CHAPTER 6
DIGRAPH LABELINGS

The concepts of D-magic and D-antimagic labelings were introduced in [10]. In this chapter we prove that $C_n + K_2$, $P_n + mK_1$, $C_3 + mK_1$, cycles with equal number of pendant vertices at each point, paths with equal number of pendant vertices at each point are D-antimagic. We also construct several classes of D-magic and D-antimagic graphs.

Theorem 6.1 $P_n + mK_1$ is D-antimagic where $m > 2$, $n > 1$, $m \neq n$ and $P_n$ is a path of length $n$.

Proof. Let $G = P_n + mK_1$.

Let $V(G) = \{ v_1, v_2, \ldots, v_n, v_{n+1}, u_1, u_2, \ldots, u_m \}$ where $P_n = (v_1, v_2, \ldots, v_n, v_{n+1})$.

Orient the edges of the graph such that the arc set $A$ is given by

$$A = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_{n-1}, v_n), (v_n, v_{n+1})\}$$

$$\cup \{(u_1, v_i), (v_i, u_2), (u_2, v_i), \ldots, (v_i, u_m) / 1 \leq i \leq n+1, i \text{ is odd}\}$$

$$\cup \{(v_i, u_1), (u_2, v_i), (v_i, u_3), \ldots, (u_m, v_i) / 1 \leq i \leq n+1, i \text{ is even}\}.$$ 

Define $f : A \rightarrow N$ as follows

$$f(v_j, v_{j+1}) = (n + 1)(m + 1) - j \text{ if } 1 \leq j \leq n.$$
\[ f(u_i,v_j) = (m - i + 1)(n + 1) + 1 - j \quad \text{if} \quad 1 \leq i \leq m - 1, \ 1 \leq j \leq n + 1 \]

and \[ f(u_m,v_j) = j, \ 1 \leq j \leq n + 1. \]

Then, \[ |f^+(v_j) - f^-(v_j)| = (n + 1) a_m \] where \( a_m \) is defined by means of the recurrence relation \( a_3 = 9 \) and \( a_i = a_{i-1} + (i + 1), 4 \leq i \leq m. \)

For \( 2 \leq j \leq n, \)

\[ |f^+(v_j) - f^-(v_j)| = (n + 1) a_m - mj + m \] where \( a_3 = 13 \) and \( a_i = a_{i-1} + (i + 2) \) and \[ |f^+(v_{n+1}) - f^-(v_{n+1})| = (n + 1) a_m + m \] where \( a_3 = 7 \) and \( a_i = a_{i-1} + i, 4 \leq i \leq m. \)

Also, \[ |f^+(u_i) - f^-(u_i)| = \frac{(2m - 2i + 1)(n + 1)^2 + (n + 1)}{2} \quad 1 \leq i \leq m. \]

Hence \( |f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)| \) for any two distinct vertices \( u \) and \( v \) so that \( P_n + mK_1 \) is \( D \)- antimagic. \( \blacksquare \)

**Example 6.2** \( D \)-antimagic labeling of \( P_3 + 5K_1 \) is given in Figure 6.1.

**Theorem 6.3** \( C_n + \overline{K_2} \) is \( D \)-antimagic if \( n \neq 10, 13, 14. \)

**Proof.** Let \( G = C_n + \overline{K_2} \) with
\[ V(G) = \{ v_1, v_2, \ldots, v_n, u_1, u_2 \} \quad \text{and} \quad C_n = (v_1, v_2, \ldots, v_n, v_1) \]

Orient the edges of \( G \) in such a way that the arc set \( A \) is given by
\[
A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1)\} \cup \{(v_i, u_1), (v_i, u_2) \mid 1 \leq i \leq n \text{ and } i \text{ is odd}\} \cup \{(u_1, v_i), (u_2, v_i) \mid 1 \leq i \leq n \text{ and } i \text{ is even}\}.
\]

Define \( f: A \to \mathbb{N} \) by
\[
f(v_1, v_2) = 2n + 1, \quad f(v_i, v_{i+1}) = 3n + 2 - i \quad \text{if} \quad 2 \leq i \leq n - 1,
\]
\[
f(v_n, v_1) = 2n + 2, \quad f(u_1, v_1) = \left\lfloor \frac{i + 1}{2} \right\rfloor \quad \text{if} \quad i \text{ is odd and } 1 \leq i \leq n
\]
\[
f(u_1, v_i) = \left\lfloor \frac{n + i}{2} \right\rfloor \quad \text{if} \quad i \text{ is even and } 1 \leq i \leq n
\]
and \( f(u_2, v_i) = n + i \) if \( 1 \leq i \leq n \).

Then, \( |f^+(v_1) - f^-(v_1)| = 5n + 5 \)
\[
|f^+(v_2) - f^-(v_2)| = \left\lfloor \frac{13n + 9}{2} \right\rfloor
\]
\[
|f^+(v_i) - f^-(v_i)| = 7n + 5 - \left( \frac{i - 1}{2} \right) \quad \text{if} \quad 3 \leq i \leq n \text{ and } i \text{ is odd}
\]
\[
|f^+(v_i) - f^-(v_i)| = \left\lfloor \frac{15n + 6}{2} \right\rfloor + 2 - \frac{i}{2} \quad \text{if} \quad 4 \leq i \leq n \text{ and } i \text{ is even}
\]
\[
|f^+(u_1) - f^-(u_1)| = \frac{n(n + 1)}{2}
\]
and \( |f^+(u_2) - f^-(u_2)| = \frac{3n^2 + n}{2} \).

Hence \( |f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)| \) for any two vertices \( u \) and \( v \) so that \( f \) is a D-antimagic labeling of \( G \).
Example 6.4 The $D$-antimagic labeling of $C_{7} + ar{K}_{2}$ is given in Figure 6.2.

Theorem 6.5 $C_{3} + mK_{1}$ is $D$-antimagic.

Proof. Let $G = K_{3} + mC_{1}$ with $C_{3} = (v_{1}, v_{2}, v_{3}, v_{1})$.

Let $V(G) = \{v_{1}, v_{2}, v_{3}, u_{1}, u_{2}, ..., u_{m}\}$.

Orient the edges of $G$ in such a way that the arc set $A$ is given by

\[ A = \{(v_{1}, v_{2}), (v_{2}, v_{3}), (v_{3}, v_{1})\} \]
\[ \cup \{(v_{1}, u_{i}), (u_{i}, v_{2}), (u_{i}, v_{3}) \mid 1 \leq i \leq n \text{ and } i \text{ is odd}\} \]
\[ \cup \{(u_{i}, v_{1}), (v_{2}, u_{i}), (v_{3}, u_{i}) \mid 1 \leq i \leq n \text{ and } i \text{ is even}\}. \]

Define $f: A \rightarrow N$ as follows.

\[ f(v_{1}, v_{2}) = 3m + 1, \quad f(v_{2}, v_{3}) = 3m + 3, \quad f(v_{3}, v_{1}) = 3m + 2, \]
\[ f(v_{1}, u_{i}) = 3i - 2 \quad \text{if} \quad 1 \leq i \leq m, \]
\[ f(v_{2}, u_{i}) = 3i - 1 \quad \text{if} \quad 1 \leq i \leq m. \]
and \( f(v_i, u_i) = 3i \) if \( 1 \leq i \leq m \). Then,

\[
|f^+(v_1) - f^-(v_1)| = (3m + 1) + 3m + 2 + 1 + 4 + \ldots + (3m - 2) = \frac{3m^2 + 11m + 6}{2}.
\]

Similarly \( |f^+(v_2) - f^-(v_2)| = \frac{3m^2 + 13m + 8}{2} \),

\[
|f^+(v_3) - f^-(v_3)| = \frac{3m^2 + 15m + 10}{2}
\]

and \( |f^+(u_i) - f^-(u_i)| = 9i - 3 \) if \( 1 \leq i \leq m \).

Hence \( |f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)| \) for any two distinct vertices \( u \) and \( v \) so that \( f \) is a \( D \)-antimagic labeling of \( G \).

**Example 6.6** D-antimagic labeling of \( C_3 + 5K_1 \) is given in Figure 6.3.

![Figure 6.3](image)

**Theorem 6.7** \( C_n \odot K_1 \) is \( D \)-antimagic if \( n > 3 \).

**Proof.** Let \( G = C_n \odot K_1, V(G) = \{ v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n \} \) and \( C_n = (v_1, v_2, \ldots, v_n, v_1) \).
Orient the edges of $G$ in such a way that the arc set $A$ is given by

$$A = \{ (v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1), (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n) \}.$$ 

Define $f : A \rightarrow N$ as follows.

$$f(v_i, v_{i+1}) = i \text{ if } 1 \leq i \leq n-1,$$

$$f(v_n, v_1) = n \text{ and } f(u_i, v_i) = 2n - i + 1 \text{ if } 1 \leq i \leq n.$$ 

Then,

$$|f^+(v_1) - f^-(v_1)| = 3n + 1,$$

$$|f^+(v_i) - f^-(v_i)| = 2n + i \text{ if } 2 \leq i \leq n \text{ and}$$

$$|f^+(u_i) - f^-(u_i)| = 2n - i + 1 \text{ if } 1 \leq i \leq n.$$ 

Hence $|f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)|$ for any two distinct vertices $u$ and $v$ so that $f$ is a $D$-antimagic labeling of $G$. $\blacksquare$

**Example 6.8** $D$-antimagic labeling of $C_6 \Theta K_1$ is given in Figure 6.4.

![Fig 6.4](image)

**Theorem 6.9** $C_n \times K_2$ is $D$-antimagic if $n > 3$. 
Proof. Let $G = C_n \times K_2$,

$V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $C_n = (v_1, v_2, \ldots, v_n, v_1)$.

Orient the edges of $G$ in such a way that the arc set $A$ is given by

$A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1), (u_n, u_{n-1}), (u_{n-1}, u_{n-2}), \ldots, (u_2, u_1), (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$

Define $f: A \rightarrow \mathbb{N}$ as follows.

$f(v_i, v_{i+1}) = i$ if $1 \leq i \leq n - 1, f(v_n, v_1) = n,$

$f(u_i, v_i) = 2n - i + 1$ if $1 \leq i \leq n,$

$f(u_i, u_{i+1}) = 2n + i$ if $1 \leq i \leq n - 1$ and $f(u_n, u_1) = 3n.$

Then, $|f^+(v_1) - f^-(v_1)| = 3n + 1,$

$|f^+(v_i) - f^-(v_i)| = 2n + i$ if $2 \leq i \leq n,$

$|f^+(u_1) - f^-(u_1)| = 7n + 1$

and $|f^+(u_i) - f^-(u_i)| = 6n + i, 2 \leq i \leq n$

Hence $|f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)|$ for any two distinct vertices $u$ and $v$ so that $f$ is a $D$-antimagic labeling of $G$. \(\blacksquare\)

Example 6.10 $D$-antimagic labeling of $C_7 \times K_2$ is given in Figure 6.5.
Theorem 6.11  Let $G$ be the graph obtained by attaching $m$ pendant edges to each vertex of the path $P_n = (v_1, v_2, \ldots, v_{n+1})$. Then $G$ is $D$-antimagic.

Proof. Let $\{v_{1i}, v_{2i}, \ldots, v_{ni}, v_{n+1,i} / 1 \leq i \leq m \}$ be the pendant vertices adjacent to $v_j$, $1 \leq j \leq n + 1$. Orient the edges of $G$ in such a way that the arc set $A$ is given by.

$$A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_{n+1}) \}
\cup \{(v_{1i}, v_1), (v_{2i}, v_2), \ldots, (v_{n+1,i}, v_{n+1}) / 1 \leq i \leq m \text{ and } i \text{ is odd} \}
\cup \{(v_1, v_{1i}), (v_2, v_{2i}), \ldots, (v_{n+1}, v_{n+1,i}) / 1 \leq i \leq m \text{ and } i \text{ is even} \}.$$

Define $f : A \to \mathbb{N}$ as follows.

$$f(v_j, v_{ji}) = (j - 1)m + i \quad \text{if} \quad 1 \leq i \leq m \text{ and } 1 \leq j \leq n + 1.$$  

$$f(v_j, v_{ji+1}) = (n + 1)m + j \quad \text{if} \quad 1 \leq j \leq n.$$  

Then, $|f^+(v_1) - f^-(v_1)| = 1 + 2 + \ldots + m + (n + 1)m + 1 = m(3 + m + 2n) + 2$.  

$$= \frac{m(3 + m + 2n) + 2}{2}$$
Similarly, $|f^+(v_j) - f^-(v_j)| = \frac{m[5 + (2j - 1)m + 4n] + (4j - 2)}{2}$ if $2 \leq j \leq n$
and $|f^+(v_{n+1}) - f^-(v_{n+1})| = \frac{m^2(1 + 2n) + 2n(m + 1) + 3m}{2}$.

Also $|f^+(v_i) - f^-(v_i)| = (j - 1)m + i$ if $1 \leq i \leq m$ and $1 \leq j \leq n + 1$.

Hence $|f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)|$ for any two distinct vertices $u$ and $v$ so that $f$ is a $D$-antimagic labeling of $G$.

**Example 6.12** D-antimagic labeling of $G$ which is obtained by attaching 5 pendant edges at each vertex of $P_5$ is given in Figure 6.6.

![Figure 6.6](image)

**Theorem 6.13** Let $G$ be the graph obtained by attaching $m$ pendant edges to each vertex of the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Then $G$ is $D$-antimagic.

**Proof.** Let $\{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ be the pendant vertices adjacent to each vertex of the cycle $C_n$.

Orient the edges of $G$ in such a way that the arc set $A$ is given by

$A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$

$\cup \{(v_{1i}, v_1), (v_{2i}, v_2) \ldots (v_{ni}, v_n) / 1 \leq i \leq m \text{ and } i \text{ is odd}\}$

$\cup \{(v_1, v_{1i}), (v_2, v_{2i}) \ldots, (v_n, v_{ni}) / 1 \leq i \leq m \text{ and } i \text{ is even}\}$
Define \( f: A \to N \) as follows.

\[
f(v_j, v_{i+1}) = (j-1)m + i \quad \text{if} \quad 1 \leq i \leq m \quad \text{and} \quad 1 \leq j \leq n,
\]

\[
f(v_i, v_{i+1}) = nm + i \quad \text{if} \quad 1 \leq i \leq n-1 \quad \text{and} \quad f(v_n, v_1) = nm + n.
\]

Then,

\[
|f^+(v_1) - f^-(v_1)| = nm + 1 + nm + 2 + 1 + 2 + \ldots + m
\]

\[
= \frac{m(1 + m + 4n) + 2(n + 1)}{2}
\]

and

\[
|f^+(v_j) - f^-(v_j)| = \frac{m[1 + (2j-1) m + 4n] + (4j - 2)}{2}, \quad 2 \leq j \leq n.
\]

Also

\[
|f^+(v_j) - f^-(v_j)| = (j - 1)m + i \quad \text{if} \quad 1 \leq i \leq m \quad \text{and} \quad 1 \leq j \leq n.
\]

Hence \( |f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)| \) for any two distinct vertices \( u \) and \( v \) so that \( f \) is a \( D \)-antimagic labeling of \( G \).

**Example 6.14** D-antimagic labeling of \( G \) when 3 pendant edges are attached to each vertex of \( C_8 \) is given in Figure 6.7.

![Fig 6.7](image)

**Theorem 6.15** If two cycles \( C_m \) and \( C_n \) are attached at the two ends of \( K_2 \) then the resulting graph \( G \) is \( D \)-antimagic.
Proof. Without loss of generality we assume that $m \leq n$.

Let $V(G) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$, $C_m = (v_1, v_2, \ldots, v_m, v_1)$, $C_n = (u_1, u_2, \ldots, u_n, u_1)$ and let $v_1, u_1$ be adjacent.

Orient the edges of $G$ in such a way that the arc set $A$ is given by $A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_m, v_1)(v_1, u_1)(u_1, u_2), \ldots, (u_n, u_1)\}$.

Define $f: A \to \mathbb{N}$ as follows.

$$f(v_i, v_{i+1}) = i \quad \text{if} \quad 1 \leq i \leq m - 1,$$

$$f(v_m, v_1) = m,$$

$$f(v_1, u_1) = m + 1,$$

$$f(u_i, u_{i+1}) = m + 1 + i \quad \text{if} \quad 1 \leq i \leq n - 1$$

and $f(u_n, u_1) = m + n + 1$.

Then, $|f^+(v_1) - f^-(v_1)| = 2m + 2$,

$$|f^+(v_i) - f^-(v_i)| = 2i - 1 \quad \text{if} \quad 2 \leq i \leq m,$$

$$|f^+(u_1) - f^-(u_1)| = 3m + n + 4$$

and $|f^+(u_i) - f^-(u_i)| = 2m + 2i + 1 \quad \text{if} \quad 2 \leq i \leq n$. 
Hence \(|f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)|\) for any two distinct vertices \(u\) and \(v\) so that \(f\) is a \(D\)-antimagic labeling of \(G\). \(\blacksquare\)

**Example 6.16** If \(C_5\) and \(C_8\) are attached at two ends of \(K_2\), the \(D\)-antimagic labeling is given in Figure 6.8.

![Figure 6.8](image)

**Theorem 6.17** A cycle \(C_n\) with a \(P_k\)-chord is \(D\)-antimagic.

**Proof.** Let \(V(G) = \{v_1, v_2, \ldots, v_n\}\) where \(C_n = (v_1, v_2, \ldots, v_n, v_1)\) and let \(v_1 v_k \in E(G)\).

Orient the edges of the graph in such a way that the arc set \(A\) is given by

\[A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1), (v_1, v_k)\}\].

Define \(f : A \rightarrow N\) as follows

\[f(v_1, v_k) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}\]

\[f(v_k, v_{k+1}) = 1, f(v_{k+1}, v_{k+2}) = 2, \ldots, f(v_n, v_1) = n+1-k, \]

\[f(v_1, v_2) = n+2-k, f(v_2, v_3) = n+3-k, \ldots, f(v_{k-2}, v_{k-1}) = n-1\]
and \( f(v_{k-1}, v_k) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even} . \end{cases} \)

Then, \( |f^+(v_1) - f^-(v_1)| = \begin{cases} 3n + 4 - 2k & \text{if } n \text{ is odd} \\ 3n + 3 - 2k & \text{if } n \text{ is even} , \end{cases} \)

\[ |f^+(v_i) - f^-(v_i)| = 2n + 2i + 1 - 2k \text{ if } 2 \leq i \leq k - 1 , \]

\[ |f^+(v_k) - f^-(v_k)| = 2n + 2 \]

and \( |f^+(v_i) - f^-(v_i)| = 2i + 1 - 2k \text{ if } k + 1 \leq i \leq n . \)

Hence \( |f^+(u) - f^-(u)| \neq |f^+(v) - f^-(v)| \) for any two distinct vertices \( u \) and \( v \) so that \( f \) is a D-antimagic labeling of \( G . \)

**Example 6.18** D-antimagic labeling of \( C_9 \) with \( P_5 \)-chord and \( C_{12} \) with \( P_6 \)-chord are given in Figure 6.9.