CHAPTER 1

PRELIMINARIES

In this chapter we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to Harary [10].

Definition 1.1 A graph \( G \) is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of \( G \), called edges. The vertex set and the edge set of \( G \) are denoted by \( V(G) \) or simply \( V \) and \( E(G) \) or simply \( E \) respectively.

If \( e = \{u, v\} \) is an edge, we write \( e = uv \); we say that \( e \) joins the vertices \( u \) and \( v \); \( u \) and \( v \) are adjacent vertices; \( u \) and \( v \) are incident with \( e \).

If two vertices are not joined, then we say that they are non-adjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other.

\[ p = | V(G) | = | V | \] is called the order of \( G \) and \( q = | E(G) | \) is called the size of \( G \). A graph of order \( p \) and size \( q \) is called a \((p, q)\) - graph.

Definition 1.2 A graph \( H \) is called a subgraph of \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). A spanning subgraph of \( G \) is a subgraph \( H \) with \( V(H) = V(G) \). For any set \( S \) of
vertices of $G$, the induced subgraph $G[S]$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $G[S]$ if and only if they are adjacent in $G$.

Let $v$ be a vertex of a graph $G$. The induced subgraph $G[V(G) - \{v\}]$ is denoted by $G - v$; it is the subgraph of $G$ obtained by the removal of $v$ and edges incident with $v$.

**Definition 1.3** The degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and is denoted by $\deg_G v$ or $\deg v$. A vertex of degree 0 in $G$ is called an isolated vertex; a vertex of degree 1 is called a pendant vertex or an end vertex of $G$.

A graph is regular of degree $k$ if every vertex of $G$ has degree $k$. Such graphs are called $k$-regular graphs.

**Definition 1.4** A graph $G$ is complete if every pair of vertices in $G$ are adjacent. A complete graph on $p$ vertices is denoted by $K_p$. A clique of a graph is a maximal complete subgraph.

**Definition 1.5** A bipartite graph $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ joins $V_1$ with $V_2$; $(V_1, V_2)$ is called a bipartition of $G$. If $G$ contains every edge joining $V_1$ and $V_2$, then $G$ is called a complete bipartite graph. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. A star is a complete bipartite graph $K_{1,n}$.

**Definition 1.6** Let $u$ and $v$ be vertices of a graph $G$. A $u$-$v$ walk of $G$ is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, e_n, u_n = v$ of vertices and edges beginning
with vertex \( u \) and ending with vertex \( v \) such that \( e_i = u_{i-1}u_i, \ i = 1, 2, ..., n \). The number \( n \) is called the \textit{length} of the walk. The walk is said to be \textit{open} if \( u \) and \( v \) are distinct vertices; it is \textit{closed} otherwise. A walk \( u_0, e_1, u_1, e_2, u_2, ..., e_n, u_n \) is determined by the sequence \( u_0, u_1, u_2, ..., u_n \) of its vertices and hence we specify this walk by \( u_0, u_1, u_2, ..., u_n \). A walk in which all the vertices are distinct is called a path. A closed walk \( u_0, u_1, u_2, ..., u_n \) in which \( u_0, u_1, u_2, ..., u_{n-1} \) are distinct is called a \textit{cycle}. A path on \( n \) vertices is denoted by \( P_n \) and a cycle on \( n \) vertices is denoted by \( C_n \).

**Definition 1.7** A graph \( G \) is said to be \textit{connected} if any two distinct vertices of \( G \) are joined by a path. A maximal connected subgraph of \( G \) is called a \textit{component} of \( G \).

**Definition 1.8** A \textit{cut-vertex} of a graph \( G \) is a vertex whose removal increases the number of components. A \textit{non separable} graph is connected, nontrivial and has no cut vertices. A \textit{block} of a graph is a maximal non separable subgraph. A graph in which each block is complete is called a \textit{block graph}.

For a cut-vertex \( v \) in a connected graph \( G \) and a component \( H \) of \( G - v \), the subgraph \( H \) and the vertex \( v \) together with all edges joining \( v \) and \( V(H) \) is called a \textit{branch of \( G \) at \( v \)}.. An \textit{end-block} of \( G \) is a block containing exactly one cut-vertex of \( G \). Thus every end-block is a branch of \( G \).

**Definition 1.9** The \textit{neighborhood} of a vertex \( v \) is the set \( N(v) \) consisting of all vertices \( u \) which are adjacent with \( v \). A vertex \( v \) is a \textit{simplicial vertex} or \textit{extreme vertex} if the subgraph induced by its neighbors is complete.

**Definition 1.10** A graph \( G \) is called \textit{acyclic} if it has no cycles. A connected acyclic graph is called a \textit{tree}. A \textit{caterpillar} is a tree for which the removal of all the end vertices gives a path. A \textit{double star} is a tree of diameter 3.
Theorem 1.11[10] Let $v$ be a vertex of a connected graph $G$. The following statements are equivalent:

(i) $v$ is a cut vertex of $G$.

(ii) There exist vertices $u$ and $w$ distinct from $v$ such that $v$ is on every $u$-$w$ path.

(iii) There exists a partition of the set of vertices $V - \{v\}$ into subsets $U$ and $W$ such that for any vertices $u \in U$ and $w \in W$, the vertex $v$ is on every $u$-$w$ path.

Theorem 1.12[10] Let $G$ be a connected graph with at least three vertices. The following statements are equivalent:

(i) $G$ is a block.

(ii) Every two vertices of $G$ lie on a common cycle.

Theorem 1.13[10] Every nontrivial connected graph has at least two vertices which are not cut vertices.

Definition 1.14 Let $G$ be a connected graph. The distance $d(u,v)$ between two vertices $u$ and $v$ in $G$ is the length of a shortest $u$-$v$ path in $G$.

The eccentricity $e(u)$ of a vertex $u$ is defined by $e(u) = \max \{d(u,v) : v \in V\}$. Each vertex in $V$ at which the eccentricity function is minimized is called a central vertex of $G$ and the set of all central vertices of $G$ is called the center of $G$ and is denoted by $Z(G)$.

The radius $r$ and diameter $d$ of $G$ are defined by $r = \min \{e(v) : v \in V\}$ and $d = \max \{e(v) : v \in V\}$ respectively.
Definition 1.15 A vertex $v$ in a graph $G$ is called an eccentric vertex of a vertex $u$ if $d(u,v) = e(u)$. In general we call a vertex $v$ an eccentric vertex if it is an eccentric vertex of some vertex $u$ and call it a non-eccentric vertex otherwise.

A vertex $v$ is a peripheral vertex of $G$ if $e(v) = d$. The set of all peripheral vertices of $G$ is called the periphery of $G$ and is denoted by $P(G)$.

Definition 1.16 For vertices $u$ and $v$ in a connected graph $G$, a $u-v$ path of length $d(u,v)$ is called an $u-v$ geodesic. A vertex $y$ is said to lie on a $u-v$ geodesic $P$ if $y$ is a vertex of $P$ including the vertices $u$ and $v$.

Definition 1.17 The closed interval $I [u,v]$ consists of all vertices lying on some $u-v$ geodesic of $G$, while for $S \subseteq V$, $I [S] = \bigcup_{u,v \in S} I[u,v]$. A set $S$ of vertices is a geodetic set if $I [S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a $g$-set.

A pair $u, v$ of distinct vertices of $G$ is said to openly geodominate a vertex $y$ if $y$ is an internal vertex of a $u-v$ geodesic in $G$. A set $S$ is an open geodominating set of $G$ if for each vertex $y$, either $y$ is a simplicial vertex and $y \in S$ or $y$ is openly geodominated by some pair of vertices of $S$. An open geodominating set of minimum cardinality is an og-set, and this cardinality is the open geodomination number $og(G)$.

Consider the graph $G$ of Figure 1.1. For the vertices $w$ and $y$ in $G$, $d(w,y) = 2$ and every vertex of $G$ lies on a $w-y$ geodesic in $G$. Thus $\{w,y\}$ is the unique minimum geodetic set of $G$ and so $g(G) = 2$. 

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The closed intervals in a connected graph $G$ were studied and characterized by Nebeský [14,15] and were also investigated extensively in the book by Mulder [13], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The geodetic number of a graph was introduced in [1,11] and further studied in [5]. It was shown in [11] that determining the geodetic number of a graph is an NP-hard problem. Geodetic concepts were first studied from the point of view of domination by Chartrand, Harary, Swart and Zhang in [4], where a pair $x, y$ of vertices in a nontrivial connected graph $G$ is said to geodominate a vertex $v$ of $G$ if $v \in I[x,y]$, that is, $v$ lies on an $x-y$ geodesic of $G$. In [4], geodetic sets and the geodetic number were referred to as geodominating sets and geodomination number and it is this terminology that we adopt in this thesis.

![Figure 1.1](image)

\textbf{Theorem 1.18}[4] Every geodominating set of a graph $G$ contains every simplicial vertex of $G$. In particular, if the set $W$ of simplicial vertices is a geodominating set of $G$, then $W$ is the unique $g$-set and the unique $og$-set of $G$ and so $g(G) = og(G) = |W|$.

\textbf{Theorem 1.19}[1] No cut vertex of $G$ belongs to any minimum geodetic set of $G$. 
Theorem 1.20[11] For the wheel $W_{1,n}$, $g(W_{1,n}) = \begin{cases} 4 & \text{for } n = 3, \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{for } n \geq 4. \end{cases}$

Theorem 1.21[1] Let $G$ be a connected graph. Then

(i) $g(G) = p$ if and only if $G = K_p$.

(ii) $g(G) = 2$ if and only if there exist peripheral vertices $u$ and $v$ such that every vertex of $G$ is on a diametral path joining $u$ and $v$.

Theorem 1.22[2] Let $G$ be a connected graph of order $p \geq 3$. Then $g(G) = p - 1$ if and only if $G = K_1 + \bigcup m_jK_j$, where $\sum m_j \geq 2$.

Theorem 1.23[5] For integers $m, n \geq 2$, $g(K_{m,n}) = \min \{m, n, 4\}$.

Definition 1.24[8] For a connected graph $G$ and a set $W \subseteq V(G)$, a tree $T$ contained in $G$ is a Steiner tree with respect to $W$ if $T$ is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all vertices in $G$ that lie on some Steiner tree with respect to $W$. The set $W$ is a Steiner set for $G$ if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of $G$ is the Steiner number $s(G)$.

Theorem 1.25[8] Let $G$ be a connected graph of order $p \geq 2$. Then $s(G) = p$ if and only if $G = K_p$.

Theorem 1.26[8] Let $G$ be a connected graph of order $p \geq 3$. Then $s(G) = p - 1$ if and only if $G$ contains a cut-vertex of degree $p - 1$.

Theorem 1.27[8] Let $G$ be a connected graph of order $p \geq 2$. Then $s(G) = 2$ if and only if $g(G) = 2$. 

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Definition 1.28[3] For vertices $u$ and $v$ in a connected graph $G$, the detour distance $D(u,v)$ is the length of a longest $u$-$v$ path in $G$.

The detour eccentricity $e_D(u)$ of a vertex $u$ is defined by $e_D(u) = \max \{ D(u,v) : v \in V \}$. Each vertex in $V$ at which the detour eccentricity function is minimized is called a detour central vertex of $G$ and the set of all detour central vertices of $G$ is called the detour center of $G$ and is denoted by $Z_D(G)$.

The detour radius $R$ and detour diameter $D$ of $G$ are defined by $R = \min \{ e_D(v) : v \in V \}$ and $D = \max \{ e_D(v) : v \in V \}$ respectively.

Definition 1.29[3] A vertex $v$ in a graph $G$ is called a detour eccentric vertex of a vertex $u$ if $D(u,v) = e_D(u)$. In general we call a vertex $v$ a detour eccentric vertex if it is a detour eccentric vertex of some vertex $u$ and call it a non-detour eccentric vertex otherwise.

A vertex $v$ is a detour peripheral vertex of $G$ if $e_D(v) = D$. The set of all detour peripheral vertices of $G$ is called the detour periphery of $G$ and is denoted by $P_D(G)$.

Definition 1.30 For vertices $u$ and $v$ in a connected graph $G$, the closed interval $I_D[u,v]$ consists of all vertices lying on some $u$-$v$ detour of $G$, while for $S \subseteq V$, $I_D[S] = \bigcup_{u,v \in S} I_D[u,v]$. A set $S$ of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a minimum detour set.

Consider the graph $G$ of Figure 1.2. For vertices $u$ and $v$ in $G$, $D(u,v) = 5$, where the hamiltonian path $u, z, y, w, x, v$ is a $u$-$v$ detour in $G$. Thus $\{ u, v \}$ is a
minimum detour set and so $dn(G) = 2$. The detour number of a graph was introduced in [6] and further studied in [7].

![Graph G](image)

**Figure 1.2**

**Theorem 1.31**[3] For every connected graph $G$, $rad_D G \leq diam_D G \leq 2 rad_D G$.

**Theorem 1.32**[6] Every end-vertex of a non-trivial connected graph $G$ belongs to every detour set of $G$. Moreover, if the set $S$ of all end-vertices of $G$ is a detour set, then $S$ is the unique minimum detour set for $G$.

**Theorem 1.33**[6] If $T$ is a tree with $k$ end-vertices, then $dn(T) = k$.

**Notation 1.34** For any real number $x$, $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.

Throughout the following, $G$ denotes a connected graph with at least two vertices.