PART-I: Models with Infinite Capacity

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CHAPTER-II

BATCH ARRIVAL POISSON QUEUE UNDER
SINGLE VACATION POLICY

2.1 Introduction:

Most of the queueing systems encountered in real life situations can be connected with vacation type model. The single server queueing systems with vacations, where the server is not available for some occasional random periods of time (referred to as vacation), arise as models of many computer, communication and production systems. The server may take exactly one vacation or sequence of vacations between two successive busy periods. Levy and Yechiaily (1975) first studied such type of vacation model under the assumption that the server takes only one vacation between two successive busy periods, known as single vacation policy. Recent progresses on these models have been surveyed by many researchers notable among them are Chae and Lee (1995), Dshalalow (1997) and Medhi (1997). The queueing models of this nature have been studied by Doshi (1986) and Takagi (1991). Their works dealt with single unit arrival systems. In fact, some aspects of batch arrival queues under single vacation policy have also been studied by Chae and Lee (1995) and Medhi (1997). Recently, Borthakur and Choudhury (1997b) and Choudhury (1998, 2002a) have studied the steady state behaviour of the batch arrival Markovian queues with generalized vacation with random set up time and with single vacation respectively.
Even though these authors have discussed some aspects of this type of model, yet some questions relevant to such models are need to be addressed. In this chapter, a single server queueing model with batch arrival and under single vacation policy is investigated under the following assumptions:

1. Customers arrive in batches of random size $'X'$ according to compound Poisson process with rate $'\lambda'$.
2. The service discipline is FCFS.
3. The service time distribution is negative exponential with mean $1/\mu$.
4. The service rule is assumed to operate as follows:
   As soon as the system becomes empty, the server goes on a vacation of random length (vacation period). After returning from that vacation to the system, if the server finds any unit waiting in the system then he starts service till the system becomes empty again (Busy period), i.e. he performs exhaustive service. However, if the server finds no unit waiting in the queue after returning from the vacation, he does not take further vacation, rather he waits till the arrival of the next unit (Dormant period).
5. The inter arrival time and service times are mutually independent of each other.
6. The vacation period $'V'$ is a random variable, which has the generic representation $\{V_1,V_2,\ldots\}$ and is assumed to follow a general law of distribution with probability distribution function $V(y)$.
The following results have been obtained under the present study of this chapter.

1. The queue size distribution at busy period epoch.
2. The queue size distribution due to idle period.
3. The queue size distribution at departure point of time.
4. Expected busy period and some other performance measures.
5. Distribution of additional delay.
6. Queue waiting time distribution.

2.2 Notations:

Let us define

\( X = \) Arrival batch size random variable.

\( a_k = \Pr[X = k]; k = 1, 2, 3, ... \)

\( a_j^{(k)} = \Pr[Y_j = j] \) is the k-fold convolution of \( \{a_j\} \) with itself and

\( \{a_j^{(0)}\} = 1. \)

\( Y_k = X_1 + X_2 + \cdots + X_k \)

\( V = \) vacation time random variable.

\( b_k = \Pr[\text{a batch of } k \text{ units arrive during vacation time } V] \).

\[
= \int_0^{\infty} \frac{x^k e^{-\lambda t} (\lambda t)^k}{k!} dv(t)
\]

\( V(t) = \) Probability distribution function of \( V. \)

Also we define,
\[ X(z) = \sum_{j=1}^{\infty} z^j a_j \quad \text{is the PGF of } \{a_j; j \geq 1\} \]

Notationally our model can be denoted as \( M^X/M/1/V_s \).

### 2.3 Queue size distribution at busy period initiation epoch:

Here the PGF of queue size distribution at busy period initiation epoch is obtained under steady state. Let us denote

\[ \alpha_n = \Pr [\text{an arbitrary customer finds a batch of 'n' customers in the system at busy period initiation epoch}]; \quad n \geq 1. \]

Then utilizing the argument of PASTA [Wolff (1982)] and conditioning the number of units that arrive during the first vacation, we can write the following state equation:

\[ \alpha_n = \sum_{k=1}^{n} b_k a_n^{(k)} + b_0 a_n \quad ; n \geq 1 \quad \text{(2.3.1)} \]

Multiplying equation (2.3.1) by appropriate powers of \( z \) and then taking summation over all possible values of 'n', we get

\[ \alpha(z) = \sum_{n=1}^{\infty} \alpha_n z^n \]

\[ = \sum_{n=1}^{\infty} \sum_{k=1}^{n} z^n b_k a_n^{(k)} + \sum_{n=1}^{\infty} z^n a_n b_0 \]

\[ = [V'(\lambda - \lambda X(z)) - V'(\lambda)] + V'(\lambda) X(z) \]

\[ = V'(\lambda - \lambda X(z)) + V'(\lambda) [X(z) - 1] \quad , \quad \text{(2.3.2)} \]

where \( V'(\theta) \) is the LST of vacation time distribution \( V(t) \).
Thus (2.3.2) gives the PGF of the number of customers that arrive during the busy period initiation epoch.

In particular, if the vacation time follows exponential distribution with probability distribution function \( V(t) = 1 - e^{-\nu t}; t \geq 0 \) then we have

\[
b_k = \int_0^{\infty} e^{-x} (\lambda t)^k \frac{1}{k!} e^{-\nu t} dt
\]

\[
= (1 - \phi) \phi^k ; k \geq 0 ,
\]

where \( \phi = \frac{\lambda}{\lambda + \nu} \quad (<1) \)

Then from equation (2.3.1) we have

\[
\alpha_n = (1 - \phi) \sum_{k=1}^{n} \phi^k a_n^{(k)} + (1 - \phi) a_n ; n \geq 1 \tag{2.3.3}
\]

Now, \( \{\alpha_n, n \geq 1\} \) can be computed from equation (2.3.3) for the various arrival size random distributions.

### 2.4 Additional queue size distribution due to idle period:

Utilizing the argument of length biasing in renewal theory, the PGF of the queue size distribution can be obtained from equation (2.3.1).

Let us first denote,

\[\alpha_n = \Pr[ \text{a batch of 'n' customers arrive before a tagged customer during the forward recurrence time (residual life) of the idle period in which the tagged customer arrives}]\]
Now, since the batch of arriving customers associated with the tagged customer is randomly chosen from the arriving batch that occurs at the completion epoch of the idle period, i.e., busy period initiation epoch, therefore we may write, \textbf{[Burke (1975)]}

$$\beta_n = \sum_{k=n+1}^{\infty} \frac{v_k}{k}; \quad n = 0, 1, 2, \ldots$$

where \{ $v_k ; \ k \geq 1$ \} is the probability that $k^{\text{th}}$ batch which starts a busy period to which the tagged arrival belongs chosen randomly. This follows directly from equation (2.3.1), by applying standard length biasing argument of renewal theory,

$$v_k = \frac{k\alpha_k}{\sum_{k=1}^{\infty} k\alpha_k}; \quad k = 1, 2, \ldots$$

Thus $\beta(z)$, the PGF of \{ $\beta_n; n \geq 0$ \} is given by

$$\beta(z) = \sum_{n=0}^{\infty} z^n \beta_n$$

$$= \frac{[1 - \alpha(z)]}{E(\alpha)(1 - z)} \quad (2.4.1)$$

Now, using equation (2.3.2) and $E(\alpha) = \left. \frac{d}{dz} \alpha(z) \right|_{z=1} = E(X)[V^*(\lambda) + \lambda E(V)]$

in (2.4.1), we get

$$\beta(z) = \frac{[1 - V^*(\lambda - \lambda X(z)) + V^*(\lambda)\{1 - X(z)\}]}{[E(X)V^*(\lambda) + \lambda E(X)E(V)](1 - z)} \quad (2.4.2)$$
which is the PGF of the number of units that arrive during the forward
recurrence time or residual life of the idle period. More specifically, we may call
it additional queue size distribution due to idle period. This result is in
agreement with that of Choudhury and Kalita (2002).

In particular, if we take \( \Pr[X=1] = 1 \), then \( X(z) = z \) and \( E(X) = 1 \).
Therefore (2.4.2) reduces to

\[
\beta(z) = \frac{[1 - V^*(\lambda - \lambda z) + (1 - z)V^*(\lambda)]}{[V^*(\lambda) + \lambda E(V)](1 - z)}
\]

which agrees with the result of Levy and Yechiali (1975) and Doshi (1986).

2.5 Queue size distribution at departure epoch:

Let us denote, \( \{P_j : j \geq 0\} \) be the steady state probability that there is a
batch of ‘\( j \)’ customers in the system at a departure epoch of a test customer. If
\( t_0, t_1, \ldots \) are the departure epochs of the units and \( N(t) \) is the number of units in
the system at instant \( t \), then

\[
P_j = \lim_{m \to \infty} \Pr[N(t_m) = j] ; j \geq 0
\]

Then utilizing the standard argument of embedded Markov chain, it is
easy to see that \( P_j (j \geq 0) \) satisfies the following steady state equations:

\[
P_j = P_0 \sum_{k=1}^{j+1} \alpha_k S_{j-k+1} + \sum_{k=1}^{j+1} P_k S_{j-k+1} ; j \geq 0
\]

(2.5.1)

where \( S_k = \Pr\{a batch of ‘k’ customers arrive during a service time\]

\[
= (1 - \theta) \sum_{j=0}^{k} \theta^j \alpha_k^{(j)} \text{ where } \theta = \frac{\lambda}{\lambda + \mu} \text{ so that }
\]

\[ S(z) = \sum_{j=0}^{\infty} z^j \cdot S_j = \frac{1-\theta}{1-\theta X(z)} = \frac{\mu}{\mu + \lambda (1 - X(z))} \quad (2.5.2) \]

Let \( P(z) = \sum_{j=0}^{\infty} z^j P_j \) be the PGF of \( \{P_j; j \geq 0\} \).

Now multiplying equation (2.5.1) by appropriate powers of \( z \) and then taking summation over all possible values of \( j \), we get

\[ P(z) = [P_0 \alpha(z) + \{P(z) - P_0\} S(z)] z^{-1} \quad (2.5.3) \]

Now utilizing (2.3.2) and (2.5.2) in (2.5.3), we get on simplification

\[ P(z) = \frac{P_0 \mu[1-V^*(\lambda - \lambda X(z)) + V^*(\lambda)(1-X(z))]}{[\mu(1-z) - \lambda z(1-X(z))]} \]

Using the normalizing condition i.e. \( \lim_{z \to 1} P(z) = 1 \), we get

\[ P_0 = \frac{(1-\rho)}{E(X)[V^*(\lambda) + \lambda E(V)]} \]

where \( \rho = \frac{\lambda E(X)}{\mu} \) (<1) is the utilization factor of the system. Thus we have

\[ P(z) = \frac{\mu(1-\rho)[1-V^*(\lambda - \lambda X(z)) + V^*(\lambda)(1-X(z))]}{E(X)[V^*(\lambda) + \lambda E(V)][\mu(1-z) - \lambda z(1-X(z))]} \quad (2.5.4) \]

Note that the stochastic decomposition property for this \( M^X/M/1/V_s \) queueing model can be demonstrated easily, by showing

\[ P(z) = \left[ \frac{[1-V^*(\lambda - \lambda X(z)) + V^*(\lambda)(1-X(z))]}{E(X)[V^*(\lambda) + \lambda E(V)](1-z)} \right] \frac{[\mu(1-\rho)(1-z)]}{[\mu(1-z) - \lambda z(1-X(z))]} \]

\[ = \beta(z)P(M^X/M/1;z), \]
where \( \beta(z) = \frac{[1 - V'(\lambda - \lambda X(z)) + V'(\lambda)(1 - X(z))] + V'(X) + \lambda E(V)(1 - z)}{[V'(\lambda) + \lambda E(V)](1 - z)} \) is the PGF of additional queue size distribution due to the idle period.

\[ P(M^X/M/1;z) = \text{The PGF of the number of customers in the queue at a stationary point of time in steady state corresponding to a standard } M^X/M/1 \text{ queue}\ [\text{Mehdi (1991), P-183}].

2.6 Expected busy period and other performance measures:

One way to find the expected busy period is to find the distribution of busy period first and then from it the expected value. However, its derivation by using the argument of the alternating renewal process is more elegant and simple. Hence we prefer this approach here to find the expected busy period.

Let us define,

\( T_B = \text{the length of a busy period i.e. the time interval during the server is busy.} \)

\( T_0 = \text{the length of idle period.} \)

Now, \( T_0 \) and \( T_B \) generate an alternating renewal process and therefore we may write

\[
\frac{E(T_B)}{E(T_0)} = \frac{\Pr[T_B]}{1 - \Pr[T_B]}
\]  \hspace{1cm} (2.6.1)

where \( \Pr[T_B] = \text{Probability that the server is busy} = \rho. \) \hspace{1cm} (2.6.2)

Further utilizing Little's formula in \( E(\alpha) \), we get
\[
\frac{E(\alpha)}{\lambda E(X)} = E(V) + \frac{\nu^*(-\lambda)}{\lambda}
\]

= This is the expected length of idle period.

= \( E(T_0) \) \hspace{1cm} (2.6.3)

Again since equilibrium condition for queues with vacation and without vacation are same [Doshi (1986)] therefore utilizing this fact and using (2.6.3) and (2.6.2) in (2.6.1), we get the expected busy period as

\[
E(T_b) = \frac{E(X)[\lambda E(V) + \nu^*(-\lambda)]}{\mu(1 - \rho)} \hspace{1cm} (2.6.4)
\]

This result is in agreement with that of Choudhury and Kalita (2002).

In particular, if we take \( \text{Pr}[X=1] = 1 \), then \( E(X) = 1 \) and \( \rho = \frac{\lambda}{\mu} = a \), therefore the expression (2.6.4) becomes

\[
E(T_b) = \frac{\lambda E(V) + \nu^*(-\lambda)}{\mu(1 - a)}
\]

which furnishes the result obtained earlier by Takagi (1991) for single unit arrival case.

Let \( E_0 \) be the event that the system is idle. Again it is well known to us that the system is idle if and only if the server is either on vacation or on dormant periods. Now, conditioning the number of arrivals during the vacation and dormant periods and utilizing the argument of renewal reward process, we may write
Pr \{\text{given } E_0, \text{ the server is idle due to vacation}\} = \frac{E(V)}{E(T_0)}

= \frac{\lambda E(V)}{\lambda E(V) + V^*(\lambda)}

= \pi \text{ (say)} \quad (2.6.5)

which is the proportion of time that the system state is on vacation during the idle period.

Similarly, we have

\Pr\{\text{given } E_0, \text{ the server is on dormant period}\} = \frac{\alpha_0 / \lambda}{E(T_0)}, \text{ where } \alpha_0 = V^*(\lambda)

= \frac{V^*(\lambda)}{\lambda E(V) + V^*(\lambda)}

= 1 - \pi

which is the proportion of time that the system state is zero during a dormant period.

Now, let \( L_Q \) be the mean queue size of this model. Then

\[ L_Q = \frac{dP(z)}{dz} \bigg|_{z=1} \]

\[ = \frac{\lambda^2 E(V^2)E(X)}{2[\lambda E(V) + V^*(\lambda)]} + E(X_R) + \frac{\lambda[E(X^2) + E(X)]}{2\mu(1 - \rho)}, \]

where \( E(X_R) = \frac{E[X(X-1)]}{2E(X)} \) is the mean residual batch size.
2.7 Further discussion on departure point queue size distribution:

Now, after some algebraic manipulation with equation (2.5.4), we can write

\[
P(z) = \left[ \frac{\lambda E(V)}{\lambda E(V) + V^*(\lambda)} \right] \frac{\mu(1 - \rho)[1 - V^*(\lambda - \lambda X(z))]}{\lambda E(X) E(V) \{\mu(1 - z) - \lambda z[1 - X(z)]\}}
\]

\[
+ \left[ \frac{V^*(\lambda)}{\lambda E(V) + V^*(\lambda)} \right] \frac{\mu(1 - \rho)[1 - X(z)]}{E(X) \{\mu(1 - z) - \lambda z[1 - X(z)]\}}
\]

\[
= \frac{(1 - \pi)\mu(1 - \rho)[1 - X(z)]}{E(X) \{\mu(1 - z) - \lambda z[1 - X(z)]\}}
\]

\[
+ \frac{\pi \mu(1 - \rho)[1 - V^*(\lambda - \lambda X(z))]}{\lambda E(X) E(V) \{\mu(1 - z) - \lambda z[1 - X(z)]\}}
\]

(2.7.1)

Now, if we take limit \( \pi \to 1 \) in equation (2.7.1), then we have

\[
\lim_{\pi \to 1} P(z) = \frac{\mu(1 - \rho)[1 - V^*(\lambda - \lambda X(z))]}{\lambda E(X) E(V) \{\mu(1 - z) - \lambda z[1 - X(z)]\}}
\]

which is the PGF of the departure point queue size distribution of the \( M^X/M/1 \) queueing model with generalized vacation time under multiple vacation policy [Borthakur and Choudhury (1997b)].

Similarly, if we take \( \pi \to 0 \) in equation (2.7.1), then we get

\[
\lim_{\pi \to 0} P(z) = \frac{\mu(1 - \rho)[1 - X(z)]}{E(X) \{\mu(1 - z) - \lambda z[1 - X(z)]\}}
\]

which is the PGF of the departure point queue size distribution of the standard \( M^X/M/1 \) queue and it agrees with the result obtained by Chaudhry and Templeton (1983).
Remark: From equation (2.7.1), we observe that the departure point queue size
distribution of this $M^X/M/1$ queue under single vacation policy is the convex
combination of the departure point queue size of the standard $M^X/M/1$ queue
and the $M^X/M/1$ queue under multiple vacation policy.

2.8 Distribution of additional delay:

We now present a simple derivation of the LST for the distribution of the
additional delay in service due to the idle period of a test customer. By test
group here we mean the arrival group in which the test customer lies. We now
define,

$$D(y) = \text{The probability distribution function of the additional}
delay.$$  

$$D^*(s) = \int_0^\infty e^{-sy} dD(y) \text{ is the LST of } D(y)$$

Now, following argument of Burke (1975), we may write

$$D^*(s) = \sum_{j=0}^\infty \beta_j \left( \frac{\mu}{\mu + s} \right)^j = \beta \left( \frac{\mu}{\mu + s} \right) \quad (2.8.1)$$

Again utilizing the equation (2.4.2) in equation (2.8.1), we get

$$D^*(s) = \frac{1 + \left( \frac{\mu}{s} \right) \left[ 1 - V^*(\lambda) \left( \frac{\mu}{\mu + s} \right) \right] + V^*(\lambda) \left[ 1 - X^*(\frac{\mu}{\mu + s}) \right]}{E(X)[\lambda E(V) + V^*(\lambda)]} \quad (2.8.2)$$
where \( X^*(\frac{\mu}{\mu + s}) = \sum_{j=1}^{\infty} a_j \left( \frac{\mu}{\mu + s} \right)^j \) is the LST of the test group or test batch satisfying the following functional equation

\[
\frac{s}{\lambda} = 1 - X^*\left( \frac{\mu}{\mu + s} \right)
\]  

(2.83)

Further, differentiating equation (2.8.2) w.r.t. \( s \) and then taking limit \( s \to 0 \), we get

\[
E(D) = \left. \frac{dD^*(s)}{ds} \right|_{s=0} = \frac{\lambda E(V^2)}{2[\lambda E(V) + V^*(\lambda)]} + \frac{E[X(X-1)]}{2\mu E(X)}
\]

(2.8.4)

where \( E(D) \) is the expected delay in service due to the idle period.

In particular, if we take \( \Pr[X=1] = 1 \), then \( E(X) = E(X^2) = 1 \) and therefore the equation (2.8.4) reduces to

\[
E(D) = \frac{\lambda E(V^2)}{2[\lambda E(V) + V^*(\lambda)]}
\]

This agrees with the result obtained by Takagi (1991) for single unit arrival case.

Now, making a simple algebraic rearrangement with the expression (2.8.2) and utilizing the interpretation of the expression (2.6.5), we may write
\[
D^*(s) = \frac{\lambda E(V)}{\lambda E(V) + V^*(\lambda)} \left[ 1 - V^* \left( \lambda - \lambda X^* \left( \frac{\mu}{\mu + s} \right) \right) \right] \left[ 1 - \frac{X^* \left( \frac{\mu}{\mu + s} \right)}{E(X)} \right] \\
+ \frac{V^*(\lambda)}{\lambda E(V) + V^*(\lambda)} \left[ 1 - \frac{X^* \left( \frac{\mu}{\mu + s} \right)}{E(X)} \right]
\]

\[= D^*_A(s) \{1 - \pi\} + \pi D^*_V(s) \]

(2.85)

where \(D^*_A(s) = \frac{1 - X^* \left( \frac{\mu}{\mu + s} \right)}{E(X) \left[ 1 - \frac{\mu}{\mu + s} \right]} \) is the LST of the distribution of the additional delay caused by the service time of the units of the batch who are served prior to the test unit. This is equivalent to the delay distribution of the test customer within the batch.

\[D^*_V(s) = \frac{1 - V^* \left( \lambda - \lambda X^* \left( \frac{\mu}{\mu + s} \right) \right)}{E(V) \left[ \lambda - \lambda X^* \left( \frac{\mu}{\mu + s} \right) \right]} \] is the LST of the distribution of those test customers who arrive during the residual vacation time.

Now, if we take limit \(\pi \rightarrow 1\) in equation (2.8.5), we get
\[
\lim_{s \to 1} D_0^*(s) = \frac{(\mu + s) \left[ 1 - V^* \left( \lambda - \lambda X^* \left( \frac{\mu}{\mu + s} \right) \right) \right]}{s \lambda E(X)E(V)}
\]

\[
= D_0^*(s) \text{ (say) } \tag{2.8.6}
\]

which is the LST of the distribution of the additional delay in service due to the vacation in an \(M^X/M/1\) queueing system under multiple vacation policy.

Further, if we take \(\Pr[X=1] = 1\), then 

\[
X^* \left( \frac{\mu}{\mu + s} \right) = \frac{\mu}{\mu + s}
\]

and therefore

from equation (2.8.3), we have

\[
\frac{s}{\lambda} = \frac{s}{\mu + s} \tag{2.8.7}
\]

Now utilizing equation (2.8.7) in (2.8.6), we get

\[
D_0^*(s) = \frac{1 - V^*(s)}{s E(V)}
\]

so that

\[
D_0(y) = \int_0^y \frac{1 - V(x)}{E(V)} \, dx,
\]

which is the distribution function of the forward recurrence time of the vacation period (time) in the multiple vacation model [Fuhrmann and Cooper (1985)].

Similarly utilizing equation (2.8.7) in (2.8.2), we get on simplification

\[
D^*(s) = \frac{V^*(\lambda) + \lambda \left( 1 - V^*(s) \right)}{[V^*(\lambda) + \lambda E(V)]},
\]

which is the LST of the forward recurrence time or residual life of vacation (idle) period in a single vacation model [Doshi (1986)].
2.9 Queue waiting time distribution:

In this section, the LST of the queue waiting time distribution of this $M^X/M/1$ queueing system under single vacation policy is derived. To obtain it, we utilize the stochastic decomposition result of Borthakur and Choudhury (1997b).

Let us define,

$W_q(y) = \text{Distribution function of the queue waiting time of a test unit}$

and $W_q^*(s) = \int_0^\infty e^{-sy}dW_q(y)$ is the LST of $W_q(y)$.

Now, utilizing the stochastic decomposition result of Borthakur and Choudhury (1997b), we may write

$$W_q^*(s) = \beta^*\left(\frac{\mu}{\mu + s}\right)W_q^*(M^x/M/1; s),$$

(2.9.1)

where $W_q^*(M^x/M/1; s) = \frac{(1 - \rho)}{1 - \left(\frac{\lambda}{s}\left(1 - X^*\left(\frac{\mu}{\mu + s}\right)\right)\right)}$ is the LST of the queue waiting time distribution of the test unit in a standard $M^x/M/1$ queue without vacation [Medhi (1991), p-186].

Now utilizing the expressions of (2.8.1) and (2.8.2) in the above relation (2.9.1), we get
Let $E(W_Q)$ be the mean waiting time of a test unit for this $M^X/M/1$ queueing system. Then

\[
E(W_Q) = \left. \frac{-dW_Q^*(s)}{ds} \right|_{s=0} = E(D) + \frac{E(X^2) + E(X)}{2 \mu (1 - \rho) E(X)} \tag{2.9.3}
\]

Suppose that, $E(D) = 0$, then the expression (2.9.3) simply reduces to

\[
E(W_Q) = \frac{[E(X^2) + E(X)]}{2 \mu (1 - \rho) E(X)}
\]

which is the mean waiting time of a test unit in a standard $M^X/M/1$ queueing system [Medhi (1991), P-188].

Also we note that similar type of decomposition results for mean waiting time had been established by Chae and Lee (1995) and Medhi (1997) through transformation-free method. Their derivation is mainly based on residual life analysis.
2.10 Particular Cases:

In this section, we discuss very briefly some particular cases of waiting time distribution for this queueing model.

Now, if we take Pr[X=1] = 1, then from equation (2.8.7) we have

\[ \frac{s}{\lambda} = \frac{s}{\mu + s} \]

and therefore the expression (2.8.9) becomes

\[ W^{\ast}(s) = \frac{(1-a)(s+\mu)\left[V^{\ast}(\lambda) + \lambda\left(1 - \frac{V^{\ast}(s)}{s}\right)\right]}{s(s-\lambda + \mu)\left[V^{\ast}(\lambda) + \lambda E(V)\right]} \]  

(2.10.1)

which is the LST of the queue waiting time distribution of M/M/1 queue under single vacation policy.

Similarly, from equation (2.9.1) we have

\[ E(W^{\ast}_Q) = \frac{\lambda E(V^2)}{2[\lambda E(V) + V^{\ast}(\lambda)]} + \frac{1}{\mu(1-a)} \]

which verifies the result obtained by Takagi (1991) for unit arrival case (in Markovian service time distribution).

Now, using interpretation of (2.6.5), we can write the expression (2.10.1) as follows

\[ W^{\ast}_Q(s) = \frac{(1-a)(s+\mu)}{s(s-\lambda + \mu)\left[\pi + (1-\pi)D_0^{\ast}(s)\right]} \]  

(2.10.2)

Now, taking Laplace inverse of the equation (2.10.2) and using the convolution property of it, we get on simplification...
\[ W_Q(y) = \pi\left[1 - ae^{-\mu(l-a)y}\right] + (1 - \pi)\int_{0}^{y} \left[1 - \sum_{n=1}^{\infty} \frac{(a\mu)^n(t-y)^{n-1}e^{-\mu(t-y)}}{(n-1)!}\right]\left[\frac{1-V(t)}{E(V)}\right]dt \]

(2.10.3)

Now, if we take limit \( \pi \to 1 \), then

\[ \lim_{\pi \to 1} W_Q(y) = 1 - ae^{-\mu(l-a)y} ; y \geq 0 \]

which is the expression for the distribution function of the waiting time of a test unit in a simple M/M/1 queueing model. Similarly, if we take limit \( \pi \to 0 \) in equation (2.10.3), then

\[ \lim_{\pi \to 0} W_Q(y) = \int_{0}^{y} \left[1 - \sum_{n=1}^{\infty} \frac{(a\mu)^n(t-y)^{n-1}e^{-\mu(t-y)}}{(n-1)!}\right]\left[\frac{1-V(t)}{E(V)}\right]dt \]

which is the expression for the distribution function of the waiting time of a test unit in an M/M/1 under multiple vacation policy.

**Note:** The results of this chapter except the section 2.3 and section 2.5 are published in a paper entitled "Analysis of a Batch Arrival Poisson Queue under Single Vacation Policy" in 'Calcutta Statistical Association Bulletin', vol. 53, No. 209-210, 2002 [Choudhury and Kalita (2002)].