CHAPTER 5

THE VERTEX DETOUR NUMBER OF A GRAPH*

In this chapter we introduce the concept of vertex detour number $d_v(G)$ of a graph $G$ at a vertex $x$ and investigate its properties. We determine bounds for it and find the same for some special classes of graphs. We define an $x$-detour superior vertex of a graph and characterize graphs $G$ for which $d_v(G) = 1$ in terms of $x$-detour superior vertices. The relationship between the vertex detour number $d_v(G)$ at any vertex $x$ and the detour number $dn(G)$ of a graph $G$ is found to be $dn(G) \leq d_v(G) + 1$ and we give a realization theorem for this inequality. It is shown that if $G$ is a graph of order $p$, then $d_v(G) \leq p - e_D(x)$ for any vertex $x$ in $G$. Connected graphs of order $p$ with vertex detour numbers $p - 1$ or $p - 2$ for every vertex are characterized. It is proved that for every non-trivial tree $T$, $d_v(T) = p - D$ or $d_v(T) = p - D + 1$ for every vertex $x$ of $T$ if and only if $T$ is a caterpillar. For positive integers $R$, $D$ and $n \geq 2$ with $R < D \leq 2R$, there exists a connected graph $G$ with $rad_D G = R$, $diam_D G = D$ and $d_v(G) = n$ or $d_v(G) = n - 1$ for every vertex $x$ of $G$. Also, for each triple $D$, $n$ and $p$ of integers with $1 \leq n \leq p - D + 1$ and $D \geq 4$, there is a connected graph $G$ of order $p$, detour diameter $D$ and $d_v(G) = n$ or $d_v(G) = n - 1$ for every vertex $x$ of $G$.

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Definition 5.1  Let \( x \) be a vertex of a connected graph \( G \). A set \( S \) of vertices of \( G \) is an \( x \)-detour set if each vertex \( v \) of \( G \) lies on an \( x \)-\( y \) detour in \( G \) for some element \( y \) in \( S \). The minimum cardinality of an \( x \)-detour set of \( G \) is defined as the \( x \)-detour number of \( G \) and is denoted by \( d_x(G) \) or simply \( d_x \). An \( x \)-detour set of cardinality \( d_x(G) \) is called a \( d_x \)-set of \( G \).

Result 5.2  For any vertex \( x \) in \( G \), \( x \) does not belong to any \( d_x \)-set of \( G \).

Proof. Suppose that \( x \) belongs to a \( d_x \)-set, say \( S_x \) of \( G \). Since \( G \) is a connected graph with at least two vertices, it follows from the definition of an \( x \)-detour set that \( S_x \) contains a vertex \( v \) different from \( x \). Since the vertex \( x \) lies on every \( x \)-\( v \) detour in \( G \), it follows that \( T = S_x \setminus \{x\} \) is an \( x \)-detour set of \( G \), which is a contradiction to \( S_x \) a minimum \( x \)-detour set of \( G \).

Example 5.3

(i) \( d_x(K_p) = 1 \) for every vertex \( x \) in \( K_p \).

(ii) For the graph \( G \) given in Figure 5.1, the minimum vertex detour sets and the vertex detour numbers are given in Table 5.1.

![Figure 5.1](image-url)
<table>
<thead>
<tr>
<th>Vertex</th>
<th>Minimum Vertex Detour Sets</th>
<th>Vertex Detour Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>{y, w}, {z, w}, {u, w}</td>
<td>2</td>
</tr>
<tr>
<td>y</td>
<td>{w}</td>
<td>1</td>
</tr>
<tr>
<td>z</td>
<td>{w}</td>
<td>1</td>
</tr>
<tr>
<td>u</td>
<td>{w}</td>
<td>1</td>
</tr>
<tr>
<td>v</td>
<td>{y, w}, {z, w}, {u, w}</td>
<td>2</td>
</tr>
<tr>
<td>w</td>
<td>{y}, {z}, {u}</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 5.1**

**Remark 5.4** Let \(x\) be any vertex of \(G\). Then for any vertex \(y\) belonging to a \(d_x\)-set \(S_x\) of \(G\), the internal vertices of an \(x-y\) detour may belong to \(S_x\). For the graph \(G\) given in Figure 5.1, \(S_x = \{u, w\}\) is a \(d_x\)-set of \(G\) and \(u\) belonging to \(S_x\) is an internal vertex of the \(x-w\) detour: \(x, z, u, v, w\). Also, \(S_x = \{y, w\}\) is a \(d_x\)-set of \(G\) such that \(y\) is not an internal vertex of any \(x-w\) detour and \(w\) is not an internal vertex of any \(x-y\) detour in \(G\).

**Theorem 5.5** Let \(x\) be any vertex of a connected graph \(G\).

(i) Every end-vertex of \(G\) other than the vertex \(x\) (whether \(x\) is end-vertex or not) belongs to every \(x\)-detour set.

(ii) No cut vertex of \(G\) belongs to any \(d_x\)-set.
Proof. (i) Let $x$ be any vertex of $G$. Let $v \neq x$ be an end-vertex of $G$. Then $v$ is the terminal vertex of an $x$-$v$ detour and $v$ is not an internal vertex of any detour so that $v$ belongs to every $x$-detour set of $G$.

(ii) Let $y$ be a cut vertex of $G$. Then by Theorem 1.11, there exists a partition of the set of vertices $V - \{y\}$ into subsets $U$ and $W$ such that for any vertex $u \in U$ and $w \in W$, the vertex $y$ is on every $u$-$w$ path. Hence, if $x \in U$, then for any vertex $w$ in $W$, $y$ lies on every $x$-$w$ path so that $y$ is an internal vertex of an $x$-$w$ detour. Let $S_x$ be any $d_x$-set of $G$. Suppose $S_x \cap W = \emptyset$. Let $w_1 \in W$. Since $S_x$ is an $x$-detour set, there exists an element $z$ in $S_x$ such that $w_1$ lies in some $x$-$z$ detour $P : x = z_0, z_1, \ldots, w_1, \ldots, z_n = z$ in $G$. Then the $x$-$w_1$ subpath of $P$ and $w_1$-$z$ subpath of $P$ both contain $y$ so that $P$ is not a path in $G$. Hence $S_x \cap W \neq \emptyset$. Let $w_2 \in S_x \cap W$. Then $y$ is an internal vertex of an $x$-$w_2$ detour. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on an $x$-$y$ detour also lies on an $x$-$w_2$ detour. Hence it follows that $S$ is an $x$-detour set of $G$, which is a contradiction to $S_x$ is a minimum $x$-detour set of $G$. Thus $y$ does not belong to any $d_x$-set. Similarly if $x \in W$, $y$ does not belong to any $d_x$-set. If $x = y$, then by Result 5.2, $y$ does not belong to any $d_x$-set.

Note 5.6 Even if $x$ is an end-vertex of $G$, $x$ does not belong any $d_x$-set by Result 5.2.

Corollary 5.7 Let $T$ be a tree with number of end-vertices $t$. Then $d_0(T) = t - 1$ or $d_e(T) = t$ according as $x$ is an end-vertex or not. In fact, if $W$ is the set of all end-vertices of $T$, then $W - \{x\}$ is the unique $d_x$-set of $T$.

Proof. Let $W$ be the set of all end-vertices of $T$. It follows from Result 5.2 and Theorem 5.5 that $W - \{x\}$ is the unique $d_x$-set of $T$ for any end-vertex $x$ in $T$ and $W$ is
the unique $d_i$-set of $T$ for any cut vertex $x$ in $T$. Thus $W - \{x\}$ is the unique $d_i$-set of $T$ for any vertex $x$ in $T$. 

**Corollary 5.8** Let $P_n$ be a non-trivial path. Then $d_i(P_n) = 1$ or $d_i(P_n) = 2$ according as $x$ is an end-vertex or not.

**Corollary 5.9** For any star $K_{1,n} (n \geq 2)$, $d_i(K_{1,n}) = n - 1$ or $d_i(K_{1,n}) = n$ according as $x$ is an end-vertex or not.

**Theorem 5.10** For any hamiltonian graph $G$, $d_i(G) = 1$ for every vertex $x$ in $G$.

**Proof.** Let $C$ be a hamiltonian cycle of $G$. Let $x$ be any vertex of $G$ and let $y$ be any adjacent vertex of $x$ in $C$. Clearly every vertex of $G$ lies on a detour joining $x$ and $y$. Thus $d_i(G) = 1$ for every vertex $x$ in $G$. 

**Corollary 5.11** For the $n$-cube $Q_n (n \geq 2)$, $d_i(Q_n) = 1$ for every vertex $x$ in $Q_n$.

**Remark 5.12** The converse of Theorem 5.10 is false. For the graph $G$ given in Figure 5.2, $d_i(G) = 1$ for every vertex $x$ in $G$. But $G$ is not hamiltonian.

![Figure 5.2](image-url)
Corollary 5.13 For any cycle $C$, $d_x(C) = 1$ for every vertex $x$ in $C$.

Corollary 5.14 For the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 4$), $d_x(W_n) = 1$ for every vertex $x$ in $W_n$.

Theorem 5.15 If a connected graph $G$ has a hamiltonian path, then $d_x(G) = 1$ for at least two vertices.

Proof. Let $P$ be a hamiltonian path with end-vertices $x$ and $y$. Then it is clear that $d_x(G) = d_y(G) = 1$.

The following theorem is an easy consequence of the definition of the vertex detour number.

Theorem 5.16

(i) For $m = n = 1$, $d_x(K_{m,n}) = 1$ for every vertex $x$ in $G$.

(ii) For $m, n \geq 2$, $d_x(K_{m,n}) = 1$ for every vertex $x$ in $G$.

(iii) For $m = 1$ and $n \geq 2$, $d_x(K_{m,n}) = n$ or $d_x(K_{m,n}) = n - 1$ for every vertex $x$ of $G$.

Theorem 5.17 Let $G$ be a connected graph with cut vertices and let $S_x$ be an $x$-detour set of $G$. Then every branch of $G$ contains an element of $S_x \cup \{x\}$.

Proof. Suppose that there is a branch $B$ of $G$ at a cut vertex $v$ such that $B$ contains no vertex of $S_x \cup \{x\}$. Then clearly, $x \in V - (S_x \cup V(B))$. Let $u \in V(B) - \{v\}$. Since $S_x$ is an $x$-detour set, there exists an element $y \in S_x$ such that $u$ lies in some $x$-$y$ detour $P : x = u_0, u_1, \ldots, u, \ldots, u_n = y$ in $G$. By Theorem 1.11 the $x$-$u$ subpath of $P$ and
$u-y$ subpath of $P$ both contain $v$, and it follows that $P$ is not a path, contrary to assumption.

Since every end-block $B$ is a branch of $G$ at some cut-vertex, it follows by Theorems 5.5 and 5.17 that every $d_x$-set of $G$ together with the vertex $x$ contains at least one vertex from $B$ that is not a cut-vertex. Thus the following corollaries are consequences of Theorem 5.17.

**Corollary 5.18** If $G$ is a connected graph with $k$ end-blocks, then $d_x(G) \geq k - 1$ for every vertex $x$ in $G$. In particular, if $x$ is a cut vertex of $G$, then $d_x(G) \geq k$.

**Corollary 5.19** If $k$ is the maximum number of blocks to which a vertex in a graph $G$ belongs, then $d_x(G) \geq k - 1$ for every vertex $x$ in $G$. In particular, if $x$ is a cut vertex of $G$, then $d_x(G) \geq k$.

**Theorem 5.20** For any vertex $x$ in $G$, $1 \leq d_x(G) \leq p - 1$.

**Proof.** It is clear from the definition of $d_x$-set that $d_x(G) \geq 1$. Also since the vertex $x$ does not belong to any $d_x$-set, it follows that $d_x(G) \leq p - 1$.

**Remark 5.21** The bounds for $d_x(G)$ in Theorem 5.20 are sharp. For the cycle $C_n$, $d_x(C_n) = 1$ for every vertex $x$ in $C_n$. Also for any non-trivial path $P_n$, $d_x(P_n) = 1$ for any end-vertex $x$ in $P_n$. For the graph $K_2$, $d_x(K_2) = p - 1$ for every vertex $x$ in $K_2$.

Now we proceed to characterize graphs for which the lower bound in Theorem 5.20 is attained. For this, we introduce the following definition.

**Definition 5.22** Let $x$ be any vertex in $G$. A vertex $y$ in $G$ is said to be an $x$-detour superior vertex if for any vertex $z$ with $D(x,y) < D(x,z)$, $z$ lies on an $x-y$ detour.
Example 5.23

(i) In the even cycle $C_{2n}$, both eccentric and detour eccentric vertices of $x$ are $x$-detour superior vertices.

(ii) For the graph $G$ given in Figure 5.3, the vertex detour superior vertices are given in Table 5.2.

![Diagram of graph G]

**Figure 5.3**

<table>
<thead>
<tr>
<th>Vertex</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$x_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertex detour superior vertices</td>
<td>$x_9, x_{10}$</td>
<td>$x_9, x_{10}$</td>
<td>$x_2, x_4, x_7$</td>
<td>$x_9, x_{10}$</td>
<td>$x_9, x_{10}$</td>
<td>$x_7$</td>
<td>$x_9, x_{10}$</td>
<td>$x_2, x_4$</td>
<td>$x_2, x_4$</td>
<td>$x_2, x_4$</td>
</tr>
</tbody>
</table>

**Table 5.2**

We give below a property related with detour eccentric vertex of $x$ and $x$-detour superior vertex in a graph $G$. 

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**Theorem 5.24** Let $x$ be any vertex in $G$. Then every detour eccentric vertex of $x$ is an $x$-detour superior vertex.

**Proof.** Let $y$ be a detour eccentric vertex of $x$ so that $e_D(x) = D(x,y)$. If $y$ is not an $x$-detour superior vertex, then there exists a vertex $z$ in $G$ such that $D(x,y) < D(x,z)$ and $z$ does not lie on any $x$-$y$ detour and hence $e_D(x) < D(x,z)$, which is a contradiction. ■

**Note 5.25** The converse of Theorem 5.24 is not true. For the even cycle $C_{2n}$, the eccentric vertex of $x$ is an $x$-detour superior vertex but it is not a detour eccentric vertex of $x$.

**Theorem 5.26** Let $G$ be a connected graph. For a vertex $x$ in $G$, $d_s(G) = 1$ if and only if there exists an $x$-detour superior vertex $y$ in $G$ such that every vertex of $G$ is on an $x$-$y$ detour.

**Proof.** Let $d_s(G) = 1$ and $S_x = \{y\}$ be a $d_s$-set of $G$. If $y$ is not an $x$-detour superior vertex, then there is a vertex $z$ in $G$ with $D(x,y) < D(x,z)$ and $z$ does not lie on any $x$-$y$ detour. Thus $S_x$ is not a $d_s$-set of $G$, which is a contradiction. The converse is clear from the definition. ■

In the following theorem, we establish the relationship between the vertex detour number of a vertex and the detour number of a graph.

**Theorem 5.27** For any vertex $x$ in $G$, $dn(G) \leq d_s(G) + 1$.

**Proof.** Let $x$ be any vertex of $G$ and let $S_x$ be a $d_s$-set of $G$. Then every vertex of $G$ lies on an $x$-$y$ detour for some $y$ in $S_x$. Thus $S_x \cup \{x\}$ is a detour set of $G$. Since $dn(G)$ is the minimum cardinality of a detour set, it follows that $dn(G) \leq d_s(G) + 1$. ■

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Note 5.28 The bound in Theorem 5.27 is sharp. For the complete graph $K_p$, $dn(K_p) = d_x(K_p) + 1$ for every vertex $x$ in $K_p$.

Theorem 5.29 For any two integers $a$ and $b$ with $2 \leq a \leq b + 1$, there exists a connected graph $G$ with $dn(G) = a$ and $d_x(G) = b$ for some vertex $x$ in $G$.

Proof. For $2 \leq a = b + 1$, let $G$ be any tree with $a$ end-vertices. By Theorem 1.33, $dn(G) = a$ and by Corollary 5.7, $d_x(G) = b$ for an end-vertex $x$ in $G$. Then assume that $2 \leq a < b + 1$. Let $F = (K_3 \cup P_2 \cup (b - a + 1)K_1) + K_2$, where $U = V(K_3) = \{u_1, u_2, u_3\}$, $W = V(P_2) = \{w_1, w_2\}$, $X = V((b - a + 1)K_1) = \{x_1, x_2, \ldots, x_{b-a+1}\}$ and $V(K_2) = \{x, y\}$. Let $G$ be the graph obtained from $F$ by adding $a - 1$ new vertices $z_1, z_2, \ldots, z_{a-1}$ and joining each $z_i$ ($1 \leq i \leq a - 1$) to $u_1$. The graph $G$ is shown in Figure 5.4. Let $Z = \{z_1, z_2, \ldots, z_{a-1}\}$ be the set of end vertices of $G$.

![Figure 5.4](attachment:image.png)

First, we show that $dn(G) = a$. By Theorem 1.32, every detour set of $G$ contains $Z$. Since $I_d[Z] = Z \cup \{u_1\} \neq V(G)$, it follows that $Z$ is not a detour set of $G$. 

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and so $dn(G) > |Z| = a - 1$. On the other hand, let $S = Z \cup \{w_1\}$. Then $D(z_1, w_1) = 8$
and for each $i$ with $1 \leq i \leq b - a + 1$, the path $z_1, u_1, u_2, u_3, y, x_i, x, w_2, w_1$ is a $z_1$-$w_1$
detour in $G$. Hence $S$ is a detour set of $G$ and so $dn(G) \leq |S| = a$. Therefore, $dn(G) = a$.

Next we show that $d_s(G) = b$ for the vertex $x$. Let $S_x$ be a minimum $x$-detour set
of $G$. By Theorem 5.5(i), $Z \subseteq S_x$. Since $D(x, z) = 7$ and no $x_i$ $(1 \leq i \leq b - a + 1)$ lies
on an $x-z$ detour for any $z \in Z$, $Z$ is not an $x$-detour set of $G$. Now we claim that $X \subseteq S_x$. Assume, to the contrary, $X \not\subseteq S_x$. Then there exists an $x$, such that $x_i \not\in S_x$ $(1 \leq i \leq b - a + 1)$. Now this $x_i$ does not lie on any $x-v$ detour for $v \neq x_i$ and $v \in S_x$, this is a
contradiction to $S_x$ is a minimum $x$-detour set. Thus $X \subseteq S_x$. It is clear that $X \cup Z$ is
an $x$-detour set. Hence it follows that $X \cup Z$ is a minimum $x$-detour set so that $d_s(G) = a - 1 + b - a + 1 = b$.

Bounds for the Vertex Detour Number of a Graph

We have seen that if $G$ is a connected graph of order $p \geq 2$, then $1 \leq d_s(G) \leq p - 1$ for any vertex $x$ in $G$. Also we have for a vertex $x$ in $G$, $d_s(G) = 1$ if and only if
there is an $x$-detour superior vertex $y$ such that every vertex of $G$ is on an $x-y$ detour.

In the following theorem we give an improved upper bound for the vertex detour
number of a graph.

**Theorem 5.30** For any vertex $x$ in a connected graph $G$ of order $p$, $d_s(G) \leq p - e_D(x)$.

**Proof.** Let $x$ be any vertex of $G$ and $v$ a detour eccentric vertex of $x$. Then $D(x, v) = e_D(x)$. Let $P : x = x_0, x_1, \ldots, x_k = v$ be an $x-v$ detour in $G$. Let $S = V(G) - \{x_0, x_1, \ldots, x_k\}$. Since each $x_i$ $(0 \leq i \leq k - 1)$ lies on an $x-v$ detour, $S$ is an $x$-detour set of $G$ so that $d_s(G) \leq p - e_D(x)$. ■
Remark 5.31 The bound in Theorem 5.30 is sharp. For the cycle $C_p$, $d_x(C_p) = 1 = p - e_D(x)$ for every vertex $x$ in $C_p$. Also for the graph $G$ given in Figure 5.3, $p = 10$, $e_D(x_7) = 7$ and $S = \{x_4, x_9, x_{10}\}$ is a $d_x$-set so that $d_{x_i}(G) = 3$. Thus $d_{x_i}(G) = p - e_D(x_7)$. The inequality in Theorem 5.30 can also be strict. For the same graph $G$ given in Figure 5.3, $e_D(x_3) = 5$ and $S = \{x_4, x_7, x_9, x_{11}\}$ is a $d_x$-set so that $d_{x_i}(G) = 4$. Thus $d_{x_i}(G) < p - e_D(x_3)$.

Corollary 5.32 If $G$ is a connected graph of order $p$ and detour diameter $D$, then $d_x(G) \leq p - D/2$ for every vertex $x$ in $G$.

**Proof.** Since $R \leq e_D(x)$ for every vertex $x$ in $G$, it follows from Theorem 1.31 and Theorem 5.30 that $d_x(G) \leq p - D/2$. 

Remark 5.33 The bound in Corollary 5.32 is sharp. For the star $K_{1,p-1}(p \geq 3)$, by Corollary 5.9, $d_x(K_{1,p-1}) = p - 1 = p - D/2$ for the cut vertex $x$ in $K_{1,p-1}$. Also, the inequality in Corollary 5.32 can be strict. For the star $K_{1,p-1}(p \geq 3)$, by Corollary 5.9, $d_x(K_{1,p-1}) = p - 2 < p - D/2$ for an end vertex $x$ in $K_{1,p-1}$.

Theorem 5.34 Let $G$ be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $d_x(G) = p - 1$ for every vertex $x$ in $G$.

**Proof.** If $G = K_2$, then $d_x(G) = 1 = p - 1$ for every vertex $x$ in $K_2$. Conversely, let $d_x(G) = p - 1$ for every vertex $x$ in $G$. If $D \geq 2$, then there exists a vertex $x$ in $G$ such that $e_D(x) \geq 2$. By Theorem 5.30, $d_x(G) \leq p - e_D(x) \leq p - 2$, which is a contradiction. Thus $D = 1$ so that $G = K_2$. 

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**Theorem 5.35** Let \( G \) be a connected graph of order \( p \geq 2 \) and \( G \neq K_3 \). Then \( G = K_{1,p-1} \) if and only if \( d_x(G) = p - 1 \) or \( d_x(G) = p - 2 \) for every vertex \( x \) of \( G \).

**Proof.** If \( G = K_{1,1} = K_2 \), then by Example 5.3(i), \( d_x(G) = 1 = p - 1 \) for every vertex \( x \) of \( G \). If \( G = K_{1,p-1} \) (\( p \geq 3 \)), then by Corollary 5.9, \( d_x(G) = p - 1 \) or \( d_x(G) = p - 2 \) for every vertex \( x \) of \( G \). Conversely, suppose \( d_x(G) = p - 1 \) or \( d_x(G) = p - 2 \) for every vertex \( x \) of \( G \). If \( p = 2 \), then \( G = K_2 = K_{1,1} \). If \( p = 3 \), then \( G = P_3 = K_{1,1} \). Let \( p \geq 4 \).

We prove that \( G \) is a star. Suppose \( G \) is not a star. If \( G \) is a tree, then \( G \) has at most \( p - 2 \) end-vertices. By Corollary 5.7, \( d_x(G) \leq p - 3 \) if \( x \) is an end-vertex, which is a contradiction. Now, if \( G \) is not a tree. Let \( c(G) \) be the length of a longest cycle, say \( C \), in \( G \). If \( c(G) \geq 4 \), then \( D \geq 3 \) so that \( e_D(x) \geq 3 \) for some vertex \( x \) in \( G \). Hence by Theorem 5.30, \( d_x(G) \leq p - 3 \), which is a contradiction. If \( c(G) = 3 \), let \( u, v, w, u \) be a triangle in \( G \). Since \( p \geq 4 \), there exists \( x \in V(G) - \{u, v, w\} \) such that \( x \) is adjacent to at least one of \( u, v, w \), say \( xu \in E(G) \). Then \( x, u, v, w \) is a path in \( G \) so that \( e_D(x) \geq 3 \). Then by Theorem 5.30, \( d_x(G) \leq p - 3 \), which is a contradiction. Thus \( G \) is a star.

**Theorem 5.36** Let \( G \) be a connected graph of order \( p \geq 3 \). Then \( G = K_3 \) if and only if \( d_x(G) = p - 2 \) for every vertex \( x \) in \( G \).

**Proof.** If \( G = K_3 \), then it is clear that \( d_x(G) = 1 = p - 2 \) for every vertex \( x \) in \( G \). Conversely, let \( d_x(G) = p - 2 \) for every vertex \( x \) in \( G \). If \( D \geq 3 \), then \( e_D(x) \geq 3 \) for some vertex \( x \) in \( G \). Hence by Theorem 5.30, \( d_x(G) \leq p - e_D(x) \leq p - 3 \), which is a contradiction. If \( D = 1 \), then \( G = K_2 \) and so \( d_x(G) = p - 1 \) for every vertex \( x \) in \( G \), which is a contradiction. Hence \( D = 2 \). If \( p \geq 4 \), then \( G = K_{1,p-1} \) and hence by Corollary 5.9, \( d_x(G) = p - 1 \) for the cut vertex \( x \) in \( G \), which is a contradiction. Thus \( p = 3 \) and so \( G \) is either \( P_3 \) or \( K_3 \). If \( G = P_3 \), then by Corollary 5.7, \( d_x(G) = 2 = p - 1 \) for the cut.
vertex \( x \) in \( G \), which is a contradiction. If \( G = K_3 \), then \( d_x(G) = 1 = p - 2 \) for every vertex \( x \) in \( G \). Thus \( G = K_3 \) is the only graph which satisfies the requirement of the theorem.

**Theorem 5.37** Let \( G \) be a connected graph of order \( p \geq 5 \). Then \( d_x(G) = p - 2 \) or \( d_x(G) = p - 3 \) for every vertex \( x \) of \( G \) if and only if \( G \) is a double star or \( K_{1,p-1} + e \).

**Proof.** It is straightforward to verify that if \( G \) is a double star or \( K_{1,p-1} + e \), then \( d_x(G) = p - 2 \) or \( d_x(G) = p - 3 \) for every vertex \( x \) of \( G \). For the converse, let \( G \) be a connected graph of order \( p \geq 5 \) such that \( d_x(G) = p - 2 \) or \( d_x(G) = p - 3 \) for every vertex \( x \) of \( G \). If \( D \leq 2 \), then \( G \) is the star \( K_{1,p-1} \) and so by Corollary 5.9, \( d_x(G) = p - 1 \) for the cut vertex \( x \) in \( G \), which is a contradiction.

Let \( D = 3 \). If \( G \) is a tree, then \( G \) is a double star and the result follows from Corollary 5.7. Assume that \( G \) is not a tree. Let \( c(G) \) denote the length of a longest cycle in \( G \). Since \( D = 3 \), it follows that \( c(G) \leq 4 \). We consider two cases.

**Case 1.** Let \( c(G) = 4 \). Let \( C_4 : v_1, v_2, v_3, v_4, v_1 \) be a 4-cycle in \( G \). Since \( p \geq 5 \) and \( G \) is connected, there exists a vertex \( x \) not on \( C_4 \) such that \( x \) is adjacent to some vertex, say \( v_1 \), of \( C_4 \). Then \( x, v_1, v_2, v_3, v_4 \) is a path of length 4 in \( G \) so that \( D \geq 4 \), which is a contradiction.

**Case 2.** Let \( c(G) = 3 \). If \( G \) contains two or more triangles, then \( c(G) = 4 \) or \( D \geq 4 \), which is a contradiction. Hence \( G \) contains an unique triangle \( C_3 : v_1, v_2, v_3, v_1 \). Now, we prove that there is exactly one vertex on \( C_3 \) of degree at least 3. If there are two or more vertices of \( C_3 \) having degree 3 or more, then \( D \geq 4 \), which is a contradiction. Thus exactly one vertex in \( C_3 \) has degree 3 or more. Since \( D = 3 \), it follows that \( G = K_{1,p-1} + e \). Now, it follows from Theorems 5.5 and 5.17 that \( d_x(G) = p - 2 \) or \( d_x(G) = p - 3 \) according as \( x \) is a cut vertex or not.

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If \( D \geq 4 \), then \( e_D(x) \geq 4 \) for some vertex \( x \) in \( G \). Hence by Theorem 5.30, \( d_i(G) \leq p - e_D(x) \leq p - 4 \), which is a contradiction.

**Remark 5.38** Theorem 5.37 is not true for \( p = 4 \). For the graph \( G \) given in Figure 5.5, \( p = 4 \) and \( d_i(G) = 1 = p - 3 \) for every vertex \( x \) in \( G \). However, \( G \) is neither a double star nor \( K_{1,p-1} + e \).

![Figure 5.5](image)

**Theorem 5.39** For every non-trivial tree \( T \), \( d_i(T) = p - D \) or \( d_i(T) = p - D + 1 \) for every vertex \( x \) of \( T \) if and only if \( T \) is a caterpillar.

**Proof.** Let \( T \) be any non-trivial tree. Let \( P : u = v_0, v_1, \ldots, v_D = v \) be a diametral path. Let \( k \) be the number of end vertices of \( T \) and \( l \) be the number of internal vertices of \( T \) other than \( v_1, v_2, \ldots, v_{D-1} \). Then \( D - 1 + l + k = p \). By Corollary 5.7, \( d_i(T) = k \) or \( d_i(T) = k - 1 \) for every vertex \( x \) of \( T \) and so \( d_i(T) = p - D - l + 1 \) or \( d_i(T) = p - D - l \) for every vertex \( x \) of \( T \). Hence \( d_i(T) = p - D + 1 \) or \( d_i(T) = p - D \) for every vertex \( x \) of \( T \) if and only if \( l = 0 \), if and only if all the internal vertices of \( T \) lie on the diametral path \( P \), if and only if \( T \) is a caterpillar.

For every connected graph \( G \), \( rad_D G \leq diam_D G \leq 2 \ rad_D G \). Chartrand, Escuadro and Zhang\[3\] showed that every two positive integers \( a \) and \( b \) with
$a \leq b \leq 2a$ are realizable as the detour radius and detour diameter, respectively, of some connected graph. This theorem can also be extended so that the vertex detour number can be prescribed when $a < b \leq 2a$.

**Theorem 5.40** For positive integers $R$, $D$ and $n \geq 2$ with $R < D \leq 2R$, there exists a connected graph $G$ with $\text{rad}_D G = R$, $\text{diam}_D G = D$ and $d_v(G) = n$ or $d_v(G) = n - 1$ for every vertex $x$ of $G$.

**Proof.** If $R = 1$, then $D = 2$. Take $G = K_{1,n}$. Then by Corollary 5.9, $d_v(G) = n$ or $d_v(G) = n - 1$ for every vertex $x$ of $G$. Now, let $R \geq 2$. We construct a graph $G$ with the desired properties as follows.

Let $C_{R+1} : v_1, v_2, \ldots, v_{R+1}, v_1$ be a cycle of order $R + 1$ and let $P_{D-R+1} : u_0, u_1, \ldots, u_{D-R}$ be a path of order $D - R + 1$. Let $H$ be a graph obtained from $C_{R+1}$ and $P_{D-R+1}$ by identifying $v_1$ in $C_{R+1}$ and $u_0$ in $P_{D-R+1}$. Now, add $n - 2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to $H$ by joining each vertex $w_i (1 \leq i \leq n - 2)$ to the vertex $u_{D-R-1}$ and obtain the graph $G$ of Figure 5.6. Now $\text{rad}_D G = R$, $\text{diam}_D G = D$ and $G$ has $n - 1$ end vertices.

**Case 1.** Let $R$ be even. If $R = 2$, then $d_v(G) = n$ or $d_v(G) = n - 1$ according as $x \in \{v_1, u_1, u_2, u_3, \ldots, u_{D-R-1}\}$ or $x \in \{v_2, v_3, u_{D-R}, w_1, w_2, w_3, \ldots, w_{n-2}\}$. If $R \geq 4$, then $d_v(G) = n$ or $d_v(G) = n - 1$ according as $x \in \{v_1, v_3, v_4, \ldots, v_R, u_1, u_2, u_3, \ldots, u_{D-R-1}\}$ or $x \in \{v_2, v_{R+1}, u_{D-R}, w_1, w_2, w_3, \ldots, w_{n-2}\}$.

**Case 2.** Let $R$ be odd. If $R = 3$, then $d_v(G) = n$ or $d_v(G) = n - 1$ according as $x \in \{v_1, u_1, u_2, u_3, \ldots, u_{D-R-1}\}$ or $x \in \{v_2, v_3, v_4, u_{D-R}, w_1, w_2, w_3, \ldots, w_{n-2}\}$. If $R \geq 5$, then $d_v(G) = n$ or $d_v(G) = n - 1$ according as $x \in \{v_1, v_3, v_4, \ldots, v_{(R+1)/2}, v_{(R+3)/2}, \ldots, v_R, u_1, u_2, u_3, \ldots, u_{D-R-1}\}$ or $x \in \{v_2, v_{(R+3)/2}, v_{R+1}, u_{D-R}, w_1, w_2, w_3, \ldots, w_{n-2}\}$. Thus $d_v(G) = n$ or $d_v(G) = n - 1$ for every vertex $x$ of $G$. 

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Figure 5.6

The graph $G$ of Figure 5.6 is the smallest graph with the properties described in Theorem 5.40. We leave the following problem as an open question.

**Problem 5.41** For positive integers $R$, $D$ and $n \geq 2$ with $R = D$, does there exist a connected graph $G$ with $\text{rad}_D G = R$, $\text{diam}_D G = D$ and $d_x(G) = n$ or $d_x(G) = n - 1$ for every vertex $x$ of $G$?

In the following, we construct a graph of prescribed order, detour diameter and vertex detour number under suitable conditions.

**Theorem 5.42** For each triple $D$, $n$ and $p$ of integers with $1 \leq n \leq p - D + 1$ and $D \geq 4$, there is a connected graph $G$ of order $p$, detour diameter $D$ and $d_x(G) = n$ or $d_x(G) = n - 1$ for every vertex $x$ of $G$.

**Proof.** Let $G$ be a graph obtained from the cycle $C_D: u_1, u_2, \ldots, u_D, u_1$ of order $D$ by (i) adding $n - 1$ new vertices $v_1, v_2, \ldots, v_{n-1}$ and joining each vertex $v_i$ ($1 \leq i \leq n - 1$) to $u_1$ and (ii) adding $p - D - n + 1$ new vertices $w_1, w_2, \ldots, w_{p - D - n + 1}$ and joining each vertex $w_i$ ($1 \leq i \leq p - D - n + 1$) to both $u_1$ and $u_3$. The graph $G$ has order $p$ and detour
diameter $D$ and is shown in Figure 5.7. If $n = 1$, then $d_x(G) = n$ for every vertex $x$ in $G$.

If $n \geq 2$, then we consider two cases.

**Case 1.** Let $D$ be even. If $D = 4$, then $d_x(G) = n$ or $d_x(G) = n - 1$ according as $x = u_1$ or $x \in \{u_2, u_3, u_4, v_1, v_2, \ldots, v_{n-1}, w_1, w_2, \ldots, w_{p-D-n+1}\}$. If $D \geq 6$, then $d_x(G) = n$ or $d_x(G) = n - 1$ according as $x \in \{u_1, u_2, \ldots, u_{D/2}, u_{(D+4)/2}, \ldots, u_D, w_1, w_2, \ldots, w_{p-D-n+1}\}$ or $x \in \{u_{(D+2)/2}, u_{D/2}, v_1, v_2, \ldots, v_{n-1}\}$.

**Case 2.** Let $D$ be odd. Clearly $d_x(G) = n$ or $d_x(G) = n - 1$ according as $x \in \{u_1, u_2, \ldots, u_{D-1}, w_1, w_2, \ldots, w_{p-D-n+1}\}$ or $x \in \{u_D, v_1, v_2, \ldots, v_{n-1}\}$. Thus $d_x(G) = n$ or $d_x(G) = n - 1$ for every vertex $x$ of $G$.

**Theorem 5.43** Let $p \geq 2$ be any integer. For $1 \leq n \leq p - 1$ there exists a connected graph $G$ with order $p$ and $d_x(G) = n$ or $d_x(G) = n - 1$ for every vertex $x$ of $G$.

**Proof.** For $p = 2$, $G = K_2$ has the desired properties. For $p = 3$, $G = C_3$ or $P_3$ has the desired properties according as $n = 1$ or $n = 2$. For $p \geq 4$, we consider three cases.
Case 1. Let \( n = 1 \). Then \( G = C_p \) has the desired properties.

Case 2. Let \( 2 \leq n \leq p - 2 \). Then \( p - n + 1 \geq 3 \). The graph \( G \) is obtained from the cycle \( C_{p-n+1} : u_1, u_2, \ldots, u_{p-n+1}, u_1 \) by adding the \( n - 1 \) new vertices \( v_1, v_2, \ldots, v_{n-1} \) and joining these to \( u_1 \). The graph \( G \) is shown in Figure 5.8.

Subcase 2.1. Let \( p - n + 1 \) be even. If \( p - n + 1 = 4 \), then \( d_G(x) = n \) or \( d_G(x) = n - 1 \) according as \( x = u_1 \) or \( x \in \{u_2, u_3, u_4, v_1, v_2, \ldots, v_{n-1}\} \). If \( p - n + 1 \geq 6 \), then \( d_G(x) = n \) or \( d_G(x) = n - 1 \) according as \( x \in \{u_1, u_2, u_3, \ldots, u_{(p-n+1)/2}, u_{(p-n+5)/2}, \ldots, u_{p-n}\} \) or \( x \in \{u_2, u_{(p-n+3)/2}, u_{p-n+1}, v_1, v_2, \ldots, v_{n-1}\} \).

Subcase 2.2. Let \( p - n + 1 \) be odd. If \( p - n + 1 = 3 \), then \( d_G(x) = n \) or \( d_G(x) = n - 1 \) according as \( x = u_1 \) or \( x \in \{u_2, u_3, u_4, v_1, v_2, \ldots, v_{n-1}\} \). If \( p - n + 1 \geq 5 \), then \( d_G(x) = n \) or \( d_G(x) = n - 1 \) according as \( x \in \{u_1, u_3, u_4, \ldots, u_{p-n}\} \) or \( x \in \{u_2, u_{p-n+1}, v_1, v_2, \ldots, v_{n-1}\} \). Thus \( d_G(x) = n \) or \( d_G(x) = n - 1 \) for every vertex \( x \) of \( G \).

Case 3. Let \( n = p - 1 \). Then \( G = K_{1,p-1} \) has the desired properties.

![Figure 5.8](image)