Jacobian of the transformation $P$ at the point $(x(0), y(0))$. Besides, the initial values $u(0) = 1$, $v(0) = 0$ result $u(2\pi) = Q(1)$, $v(2\pi) = Q(2)$, and the initial values $u(0) = 0$, $v(0) = 1$ give $u(2\pi) = Q(3)$, $v(2\pi) = Q(4)$. So, with these initial values, the solution of the system (5.25) yield the values of the Jacobian elements $Q(i), i = 1, 2, 3, 4.$

5.6 The Computational Scheme to Evaluate Bifurcation Values

We now wish to outline the computational scheme which gives the bifurcation
values for the map \( P \). Initially, we put \( b = 0.25 \) so that \( P \) is a function of \( F \) alone. We want to recall the Jacobian theory which says that the sum of the eigenvalues of the Jacobian of the map \( P \) is equal to its trace and the product of the eigenvalues is \( e^{-2nb} \). A bifurcation value occurs when one of the eigenvalues equals \(-1\) and so the trace \( Q(1) + Q(4) = -1 - e^{-2nb} \). If we put \( I = Q(1) + Q(4) + 1 + e^{-2nb} \) then \( I \) turns out to be a function of \( F \). In order to obtain the bifurcation values of \( F \), we need to find the zeroes of \( I \) and this can be achieved by the Secant method. Moreover, to give an initial value of \( F \), we use the relation

\[
F_{n+2} \approx F_{n+1} + \frac{F_{n+1} - F_n}{\delta},
\]

where \( \delta \) is the Feigenbaum universal constant.

**Step I:**

Initially, we put some arbitrary initial values of \( F, x \) and \( y \) keeping an eye that the Runge-Kutta 4th order method gives the convergence of \( x \) and \( y \). Then by the trial and error method, we keep increasing the values of \( F \) in such a way that the values of \( I \) approaches to zero. For each value of \( F \) the Runge-Kutta method yields a periodic point, and this periodic point is used as initial values of \( x \) and \( y \) for the next chosen higher value of \( F \). At the same time, to check whether the program (given in the appendix as program 1 of Duffing’s equation) runs properly we notice the value of the Jacobian determinant which should be approximately equals and which is given by \( Z(9) \) in the program. The process is continued and gives a rough estimate of the first bifurcation value as \( 2.65485 \), (evaluated upto five decimal places) with a periodic point \((0.99799968, 2.36515406)\). Since, our intention is to obtain this value upto 12 decimal places, we could have continued this method to do so, but the method is too time-consuming and tedious. So, estimating approximately the bifurcation value upto 5 decimal places by this method, we then apply the Secant method to attain upto 12 decimal places.
Step II :

Secondly, we use the computer program 2 (given in the appendix as program 2 of Duffing's equation). Here, P = 2.65485, PP = 2.654855 are taken as two initial values of P; and x₁ = 0.99799968, and y₁ = 2.36515406 are taken as initial values of x and y. In this program, N is the number of divisions of the period 2π and equals 250. So the step length is $H = \frac{2\pi}{N}$. This method yields the first-bifurcation value as 2.654850042763 with periodic point

$$A_1 = (0.998012083515, 2.365154031933)$$

Step III :

Next, the period 2π is increased to 4π, but the step length $H = \frac{4\pi}{N}$ is kept fixed by putting N=500. In order to apply the trial and error method for evaluating some approximate second bifurcation value, a slightly larger value of the first bifurcation value of P, and A₁ is taken as an initial value for (x,y). This process gives 2.76459 as an approximate second bifurcation value with a periodic point ($x_2 = 1.29150051; y_2 = 2.49939170$). Then, in program 2, necessary alterations of the values of N and H are made, two initial values 2.76459 and 2.764591 are put for P and ($x_2, y_2$) is used as the an initial value for (x,y). Ultimately this procedure yields 2.764589999694 as the second bifurcation value with a periodic point $A_2 = (1.291489423615, 2.499385725247)$

Step IV :

We next repeat the step III for the periods 8π, 16π, 32π and 64π with the necessary alterations of the values of N and H and initial values of P, PP, x and y. For all periods we keep this step length H fixed by choosing N rightly in order to have higher accuracy in values. The trial and error method computes the following approximate values.
Periods  P values        PP-values  x values       y values
8\pi    P = 2.79344      PP = 2.793441  1.41105997  2.51319511
16\pi   P = 2.79948      PP = 2.799481  1.39558224  2.51845406
32\pi   P = 2.80099      PP = 2.800991  1.40583830  2.51809560
64\pi   P = 2.801208     PP = 2.8012081 1.40271141  2.51859996

Having considered the above mentioned values as initial values with an appropriate period, the Secant method determines 3rd, 4th, 5th and 6th bifurcation values respectively, as given below:

<table>
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<tr>
<th>Periods</th>
<th>Bifurcation Values</th>
<th>Periodic Points</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td>x-value</td>
</tr>
<tr>
<td>8\pi</td>
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<td>1.410972884779</td>
</tr>
<tr>
<td>16\pi</td>
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<td>1.395604777631</td>
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<tr>
<td>32\pi</td>
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<td>64\pi</td>
<td>2.801208324243</td>
<td>1.402658349000</td>
</tr>
</tbody>
</table>

Remark:
Following the same computational mechanism, we can evaluate further higher bifurcation values. However, to keep N fixed, N should be made considerably larger with the higher periods. As a result the computer program takes a very long time to produce the required results.

5.7 Conclusion:
In general, it is indeed very tough job to find a suitable Poincare map of a given nonlinear differential equation. However, if we can obtain a suitable Poincare map, we can obtain different bifurcation values with the aforesaid mechanism.

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