3. A Continuous Time Markov Chain Model

3.1 Introduction

As discussed in the previous chapter the grading pattern of the pre-degree level students is observed as a stochastic process because the grades or the marks obtained in the pre-degree examination is dependent on the grades or marks in the H.S.L.C. Examination. It is very important to study the grading pattern for the students of this level because a major decision about future career depends upon the results of pre-degree examination. On the basis of the marks scored in this examination, a student gets a chance to enter into different fields of study.

Till now we discussed the grading pattern as a discrete time Markov chain, when tests were held at a fixed interval of time. But when the students are judged on the basis of their marks scored in tests which are conducted in a fixed interval of time, our judgement is not always correct. Because, if the tests are announced earlier, the student gets enough time to prepare for the examinations and even though they score quite high in the examinations, we cannot be sure about some factors like creativity and capability of the students. That is why if the tests are taken suddenly depending on the circumstances viz., environmental conditions, availability of teachers, finishing of courses etc. it will be better to understand a student's understanding and creativity capability. Consequently we shall get a continuous time Markov chain.
In this chapter Section 2 provides the model description which is an extension of (2.2.1) to continuous time Markov chain. Section 3 deals with obtaining the probability of attaining a particular grade at time t under different initial conditions viz.,

(i) initially all the students were in grade 0
(ii) initially all the students were in grade 1
and, (iii) initially all the students were in grade 2

Section 4 deals with probabilistic analysis where different characteristics viz., expected number of visits to the state of excellence, probability of sure success in the interval \((0,t]\) etc. were obtained. In section 5, the probability of attaining a particular grade at time t is obtained with reduced state space and the probability of the student’s being successful throughout in \((0,t]\) is also obtained in the last section of this chapter.

3.2 Model

This is an extension of the model described in (2.3) to continuous time Markov model. The following assumptions are made on the model.

(i) A student enters in a institute with grade “i”, \(i = 0,1,2\).
(ii) Tests are held at random interval of time depending upon the circumstances namely availability of teachers, finishing of courses, environmental conditions etc.
(iii) At the end of each test records of grades of students were presented.
As in section (2.1) it is assumed that the grades of pre-degree level students follow a Markov chain. If $X_n = i, i = 0,1,2$ be the grades obtained by a student at the $n^{th}$ test, then $\{X_n, n \geq 0\}$ is assumed to follow a finite state continuous time Markov chain with state space $\mathcal{S} = \{0,1,2\}$ and the probability matrix.

$$
P = \begin{pmatrix}
1-a & a & 0 \\
\frac{c}{2} & 1-c & \frac{c}{2} \\
0 & b & 1-b
\end{pmatrix}
$$

and $P_r \{ X_n=j \mid X_{n-1}=i; t_n - t_{n-1} \leq t \} = p_{ij}$

characteristics viz., probability of attaining a particular grade at time $t$, under different initial conditions, expected number of visits to the state of excellence, i.e. the state 0 etc. are obtained in the subsequent sections.

3.3 Probability of attaining a particular grade at time $t$ under different initial status

In this section a theorem is established which states as follows:
**THEOREM : 3.3.1**

**Statement :** Let $P_i(t)$ be the probability that a student is in the state $i$ at time $t$. Depending upon initial conditions viz. $P_i(0) = 1$, $P_j(0) = 0$ for $i \neq j$, and $i, j = 0,1,2$, $P_i(t)$ may be represented in the matrix form as

\[
\begin{pmatrix}
0 & 1 & 2 \\
0 & \frac{1}{3}\left(1 + \frac{3}{2}e^{-3\lambda} + \frac{1}{2}e^{-3\lambda}\right) & \frac{1}{3}\left(1 - e^{-3\lambda}\right) & \frac{1}{3}\left(1 - \frac{3}{2}e^{-\lambda} + \frac{1}{2}e^{-3\lambda}\right) \\
1 & \frac{1}{3}\left(1 - e^{-3\lambda}\right) & \frac{1}{3}\left(1 + 2e^{-3\lambda}\right) & \frac{1}{3}\left(1 - 2e^{-\lambda} + \frac{1}{2}e^{-3\lambda}\right) \\
2 & \frac{1}{3}\left(1 - \frac{3}{2}e^{-\lambda} + \frac{1}{2}e^{-3\lambda}\right) & \frac{1}{3}\left(1 - e^{-3\lambda}\right) & \frac{1}{3}\left(1 + \frac{3}{2}e^{-\lambda} + \frac{1}{2}e^{-3\lambda}\right)
\end{pmatrix}
\]

\[\ldots\ldots\ (3.3.1)\]

where, $\lambda = \frac{abc}{k}$

and, $k = 2ab + ac + bc$

**Proof :** Let $X(t) = i$, $i=0,1,2$ be the grades obtained by a student in a particular test conducted at time $t$. Therefore, $\{X(t), t \geq 0\}$ is a stochastic process with state space $S = \{0,1,2\}$.

The possible transitions are,

\[i \rightarrow i+1, \ i = 0,1\]
\[i \rightarrow i-1, \ i = 1,2\]
\[i \rightarrow i, \ i = 0,1,2\]
The rate of transition from state \( i \) to state \( j \) is given by,

\[
\lambda_{ij} = \pi_i p_{ij}, \quad i, j = 0, 1, 2
\]

where, \( P = (p_{ij})_{3 \times 3} \) is the t.p.m. of the model under consideration and \( \pi_i \)'s are the limiting probabilities, \( i = 0, 1, 2 \).

With the help of the t.p.m. \( P \), \( \pi_i \)'s are calculated as follows

We have the t.p.m. of the model under consideration as,

\[
P = \begin{pmatrix}
1-a & a & 0 \\
c & 1-c & c \\
2 & 1-c & c \\
0 & b & 1-b
\end{pmatrix}
\]

The \( \pi_i \)'s are obtained by solving the following equations,

\[
\pi_0 = (1-a)\pi_0 + \frac{c}{2} \pi_1
\]

\[
\pi_1 = a\pi_0 + (1-c)\pi_1 + b\pi_2
\]

and \( \pi_0 + \pi_1 + \pi_2 = 1 \)

as,

\[
\pi_0 = \frac{bc}{k}
\]
\[ \pi_1 = \frac{2ab}{k} \]

\[ \pi_2 = \frac{ac}{k} \]

where, \( k = 2ab + ac + bc \)

with the help of the above limiting probabilities i.e. \( \pi_i \)'s (\( i = 0,1,2 \)), the rate matrix \( A \) can be calculated as,

\[ A = (a_{ij})_{3 \times 3} \]

where, \( a_{ij} = \frac{abc}{k} \), \( j = i+1, i-1 \)

\[ = 0, \text{ otherwise} \]

\( i, j = 0,1,2 \)

and, \( a_{ii} = \frac{abc}{k} \), \( i = 0,2 \)

\[ = -\frac{2abc}{k}, \text{ i.e. } i = 1 \]
Assuming $\frac{abc}{k} = \lambda$, the rate matrix can be rewritten as,

$$A = (a_{ij})_{3x3} \quad \text{... (3.3.3)}$$

Where, $a_{ij} = \lambda$, $j = i+1, i-1$ and $i = 0,1,2$

$$= 0, \text{ otherwise}$$

$$i, j = 0,1,2$$

and, $a_{ii} = -\lambda$, $i = 0,1$

$$= -2\lambda, i = 1$$

The following three cases were studied assuming $P_i(t)$ to be the probability that the system is in the state $i$ at time $t$, $i = 0,1,2$.

*Case I:*

Here the system is observed under the assumption that a student acquires grade $i$, in the final examination, $(i = 0,1,2)$, provided the student starts with grade “0” initially, i.e. the student enters an institution after passing the H.S.L.C. examination with grade “0”. So the initial status was,

$$P_0(0) = 1, P_1(0) = 0, P_2(0) = 0.$$
From the rate matrix (3.3.3), the differential difference equations can be written as,

\[ P_0'(t) = -\lambda P_0(t) + \lambda P_1(t) \quad \ldots \quad (3.3.4) \]

\[ P_1'(t) = \lambda P_0(t) - 2\lambda P_1(t) + \lambda P_2(t) \quad \ldots \quad (3.3.5) \]

\[ P_2'(t) = \lambda P_1(t) - \lambda P_2(t) \quad \ldots \quad (3.3.6) \]

Taking Laplace transforms on the both sides of the above three equations, we get,

\[ sP_0^*(s) - P_0(0) - \lambda P_0^*(s) + \lambda P_1^*(s) \quad \ldots \quad (3.3.4)' \]

\[ sP_1^*(s) = \lambda P_0^*(s) - 2\lambda P_1^*(s) + \lambda P_2^*(s) \quad \ldots \quad (3.3.5)' \]

\[ sP_2^*(s) = \lambda P_1^*(s) - \lambda P_2^*(s) \quad \ldots \quad (3.3.6)' \]

Solving (3.3.4)', (3.3.5)', (3.3.6)' we get,

\[ P_1^*(s) = \frac{\lambda}{s(s + 3\lambda)} \]

\[ = \frac{1}{3s} \left[ \frac{1}{s} - \frac{1}{s + 3\lambda} \right] \]
and, 

\[ P_2^*(s) = \frac{\lambda}{s + \lambda} P_1^*(s) \]

\[ = \frac{\lambda^2}{s(s + \lambda)(s + 3\lambda)} \]

\[ = \frac{1}{3} \left[ \frac{1}{s} - \frac{3}{2(s + \lambda)} + \frac{1}{2(s + 3\lambda)} \right] \]

Taking inverse of Laplace transforms \( P_1(t) \) and \( P_2(t) \) may be obtained as,

\[ P_1(t) = \frac{1}{3} \left[ 1 - e^{-3\lambda t} \right] \]

\[ P_2(t) = \frac{1}{3} \left[ 1 - \frac{3}{2} e^{-\lambda t} + \frac{1}{2} e^{-3\lambda t} \right] \]

\[ P_0(t) = 1 - P_1(t) - P_2(t) \]

\[ = 1 - \frac{1}{3} \left[ 1 - e^{-3\lambda t} \right] - \frac{1}{3} \left[ 1 - \frac{3}{2} e^{-\lambda t} + \frac{1}{2} e^{-3\lambda t} \right] \]

\[ = \frac{1}{3} \left[ 1 + \frac{3}{2} e^{-\lambda t} + \frac{1}{2} e^{-3\lambda t} \right] \]
Now using \( \lim_{t \to \infty} P_n(t) = \lim_{s \to 0} sP_n^*(s) = P_n \) (say)

\[
P_0 = \lim_{t \to \infty} R_0(t) = \frac{1}{3}
\]

\[
P_1 = \lim_{t \to \infty} R_1(t) = \frac{1}{3}
\]

\[
P_2 = \lim_{t \to \infty} R_2(t) = \frac{1}{3}
\]

**Case II:**

In this case the system is observed assuming that the student moved to the next grade \( i, i = 0,1,2 \), provided the student starts with grade 1 initially. That is here the initial condition is,

\[
P_0(0) = 0, \ P_1(0) = 1, \ P_2(0) = 0 \quad \ldots \quad (3.3.7)
\]

As in Case I solving the differential difference equations obtained from the rate matrix (3.3.3) and using the initial status (3.3.7) we get,

\[
P_0(t) = \frac{1}{3} \left[ 1 - e^{-3t} \right]
\]

\[
P_1(t) = \frac{1}{3} \left[ 1 + 2e^{-3t} \right]
\]
\[ P_2(t) = \frac{1}{3} \left[ 1 - e^{-3\mu} \right] \]

Again, taking limit on the \( P_i(t)' \)’s, \( i = 0,1,2 \), as \( t \to \infty \), we get,

\[ P_0 = P_1 = P_2 = \frac{1}{3} \]

**Case III:**

Here the system is observed assuming that the student moves to the next grade \( i, i = (0,1,2) \) provided the student starts with grade 2 initially. That is in this case, the initial status was,

\[ P_0(0) = 0, P_1(0) = 0, P_2(0) = 1 \quad \ldots \quad (3.3.8) \]

As in previous two cases, solving the differential difference equations arising from the rate matrix (3.3.3) and using the initial status (3.3.8), we get,

\[ P_0(t) = \frac{1}{3} \left[ 1 - \frac{3}{2} e^{-\mu} + \frac{1}{2} e^{-3\mu} \right] \]

\[ P_1(t) = \frac{1}{3} \left[ 1 - e^{-3\mu} \right] \]

\[ P_2(t) = \frac{1}{3} \left[ 1 + \frac{3}{2} e^{-\mu} + \frac{1}{2} e^{-3\mu} \right] \]
And using \( \lim_{t \to \infty} P_n(t) = P_n \) (say)

we get, \( P_0 = P_1 = P_2 = \frac{1}{3} \)

All the results obtained above, under Case I, Case II and Case III may be represented in the matrix form as given by (3.3.1).

The matrix 'P' as given in (3.3.1) is a doubly stochastic matrix.

### 3.4 Expected number of visits to the state of excellence

Let \( N_{i0}(t) \) be defined as the expected number of visits to the state of excellence, i.e. state 0 at time \( t \), provided the student starts with grade \( i \), \( i = 0,1,2 \) initially. Therefore,

\[
\begin{align*}
N_{00}(t) &= Q_{00}(t) + Q_{01}(t) \ast N_{10}(t) \\
N_{10}(t) &= Q_{10}(t) + Q_{11}(t) \ast N_{10}(t) + Q_{12}(t) \ast N_{20}(t) \\
N_{20}(t) &= Q_{21}(t) \ast N_{10}(t) + Q_{22}(t) \ast N_{20}(t)
\end{align*}
\]

where, \( \ast \) stands for convolution and \( Q_{ij}(t) \) is defined as,
\[ Q_{ij}(t) = P_r \{ X(T_1) = i, T_0 = T \leq t \} \]

\[ Q_{01}(t) = Q_{10}(t) = 1 - e^{-\lambda t} = Q_{12}(t) = Q_{21}(t) \]

\[ Q_{00}(t) = e^{-\lambda t} = Q_{22}(t) \]

\[ Q_{11}(t) = e^{-2\lambda t} \]

where, \( \lambda = \frac{abc}{k} \)

and, \( k = 2ab + ac + bc \)

Now taking Laplace transforms of \( Q_{ij}(t)'s, (i,j=0,1,2) \) we get,

\[
\begin{align*}
Q_{00}(s) &= \frac{1}{s + \lambda} = Q_{22}(s) \\
Q_{11}(s) &= \frac{1}{s + 2\lambda} \\
Q_{01}(s) &= \frac{1}{s} - \frac{1}{\lambda + s} = Q_{10}(s) = Q_{12}(s) = Q_{21}(s)
\end{align*}
\]

Again taking Laplace transform of the equations in (3.4.1) and solving we get,
\[ N_{00}(s) = Q_{00}(s) + \frac{Q_{01}(s)Q_{10}(s)(1 - Q_{22}(s))}{\{1 - Q_{11}(s)\}{1 - Q_{22}(s)} - Q_{12}(s)Q_{21}(s)} \]

\[ N_{10}(s) = \frac{Q_{10}(s)(1 - Q_{22}(s))}{\{1 - Q_{11}(s)\}{1 - Q_{22}(s)} - Q_{12}(s)Q_{21}(s)} \]  
\[ N_{20}(s) = \frac{Q_{21}(s)Q_{10}(s)(1 - Q_{22}(s))}{\{1 - Q_{22}(s)\}{1 - Q_{11}(s)} - Q_{12}(s)Q_{21}(s)} \]

Putting the values of (3.4.2) in (3.4.3) we get,

\[ N_{00}(s) = \frac{1}{\lambda + s} + \frac{\lambda^2}{s^2(\lambda + s)^2} \left\{1 - \frac{1}{s + \lambda}\right\} - \frac{\lambda^2}{s^2(\lambda + s)^2} \]

\[ = \frac{1}{\lambda + s} + \frac{\lambda^2(2\lambda + s)(\lambda + s - 1)}{(\lambda + s)((\lambda + s)(\lambda + s - 1)) - \lambda^2(2\lambda + s)} \]  
\[ N_{10}(s) = \frac{\lambda}{s(\lambda + s)} \left\{1 - \frac{1}{s + \lambda}\right\} - \frac{\lambda^2}{s^2(\lambda + s)^2} \]

\[ = \frac{\lambda s(2\lambda + s)(\lambda + s - 1)}{s^2(\lambda + s)((\lambda + s)(\lambda + s - 1)) - \lambda^2(2\lambda + s)} \]
\[ N_{20}^*(s) = \frac{s^2(\lambda + s)^2}{1 - \frac{1}{s + 2\lambda} - \frac{1}{s + \lambda} - \frac{\lambda^2}{s^2(\lambda + s)^2}} \]

\[ = \frac{\lambda^2(2\lambda + s)}{s^2(\lambda + s)(\lambda + s - 1)(2\lambda + s - 1) - \lambda^2(2\lambda + s)} \] .... (3.4.6)

\[ N_{i0}(t)'s, \ i = 0,1,2 \] may be obtained upon inversion of \( N_{i0}^*(s)'s, i=0,1,2. \)

**Probability of sure success in (0,t]**

Let \( H_{i2}(t) \) be the probability that a student does not go to state 2 in the interval (0,t], provided he or she enters with the state i initially (i=0,1,2). That is \( H_{i2}(t) \) may be called the probability of sure success in (0,t], provided the student is admitted with grade i initially.

\[
\begin{align*}
H_{o2}(t) &= Q_{00}(t) + Q_{01}(t) \text{ c. } H_{i2}(t) \\
H_{i2}(t) &= Q_{11}(t) + Q_{i0}(t) \text{ c. } H_{o2}(t)
\end{align*}
\] .... (3.4.7)

where, c. stands for convolution.
Taking Laplace transforms of the above two equations and solving we get,

\[
\begin{align*}
H_{02}(s) &= \frac{Q_{00}*(s) + Q_{01}*(s)Q_{11}*(s)}{1 - Q_{10}*(s)Q_{01}*(s)} \\
H_{12}(s) &= \frac{Q_{11}*(s) + Q_{00}*(s)Q_{00}*(s)}{1 - Q_{10}*(s)Q_{01}*(s)} 
\end{align*}
\]

\[\text{... (3.4.8)}\]

Putting the values of \(Q_{ij}*(s)\) from (3.4.2) in (3.4.8), for \(i,j=0,1,2\), we get,

\[
\begin{align*}
H_{02}(s) &= \frac{s(2\lambda s + s^2 + \lambda)(\lambda + s)}{(2\lambda + s)s^2(\lambda + s)^2 - \lambda^2} \\
H_{12}(s) &= \frac{s^2(\lambda + s)^2 + s\lambda(2\lambda + s)}{(2\lambda + s)s^2(\lambda + s)^2 - \lambda^2} 
\end{align*}
\]

\[\text{... (3.4.9)}\]

\(H_{12}(t), i=0,1,2\) may be obtained upon inversion.

Now,

\[
M_{02} = -\frac{d}{ds}H_{02}(s)\bigg|_{s=0}
\]

\[
= \frac{4\lambda^3 - 3\lambda^2 + 4\lambda - 1}{4\lambda(\lambda^2 - 1)^2}
\]

\[
M_{12} = -\frac{d}{ds}H_{12}(s)\bigg|_{s=0}
\]
\[
\frac{\lambda^3 - 8\lambda^2 + 3\lambda}{4\lambda(\lambda^2 - 1)^2}
\]

where, \(M_{i2}, i = 0,1,2\), is the mean time required for an individual to be always successful, when the tests are conducted at random interval of time provided the individual enters the test procedure with state \(i\) initially.

Clearly, \(M_{02} < M_{12}\)

Or, \(M_{02} > M_{12}\)

i.e., a student who enters the institution at the state of excellence has lower tendency to visit the unsuccessful state than a mediocre student.

3.5. Special case with reduced state space

3.5.1 Probability of an individual attaining a particular grade at time \(t\)

Here a random variable \(I(t)\) is defined such that,

\(I(t) = 1\), if the student gets either grade “0” or grade “1” at time \(t\)

\(= 0\), otherwise
Therefore, the state space $S = \{0,1,2\}$ of the Markov chain model as defined in the section 2 reduces to the state space $S' = \{s,f\}$, where,

$s=\{0,1\}$ and $f=\{2\}

And the corresponding t.p.m. $P$ as defined in (2.2.1) reduces to,

$$P'' = \begin{pmatrix}
    s & f \\
    f & s
\end{pmatrix}$$

with,

$$p_{ij} = P_s(X_n = j | X_{n-1} = i) ; \ t_n - t_{n-1} \leq t, \ i,j = s,f$$

The limiting probabilities $\pi_s$ and $\pi_f$, that is the probability that the student is successful and the probability that the student is failure respectively are calculated from the following equations.

$$\begin{cases}
\pi_s = \left(1-\frac{c}{2}\right)\pi_s + b\pi_f \\
\pi_f = \frac{c}{2}\pi_s + (1-b)\pi_f \\
\pi_s + \pi_f = 1
\end{cases}$$

The rate of transition from state $i$ to state $j$, $(i,j=s,f)$ is given by,

$$\lambda_{ij} = \pi_i p_{ij}, \ i,j = s,f$$
where, $P''=(p_y'')_{2\times2}$ is as defined in (3.5.1)

Thus the transition rate matrix $A'$ is obtained as,

$$A' = \begin{pmatrix} \frac{bc}{2b+c} & \frac{bc}{2b+c} \\ \frac{bc}{2b+c} & \frac{bc}{2b+c} \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda' & \lambda' \\ \lambda' & -\lambda' \end{pmatrix} \quad \ldots \ldots (3.5.1.3)$$

Here $p_i(t)$'s, (i=s,f) i.e. the probabilities that the individual is in the state i at time t, at the time of assessment is obtained under different initial status.

**Case I**:

When $P_s(0)=1, P_f(0)=0 \quad \ldots \ldots (3.5.1.4)$

From the matrix (3.5.1.3) the differential difference equations can be written as,

$$P_s'(t) = -\lambda'P_s(t) + \lambda'P_f(t)$$

$$P_f'(t) = \lambda'P_s(t) - \lambda'P_f(t) \quad \ldots \ldots (3.5.1.5)$$
Taking Laplace transformations of (3.5.1.5) and using the initial status (3.5.1.4) we get,

\[
\begin{align*}
    sP_s^*(s) - P_s(0) &= -\lambda'P_s^*(s) + \lambda'P_f^*(s) \\
    sP_f^*(s) &= \lambda'P_s^*(s) - \lambda'P_f^*(s)
\end{align*}
\]  
...... (3.5.1.6)

Solving the equations in (3.5.1.6) we get,

\[
P_s^*(s) = \frac{1}{2} \left[ \frac{1}{s} + \frac{1}{s + 2\lambda'} \right]
\]

And,

\[
P_f^*(s) = \frac{1}{2} \left[ \frac{1}{s} - \frac{1}{s + 2\lambda'} \right]
\]

Taking inverse Laplace transformations of \(P_s^*(s)\) and \(P_f^*(s)\), we get,

\[
P_s(t) = \frac{1}{2} \left[ \text{I} + e^{-2\lambda t} \right]
\]  
...... (3.5.1.7)

\[
P_f(t) = \frac{1}{2} \left[ \text{I} - e^{-2\lambda t} \right]
\]

**Case II:**

Now when the student started initially with state f, i.e. with grade “2”, then the initial status will be,
$P_r(0)=0, \ P_f(0)=1$ \hspace{2cm} (3.5.1.8)

In this case solving the differential difference equations as given in (3.5.1.5) and using the initial status (3.5.1.8) we get,

$$P_r(t) = \frac{1}{2} \left[ 1 - e^{-2\lambda t} \right]$$

and

$$P_f(t) = \frac{1}{2} \left[ 1 + e^{-2\lambda t} \right]$$  \hspace{2cm} (3.5.1.9)

Now these values of the $P_i(t)$'s, $i= s, f$ under Case I and Case II may be represented in the matrix form as,

$$\begin{pmatrix} 
  s & f \\
  \frac{1}{2} \left[ 1 + e^{-2\lambda t} \right] & \frac{1}{2} \left[ 1 - e^{-2\lambda t} \right] \\
  \frac{1}{2} \left[ 1 - e^{-2\lambda t} \right] & \frac{1}{2} \left[ 1 + e^{-2\lambda t} \right] 
\end{pmatrix}$$

3.5.2 Probability that a student is successful at the time of assessment

Again, let the probability $Q_{ij}(t)$ be defined as,

$$Q_{ij}(t) = P_r \{ X(T_i) = j \mid X(T_0) = i, T_0 \leq t, i,j = s,f \}$$

Therefore, from the rate matrix (3.5.1.3) we get,
\[ Q_{\beta}(t) = 1 - e^{-bc(2b+c)^{-1}} = Q_{\eta}(t) \]

\[ Q_{\eta}(t) = e^{-bc(2b+c)^{-1}} = Q_{\alpha}(t) \]

Let \( A_{ia}(t) \) be the probability that a student is successful at the time of assessment provided the student entered with state \( i \) initially, \( i=s,f \)

\[ \therefore A_{sa}(t) = e^{-\lambda_1 t} + \int_{0}^{t} \lambda_1 e^{-\lambda_1 u} du. \quad A_{\beta}(t) \]

\[ A_{\beta}(t) = (1 - e^{-\lambda_1 t}) e^{-\lambda_1 t} + \int_{0}^{t} \lambda_1 e^{-\lambda_1 u} A_{sa}(t-u) du \]

or,

\[ A_{sa}(t) = e^{-\lambda_1 t} + g \quad c. \quad A_{\beta}(t) \]

\[ \text{.... (3.5.2.1)} \]

\[ A_{\beta}(t) = (1 - e^{-\lambda_1 t}) e^{-\lambda_1 t} + g \quad c. \quad A_{sa}(t) \]

where, \( g = \text{Exp} (\lambda_1) \) and \( c. \) stands for convolution.

Taking Laplace transform on the equations in (3.5.2.1) we get,

\[ A_{sa} * (s) = \frac{1}{\lambda_1 + s} + \frac{\lambda_1}{s(\lambda_1 + s)} \quad c. \quad A_{\beta} * (s) \]

\[ A_{\beta} * (s) = \left[ \frac{1}{\lambda_1 + s} - \frac{1}{2\lambda_1 + s} \right] + \frac{\lambda_1}{s(\lambda_1 + s)} \quad c. \quad A_{sa} * (s) \]
Solving the above two equations we get,

\[ A_{ss} *(s) = \frac{(\lambda' + s)(2\lambda' + s) + \lambda^2}{(2\lambda' + s)^2 s} \]

\[ A_{sf} *(s) = \frac{\lambda'}{(\lambda' + s)(2\lambda' + s)} + \frac{\lambda'}{(\lambda' + s)} \left[ \frac{(\lambda' + s)(2\lambda' + s) + \lambda^2}{(2\lambda' + s)^2 s} \right] \]

\[ = \frac{\lambda'(2\lambda' + s)(\lambda' + 2s) + \lambda^3}{s(\lambda' + s)(2\lambda' + s)^2} \]

Now using the relation,

\[ \lim_{t \to \infty} A_{ss}(t) = \lim_{s \to 0} sA_{ss} *(s) = A_s \]

and

\[ \lim_{t \to \infty} A_{sf}(t) = \lim_{s \to 0} sA_{sf} *(s) = A_f \]

we get,

\[ A_s = \frac{3}{4} \]

\[ A_f = \frac{3}{4} \]

3.6 Probability that a student is successful throughout in \((0,t] \]

Let \( R_{is}(t) \) be the probability that the student is successful throughout in \((0,t]\), provided the student entered with grade \( i, i=s,f \) initially.
\[ R_s(t) = e^{-\lambda t} \]

\[ R_{\beta}(t) = (1 - e^{-\lambda t})e^{-\lambda t} \]

where, \( \lambda' = \frac{bc}{2b + c} \)

\[ M_s^* = \int_0^\infty R_s(t)dt \]

\[ = \int_0^\infty e^{-\lambda t}dt \]

\[ = \frac{1}{\lambda'} \]

\[ = \frac{2b + c}{bc} \]

\[ M_f^* = \int_0^\infty R_{\beta}(t)dt \]

\[ = \int_0^\infty (1 - e^{-\lambda t})e^{-\lambda t}dt \]

\[ = \frac{1}{\lambda'} - \frac{1}{2\lambda'} \]
\[ b + c = \frac{2b + c}{2bc} \]

where \( M_s^* \) and \( M_f^* \) are the mean time required for an individual to be unsuccessful provided he or she entered with grade \( i \) initially, \( i = s, f \).

Clearly, \( M_s^* > M_f^* \)

In chapter 2 and 3, we have analysed the grading pattern probabilistically. We shall try to analyse the results based on grades, in the following chapter, which will help us to judge the merit of an institution and the educational system as a whole.