CHAPTER II
2.1 INTRODUCTION:

Studies of the 'Banking Systems' with particular reference to the money reserve available with the system at any arbitrary time $t$ has to play an important role in fiscal policies of any nation. In particular in a developing country like India 'Banking System' is the backbone for nation's economy. Because of its impact on investment and hence on capital formation, for example, changes in the pattern of interest rates produces a large effect on the borrowing as well as on the investment patterns. Investment again has a useful role in capital formation with important concomitant effects changing the complexion of the economy. Thus, suitably manipulating interest rates it is possible to promote investments and also control borrowings. Thus arises the natural motivating interest to study the reserve level available with the 'Banking System' on a perspective basis. With this motivation Sarma in 1983 proposed a 'Stochastic Banking Model' and obtained analytic as well as explicit results in his Doctoral thesis. The model proposed by him is already explained in (1.2) of Chapter I. In order to obtain solutions he assumed that the inter-withdrawal times follow a negative exponential distribution. A typical realisation of the model considered by him is given in Fig. 3.
FIG. 2. THE STORAGE MODEL (Infinite capacity)
As already explained in Chapter I, the main object of this dissertation is to generalise the results obtained by Sarma in 1983, when inter-withdrawal time follows an Erlangian distribution. Such results will have the following two advantages, namely:

(i) generalising the results already available in literature and

(ii) obtaining solutions for those 'Banking Systems' where the customers arrive in phases.

Thus in this chapter analytic solution for the 'Banking System' when the inter-withdrawal times follow an Erlangian distribution is obtained in the following section.

2.2 ANALYTIC SOLUTION OF $M(x,y,t)$ FOR $E_m/M/1/FIFO/\infty$

STOCHASTIC BANKING MODEL:

First we proceed to obtain analytic solution for $M(x,y,t)$ defined in (1.2.2). For this an integral equation is formed by Sarma in 1983 and is given as follows:

$$M(x,y,t) = H(x-y) \delta (x-y-t) \int_t^\infty h(u) \, du$$

$$+ \int_0^t h(u) \, du \int_0^{y+u} g(v) M(x, y+u-v, t-u) \, dv. \quad (2.2.1)$$
Here it is assumed that the density function $h(u)$ governing the inter-withdrawal times follow an Erlangian distribution. Hence we have:

$$h(u) = \frac{-m\lambda u^{m-1} e^{-m\lambda u}}{(m-1)!} \quad \text{if } 0 < u < \infty$$

$$A > 0$$

$$m > 0$$

$$= 0 \quad \text{otherwise.} \quad (2.2.2)$$

Theorem (2.2.3)

The d.l.t. $M^*(x,s,p)$ of $M(x,y,t)$ is given by

$$M^*(x,s,p) = \frac{e^{-sx}}{(m\lambda + p-s)^m} \sum_{j=1}^{\infty} \frac{e^{-sx}}{(m\lambda + p-s)^{m-j+1}}$$

$$= \frac{m}{(m\lambda + p)^x} \sum_{j=1}^{\infty} \frac{e^{-sx}}{(m\lambda + p-s)^{m-j+1}}$$

$$= \mu(m\lambda + p)^x M^*(x, m\lambda + p, p) \sum_{j=1}^{\infty} \frac{1}{(m\lambda + p-s)^{m-j+1}}$$

$$= \frac{m}{(m\lambda + p)^x} \sum_{j=1}^{\infty} \frac{1}{(m\lambda + p-s)^{m-j+1}}$$

$$= \mu(m\lambda + p)^x M^*(x, m\lambda + p, p) \sum_{j=1}^{\infty} \frac{1}{(m\lambda + p-s)^{m-j+1}}$$

$$(2.2.4)$$
proof:

The d.l.t. \( M^*(x,s,p) \) of \( M(x,y,t) \) is given by

\[
M^*(x,s,p) = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} M(x,y,t) \, dt \tag{2.2.5}
\]

\( \text{Re}\, s, \text{Re}\, p > 0 \cdot \)

Substituting (2.2.1) in (2.2.5) we have

\[
M(x,s,p) = A + B, \tag{2.2.6}
\]

where

\[
A = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} H(x-y) \delta(x-y-t) \int_t^\infty h(u) \, du \, dt \, dy \tag{2.2.7}
\]

and

\[
B = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} \left[ \int_0^t h(u) \, du \int_0^{y+u} g(v) M(x, y+u-v, t-u) \, dv \right] \, dt \, dy. \tag{2.2.8}
\]

Now substituting (2.2.2) in (2.2.7) and after some calculations, we have

\[
A = (m\lambda)^m e^{-(m\lambda+p)x} \left[ \sum_{j=1}^m \frac{1}{(m\lambda)^j} \left\{ - \sum_{k=0}^{m-j} \frac{m-j-k}{x} \frac{1}{(m-j-k)! (m\lambda+p-s)} \right\} \right. \]

\[
+ \left. \frac{(m\lambda+p-s) x}{(m\lambda+p-s)^{m-j+1}} \right]. \tag{2.2.9}
\]
Now substituting (2.2.2) in (2.2.8) and substituting
\[ G(v) = \mu e^{-\mu v} \quad \text{if } 0 < v < \infty \]
\[ = 0 \quad \text{otherwise} \quad (2.2.10) \]

\text{after some calculations we have}
\[ B = \mu (m\lambda) \left[ \sum_{s=0}^{m-1} \frac{M(x, s, p)}{(s+\mu)(m\lambda+p-s)^m} \right. \]
\[ \left. - \sum_{k=0}^{n} \frac{M(x, m\lambda+p, p)}{(m\lambda+p-p)k+1} \right) \quad (2.2.11) \]

Now adding (2.2.9) and (2.2.11) and after some simplifications we obtain (2.2.4).

Hence the theorem.

Theorem (2.2.12)

The time dependent solution of $M(x,y,t)$ is well determined in terms of $M(x,s,p)$ using (2.2.4).

Proof:

First we observe that the R.H.S. of (2.2.4) involves an unknown constant namely $M(x, m\lambda+p, p)$, where $m$ is a
positive constant. \( M^*(x,s,p) \) is completely determined only after the evaluation of this unknown constant. We next notice that \( M^*(x,s,p) \) is analytic in 's' in the right half of the plane, \( \text{Re. } s>0 \). We then observe that the denominator of (2.2.4) namely 
\[
\left( s+\mu \right) \left( m \lambda +p-s \right)^m - \mu \left( m \lambda \right)^m
\]
has two zeros (in the 's' argument) of which one zero with positive real part. Thus \( M^*(x,s,p) \) has pole of order 'm' with positive real part. Since \( M^*(x,s,p) \) is analytic in 's' for \( \text{Re. } s>0 \), the numerator of (2.2.4) also must vanish at this pole. Thus by equating the numerator on the R.H.S. of (2.2.4) to zero the unknown constant \( M^*(x, m \lambda +p, p) \) can be determined, so that \( M^*(x, s, p) \) is solved. By taking Inverse Laplace Transform of \( M^*(x,s,p) \) with respect to the arguments 's' and 'p' successively \( M(x,y,t) \) is determined.

Hence the proof.

Now we proceed to obtain the analytic solution of \( M_1(x,y,t) \) for \( E_m / M / 1 \) / FIFO / ∞ model in the following section.
2.3 ANALYTIC SOLUTION OF $M_1(x,y,t)$ FOR $E_{m/M/1/FIFO/\infty}$ STOCHASTIC BANKING MODEL:

On the similar lines of the solution of $M(x,y,t)$ now we proceed to solve for $M_1(x,y,t)$ by considering the integral equation given in (1.2.9) in the following theorem.

Theorem (2.3.1)

The d.l.t. $M_1^*(x,s,p)$ of $M_1(x,y,t)$ is given by

$$M_1^*(x,s,p) = \frac{(s+\mu) (m\lambda + p - s)^m (m\lambda)^m}{\left( s+ \mu \right) (m\lambda + p - s)^m - \mu (m\lambda)^m} \left[ \frac{-sx}{e} \right] \left[ \frac{(m\lambda + p - s)^m}{(m\lambda + p - s)^m} \right]$$

$$- \sum_{j=1}^{m} \frac{e^{-(m\lambda + p)x}}{(m - j)!} \left( m \lambda + p - s \right)^j \left( x^{m-j} \right)$$

$$= \sum_{j=1}^{m} \frac{\mu M_1^*(x, m\lambda + p, p)}{(m\lambda + p - s)^j} \left\{ \frac{1}{\sum_{k=0}^{m-j} \frac{1}{(m\lambda + p + \mu)^{k+1}}} \right\} \left[ \frac{1}{(m\lambda + p + \mu)^{m-j}} \right]$$

... (2.3.2)

Proof:

The d.l.t. $M_1^*(x,s,p)$ of $M_1(x,y,t)$ is given by
\[ M_1^* (x,s,p) = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} M_1(x,y,t) \, dt. \quad (2.3.3) \]

Substituting (1.2.9) in (2.3.3) we have

\[ M_1^* (x,s,p) = A + B, \quad (2.3.4) \]

where

\[ A = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} H(x-y) \delta(x-y-t) h(t) \, dt \, dy \quad (2.3.5) \]

and

\[ B = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} \left[ \int_0^t \int_0^{y+u} \int_0^{x+y+u-v-t} M_1(x, y+u-v, t-u) \, dv \right] \, dt \, dy. \quad (2.3.6) \]

Now substituting \( h(t) = \frac{e^{-\lambda t}}{(m-1)!} \) if \( 0 < t < \infty \)

\[ \lambda > 0 \]

\[ m > 0 \]

\[ = 0 \] otherwise,

(2.3.7)

in (2.3.5) and after some calculations we have

\[ A = (\lambda \lambda^m) e^{-(m \lambda + p)x} \left[ -\sum_{j=1}^{m} \frac{x^{m-j}}{(m-j)! (m \lambda + p - s)^j} \right. \]

\[ + \frac{(m \lambda + p - s)x^m}{(m \lambda + p - s)^m}. \quad (2.3.8) \]
Now by substituting (2.2.2) and (2.2.10) for \( h(u) \) and \( g(v) \) in (2.3.6) and after some calculations we have:

\[
B = m \left( \frac{M^*_1(x, s, p)}{(s + \mu)(m \lambda + p - s)^m} \right)
- \sum_{j=1}^{m} \frac{1}{(m \lambda + p - s)^j} \sum_{k=0}^{m-j} \frac{M^*_1(x, m \lambda + p, p)}{(m \lambda + p + \mu)^{k+1}}.
\]

(2.3.9)

Now adding (2.3.8) and (2.3.9) and after some simplifications we obtain (2.3.2).

Hence the theorem.

Theorem (2.3.10)

The time dependent solution of \( M_1(x, y, t) \) is well determined in terms of \( M^*_1(x, s, p) \) using (2.3.2).

Proof:

The proof of this theorem runs on similar lines of the proof of the theorem (2.2.12).

Hence the proof.
It is interesting to note that by substituting \( m = 1, \ j = 1 \) and \( k = 0 \) in (2.2.4) and (2.3.2) and after simplification, we obtain the corresponding results of Sarma for \( M(x,y,t) \) and \( M_1(x,y,t) \) given in (1.2.12) and (1.2.13) respectively.

So far we have considered a 'Stochastic Banking Model' when the inter-withdrawal times follow an Erlangian distribution. The results obtained in the theorems (2.2.3) and (2.3.1) of \( M(x,y,t) \) and \( M_1(x,y,t) \) respectively will be useful when the customers arrive at counter in 'm' phases. Now we proceed to obtain the solutions for these functions when the amount of withdrawals follow an Erlangian distribution in the following chapter.