CHAPTER VI
PATH COVERS OF GRAPHS AND DIGRAPHS

Let $G$ be a simple graph. $G^{(2)}$ denotes the graph got by replacing each edge of $G$ by two parallel edges. Harary in [29] introduced the concept of path cover (or path partition) of $G$. For a graph $G$, let $E = \{\Psi : \Psi$ is a path cover of $G\}$. Clearly $E(G)$ is a path cover for any graph $G$. Hence $E \neq \emptyset$. Let $P_n(G)$ denote the minimum number of paths covering all the edges of $G$ exactly once. Stanton et al. in [45], Arumugam and Suresh Suseela in [8] found $P_n(G)$ for certain classes of graphs. In 6.1 we find $P_n(G^{(2)})$ for some standard graphs. In 6.2 we find path number of some directed graphs.

6.1. PATH COVER OF $G^{(2)}$

Harary in [29] introduced the concept of path cover (or path partition) of $G$. The path cover of $G$ is defined as the set of all non-trivial edge-disjoint paths covering all the edges of $G$ exactly once. $P_n(G)$, the path number of $G$, is defined as follows: $P_n(G) = \min \{|\Psi| : \Psi$ is a path cover of $G\}$.

If $\Psi$ is a collection of paths covering all the edges of $G$ exactly twice, then $\Psi$ is called a path double cover of $G$. The notion of path double cover was first introduced by J.A. Bondy. In [16], he posed the following conjecture.

Conjecture: PPDC (The Perfect Path Double Cover Conjecture)
Every simple graph has a path double cover $\Psi$ such that each vertex of $G$ occurs exactly twice as an end vertex of a path of $\Psi$.

Not before long, this conjecture was proved by H. Li and the conjecture becomes a theorem now. The theorem implies that every simple graph of order $p$ can be path double covered by at most $p$ paths. Obviously, the reason we need $p$ paths in a perfect path double cover is due to the requirement that every vertex must be an end vertex of a path exactly twice. If we drop this requirement, the number of paths needed is less than $p$ in general. In this section, we shall investigate the following number.

$$\gamma_2(G) = \min \{|\Psi| : \Psi \text{ is a path double cover of } G\}$$

For convenience, we call $\gamma_2(G)$ the path double cover (PDC) number of $G$. By the above observation we have $\gamma_2(G) \leq p$. As $Pn(G^{(2)}) = \gamma_2(G)$, we find $\gamma_2(G)$ for some standard graphs. Arumugam and Meena also found path double cover number for some standard graphs in [32].

6.1.1. Lemma: Let $G$ be a graph with $n$ pendant vertices. Then $\gamma_2(G) \geq n$.

**Proof:** Every pendant vertex is an end vertex of two different paths of a path double cover of $G$. Since there are $n$ pendant vertices, we have $\gamma_2(G) \geq n$. ■

6.1.2. Lemma: If $G$ is a graph with $\delta(G) \geq 2$ then $\gamma_2(G) \geq \max(\delta(G) + 1, \Delta(G))$.

**Proof:** One can observe that the total degree of each vertex $v$ of $G$ in a path double cover is $2 \deg(v)$. If $v$ is an external vertex of a path in a path double cover $\Psi$ of $G$ then $v$ is external in at least two different paths of $\Psi$. So we
have \(|\Psi| \geq \frac{2\text{deg}(v) - 2}{2} + 2 = \text{deg}(v) - 1 + 2 = \text{deg}(v) + 1 \geq \delta(G) + 1\). This is true for every path double cover of \(G\). Hence \(\gamma_2(G) \geq \delta(G) + 1\). Let \(u\) be a vertex of degree \(\Delta\) in \(G\). We always have \(|\Psi| \geq \frac{2\text{deg}(u)}{2} = \Delta\). Hence \(\gamma_2(G) \geq \max(\delta(G) + 1, \Delta(G))\). 

6.1.3. Corollary: If \(G\) is a \(k\) - regular graph then \(\gamma_2(G) \geq k + 1\) and for all other graphs \(\gamma_2(G) \geq \Delta\).

**Proof:** We know that \(\Delta(G) \geq \delta(G)\) and for a regular graph \(\delta(G) = \Delta(G)\). Hence the result follows.

As \(\gamma_2(G) \leq p\) and by the above corollary we have

6.1.4. Corollary [32]: \(\gamma_2(K_p) = p\).

The following proposition is found in [32]. So we omit the proof of it.

6.1.5. Proposition: For a tree \(T\), \(\gamma_2(T) = n\), where \(n\) is the number of pendant vertices.

**Observation:1:** Let \(T\) be a tree with \(n\) pendant vertices. Then there exists a path double cover \(\Psi\) of \(T\) such that exactly two paths of \(\Psi\) end at a given vertex of degree \(\geq 2\) and \(|\Psi| = n + 1\).

For, in a minimum path double cover \(\Psi\) of \(T\), every vertex of degree \(\geq 2\) is not an end vertex of a path in \(\Psi\) and \(|\Psi| = n\).
In [32], Arumugam and Meena found path double cover number for unicyclic graphs (see 1.69). If $m = 2$ ($m$ as in 6.1.6) and the two vertices on $C$ are of degree $> 3$ then we prove $\gamma_2(G) = n$. They proved the theorem in [32] by induction on number of pendant vertices. However, we give our proof here for the cases $m = 0$, $m = 1$, $m = 2$ separately and for $m \geq 3$, the proof is by induction on $m$.

6.1.6. **Proposition:** Let $G$ be a unicyclic graph with $n$ pendant vertices. Let $C$ be the unique cycle in $G$ and let $m$ be the number of vertices of degree greater than 2 on $C$. Then

$$\gamma_2(G) = \begin{cases} n + 3 - m & \text{if } m \leq 1 \text{ or } m = 2 \text{ and a vertex on } C \text{ is of degree 3, and} \\ n & \text{otherwise} \end{cases}$$

**Proof:** Let $V(G) = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}$. Let $C = (u_1, u_2, \ldots, u_r, u_1)$

Case (i): If $m = 0$ then $G = C$ and $n = 0$. Clearly $\gamma_2(G) \geq n + 3 - m$

Let $m = 1$. In any path double cover of $G$ all the $n$ pendant vertices and at least two vertices in $C$ are exterior points.

Hence $\gamma_2(G) \geq n + 2 = n + 3 - m$.

Let $m = 2$ and $v$ be a vertex of degree 3 on $C$. Suppose there exists a path double cover $\Psi$ with $n$ paths covering all the edges of $G$ exactly twice. Since $m = 2$, $\deg(v) = 3$ on $C$ and $|\Psi| = n$, only two paths are through the edges of $C$. Then some edges of $C$ can not be covered by the paths of $\Psi$. This contradiction proves $\gamma_2(G) \geq n + 1 = n + 3 - m$. For the other cases, $\gamma_2(G) \geq n$ by lemma 6.1.1.
Case (ii) Let $m = 0$, $G = C$ and $n = 0$.

$$\Psi = \{(u_1, u_2, \ldots, u_\tau), (u_\tau, u_1, \ldots, u_2), (u_2, u_1, u_\tau)\}$$

is a path double cover for $C$. Hence $\gamma_2(G) \leq 3 = n + 3 - m$. Let $m = 1$ and $\deg(v) \geq 3$ where $v$ is the unique vertex of degree greater than 2 on $C$. Let $u_1, u_2$ be any two points on $C$ different from $v$. Let $C_1 = <u_1, \ldots, v>$, $C_2 = <v, \ldots, u_2>$ and $C_3 = <u_2, \ldots, u_1>$ so that $C = C_1 \cup C_2 \cup C_3$. Remove all the degree two vertices on $C$ and the resulting graph is a tree $G_1$. If $\deg(v) = 1$ in $G_1$ then there is a minimum path double cover $\Psi_1$ of $G_1$ such that there are two paths in $\Psi_1$ which end at $v$ and $|\Psi_1| = n + 1$. If $\deg(v) \geq 2$ in $G_1$, then by observation 1, it is possible to find a path double cover $\Psi_1$ of $G_1$ such that there are two paths which end at $v$ and $|\Psi_1| = n + 1$. So we have two paths $P_1$ and $P_2$ end at $v$ in a path double cover $\Psi_1$ of $G_1$ and $|\Psi_1| = n + 1$. Let $\Psi = \{\Psi_1 - \{P_1, P_2\}\} \cup \{C_2 \cup C_3 \cup P_1\} \cup \{P_2 \cup C_1 \cup C_3\} \cup \{C_1 \cup C_2\}$ and $|\Psi| = |\Psi_1| - 2 + 3 = n + 2 = n + 3 - m$.

Then we have the following observation:

If $m(G) = 1$ and $u_1, u_2$ be any two points on $C$ other than $v$ then each of them can be made external in two paths of a minimum path double cover of $G$.

Let $m = 2$ and let at least one of the vertices $u, v$ on $C$ be of degree 3, say, $\deg(u) \geq 3$ and $\deg(v) = 3$ in $G$. Remove the tree $T$ incident with $v$ from $G$ and the resulting graph is a unicyclic graph $G_1$ and $\deg(v) = 2$ in $G_1$. Clearly
Let $n_1$ be the number of pendant vertices of $G_1$. By (*) there is a minimum path double cover $\Psi_1$ of $G_1$ with $|\Psi_1| = n_1 + 2$ such that $v$ is an external vertex for two paths $Q_1$ and $Q_2$ of $\Psi_1$. Now by 6.1.5, there is a minimum path double cover $\Psi_2$ of $T$ such that $|\Psi_2| = n - n_1 + 1$. Since $\text{deg}(v) = 1$ in $T$, there are two path $P_1, P_2$ in $\Psi_2$ end at $v$. Let $\Psi = \{ \Psi_1 - \{Q_1, Q_2\} \} \cup \{ \Psi_2 - \{P_1, P_2\} \} \cup \{P_1 \cup Q_1\} \cup \{P_2 \cup Q_2\}$ and $|\Psi| = n_1 + n - n_1 - 1 + 2 = n + 1$. Now, if $m = 2$ and $u, v \in V(C)$ such that $\text{deg}(u) > 3$, $\text{deg}(v) > 3$ in $G$. Remove that trees $T_1, T_2$ incident at $u, v$ respectively from $G$ so that the resulting graph is a cycle. Let $n_1$ and $n_2$ be the number of pendant vertices of $T_1$ and $T_2$ respectively. So we have $n = n_1 + n_2$. Let $C_1 = \langle u, \ldots, v \rangle$ and $C_2 = C - C_1$. By 6.1.5., let $\Psi_i$ be a minimum path double cover of $T_i$ and $|\Psi_i| = n_i$ ($i = 1, 2$). So every vertex of degree $\geq 2$ in $T_i$ is an internal vertex of a path in $\Psi_i$ ($i = 1, 2$). Let $P, Q$ be two paths in $\Psi_1$ containing $u$ as an internal vertex. Let $R, S$ be two paths in $\Psi_2$ containing $v$ as an internal vertex. Divide the paths $P, Q$ at $u$ so that $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$. Similarly, divide the paths $R, S$ at $v$ so that $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$. Now, $\Psi = \{ \Psi_1 - \{P, Q\} \} \cup \{ \Psi_2 - \{R, S\} \} \cup \{P_1 \cup C_1 \cup R_1\} \cup \{P_2 \cup C_2 \cup R_2\} \cup \{Q_1 \cup C_1 \cup S_1\} \cup \{Q_2 \cup C_2 \cup S_2\}$ and $|\Psi| = n_1 - 2 + n_2 - 2 + 4 = n_1 + n_2 = n$. For $m \geq 3$ the proof is by induction on $m$. Let $m = 3$. Let $u, v$ and $w$ lie on $C$ such that $\text{deg}(u), \text{deg}(v), \text{deg}(w) \geq 3$ in $G$. Let $n_1, n_2, n_3$ be the number of
pendant vertices of trees in $G$ incident at $u, v$ and $w$ respectively. Clearly $n = n_1 + n_2 + n_3$. Remove the trees $T_1, T_2$ and $T_3$ incident at $u, v$ and $w$ respectively from $G$ so that the resulting graph in $G$ is a cycle. Let $C_1 = \langle u, ..., v \rangle, C_2 = \langle v, ..., w \rangle$ and $C_3 = \langle w, ..., u \rangle$ be paths on $C$ such that $C = C_1 \cup C_2 \cup C_3$. By observation 1 or by 6.1.5, let $\Psi_1, \Psi_2$ and $\Psi_3$ be path double covers for $T_1, T_2$ and $T_3$ respectively so that $|\Psi_i| = n_i + 1$ ($1 \leq i \leq 3$) and there are paths $P_1, P_2 \in \Psi_1$ which end at $u$, $Q_1, Q_2 \in \Psi_2$ which end at $v$ and $R_1, R_2 \in \Psi_3$ which end at $w$. Let $\Psi = \{ \Psi_1 - \{ P_1, P_2 \} \} \cup \{ \Psi_2 - \{ Q_1, Q_2 \} \} \cup \{ \Psi_3 - \{ R_1, R_2 \} \} \cup \{ P_1 \cup C_1 \cup C_2 \cup R_2 \} \cup \{ Q_1 \cup C_2 \cup C_3 \cup P_2 \} \cup \{ R_1 \cup C_3 \cup C_1 \cup Q_2 \}$ and $|\Psi| = n_1 - 1 + n_2 - 1 + n_3 - 1 + 3 = n$. Clearly $\Psi$ is a path double cover for $G$. Hence the result is true when $m = 3$. Assume that the result is true for $m < k$. Let $G$ be a unicyclic graph with $m = k > 3$. Let $u$ be on $C$ such that $\deg(u) \geq 3$ on $G$. Let $n_1$ be number of pendant vertices of the tree $T$ incident at $u$. Remove the tree $T$ incident at $u$. Let $G_1$ be the unicyclic graph obtained by the removal of $T$ from $G$ and $\deg(u) = 2$ in $G_1$. By observation 1 or by 6.1.5, let $\Psi_1$ be a path double cover of $T$ such that $|\Psi_1| = n_1 + 1$ and there are paths $P_1, P_2$ in $\Psi_1$ end at $u$. Now $m(G_1) = k - 1 \geq 3$. By induction hypothesis, let $\Psi_2$ be a minimum path double cover of $G_1$ such that $|\Psi_2| = n - n_1 =$ Number of pendant vertices of $G_1$. Since only the pendant vertices of $G_1$ are end vertices of paths in $\Psi_2$, we have $u$ is an internal vertex of a path, say, $Q$.
in \( \Psi \). Divide \( Q \) at \( u \) into two paths \( Q_1 \) and \( Q_2 \). Let \( \Psi = \{ \Psi_1 - \{ P_1, P_2 \} \} \cup \{ \Psi_2 - \{ Q \} \} \cup \{ P_1 \cup Q_1 \} \cup \{ P_2 \cup Q_2 \} \). Clearly \( \Psi \) is a path double cover of \( G \) and \( |\Psi| = n_1 - 1 + n - n_1 - 1 + 2 = n \). This completes the induction. Hence the proof.

6.1.7. Proposition: For the complete bipartite graph \( K_{m,n} \), \( \gamma_2(K_{m,n}) = n \) if \( m < n \).

Proof: Let \( V(K_{m,n}) = (A, B) \) where \( A = \{ u_0, u_1, \ldots, u_{m-1} \} \), \( B = \{ v_0, v_1, \ldots, v_{n-1} \} \). By Corollary 6.1.3. \( \gamma_2(K_{m,n}) \geq n \). Let \( P_1 = (v_1 u_1 v_{i+1} u_2 \ldots u_{m-1} v_{i+m-1} u_0 v_{i+m}) \), where \( 0 \leq i \leq n-1 \) and the indices \( i \) are taken modulo \( n \). \( \Psi = \{ P_i : 0 \leq i \leq n-1 \} \) is clearly a path double cover for \( K_{m,n} \) with \( n \) paths. Hence \( \gamma_2(K_{m,n}) = n \) if \( m < n \).

The result \( \gamma_2(P_m \times P_n) = 4 \) if \( m, n \geq 3 \) was found in [32]. However we give the proof by the figures of \( P_2 \times P_5, P_5 \times P_4 \) and \( P_5 \times P_5 \) for the cases \( m = 2, m \geq 3, n \) even and \( m \geq 3, n \) odd (See Fig. 1, Fig. 2 and Fig. 3).

6.1.8. Proposition: Let \( m, n \geq 2 \). Then \( \gamma_2(P_m \times P_n) = \begin{cases} 3 & \text{if } m = 2, \text{ or } n = 2 \\ 4 & \text{otherwise} \end{cases} \)

Proof: By lemma 6.1.2., \( \gamma_2(P_m \times P_n) \geq 3 \) if \( m = 2 \) or \( n = 2 \) and \( \gamma_2(P_m \times P_n) \geq 4 \) if \( m, n \geq 3 \). The reverse inclusion follows from Fig. 1, Fig. 2 and Fig. 3.

6.1.9. Proposition: Let \( m \geq 3, n \geq 3 \). \( \gamma_2(C_m \times C_n) = 5 \) if at least one of the numbers \( m \) and \( n \) is odd.

Proof: Since \( C_m \times C_n \) is a 4 regular graph, we have, \( \gamma_2(C_m \times C_n) \geq 5 \) by
6.1.3. Since at least one of the numbers \( m \) and \( n \) is odd, \( C_m \times C_n \) can be decomposed into two Hamiltonian cycles \( C_1 \) and \( C_2 \) by theorem 1.51. Let \( v \in V(C_m \times C_n) \). Since \( \deg(v) = 4 \), there exist four vertices \( u_1, u_2, u_3 \) and \( u_4 \) adjacent with \( v \) and exactly two of them together with \( v \) are on \( C_1 \) and the other two together with \( v \) are on \( C_2 \). Without loss of generality assume that \( \langle u_1, v, u_2 \rangle \) and \( \langle u_3, v, u_4 \rangle \) lie on \( C_1 \) and \( C_2 \) respectively.

\[
\text{Fig. 4}
\]

Since \( \deg(u_4) = 4 \), there are vertices \( u_5, u_6, u_7 \) together with \( v \) are adjacent with \( u_4 \) as in Fig 4. As before assume that \( \langle u_5, u_4, u_6 \rangle \) and \( \langle v, u_4, u_7 \rangle \) lie on \( C_1 \) and \( C_2 \) respectively. Let \( C_i^{(1)}, C_i^{(2)} \) be the two copies of \( C_i \) \((i = 1, 2)\). If \( u_2u_6 \) is in \( C_1 \) then \( \{ (C_1^{(1)} - (u_2u_6)), (C_1^{(2)} - (u_4u_6)), (C_2^{(1)} - (vu_3)), (C_2^{(2)} - (vu_4)), (u_3vuu_6u_2) \} \) is a path double cover for \( C_m \times C_n \). If \( u_2u_6 \) is in \( C_2 \) then \( \{ (C_1^{(1)} - (u_1v)), (C_1^{(2)} - (u_4u_6)), (C_2^{(1)} - (vu_4)), (C_2^{(2)} - u_2u_6)), (u_1vuu_6u_2) \} \) is a path double cover for \( C_m \times C_n \). Hence \( \gamma_2(C_m \times C_n) = 5 \).

6.1.10. Proposition \( \gamma_2(C_m \times K_2) = 4, m \geq 3 \).
Proof: Consider $C_m \times K_2$ as in Fig 5. By 6.1.3, $\gamma_2(C_m \times K_2) \geq 4$. Now we prove the other part.

![Figure 5](image)

If $m$ is even then take $P_1 = <u_0 u_1 v_1 v_2 u_2 u_3 v_3 \ldots v_{m-2} u_{m-2} u_{m-1}>$ and $P_2 = <u_0 v_0 u_1 u_2 v_2 u_3 v_3 \ldots u_{m-2} v_{m-2} v_{m-1} u_{m-1}>$.

If $m$ is odd then take $P_1 = <u_0 u_1 v_1 v_2 u_2 u_3 v_3 \ldots u_{m-2} v_{m-2} v_{m-1} u_{m-1} >$ and $P_2 = <u_0 v_0 u_1 u_2 v_2 u_3 v_3 \ldots v_{m-2} u_{m-2} u_{m-1} v_{m-1} >$.

Let $P_3 = <u_1 u_2 \ldots u_{m-1} u_0 v_0 v_{m-1} v_{m-2} \ldots v_2 v_1>$ and $P_4 = <u_1 u_0 u_{m-1} v_{m-1} v_0 v_1>$.

Clearly, $\{P_1, P_2, P_3, P_4\}$ is a path double cover for $C_m \times K_2$ and $\gamma_2(C_m \times K_2) \leq 4$.

Hence $\gamma_2(C_m \times K_2) = 4$.

6.1.11. Theorem: If $m, n \geq 3$ then $\gamma_2(P_m \times C_n) = 4$.

Proof: By 6.1.2. $\gamma_2(P_m \times C_n) \geq 4$ and the reverse inclusion follows from Fig. 6, Fig. 7, and Fig. 8 of $P_4 \times C_3, P_5 \times C_4$ and $P_5 \times C_5$ respectively for the cases $m \geq 3, n = 3$; $m \geq 3, n$ even and $m \geq 3, n$ odd.
To prove 6.1.12 we describe the following. We view the graph $G \circ H$ as follows: Let $v_1, v_2, \ldots, v_n$ be the vertices of $G$ and let $u_1, u_2, \ldots, u_m$ be the vertices of $H$. Then $V(G \circ H) = V(G) \times V(H) = \bigcup_{i=1}^{n} \{v_i \times V(H)\} = \bigcup_{i=1}^{n} \{v_i \times \{u_1, u_2, \ldots, u_m\}\}$.

For our convenience, we denote $v_i \times V(H)$ by $V_i = \{u_1^i, u_2^i, \ldots, u_m^i\}$, $1 \leq i \leq n$, where $u_j^i$ stands for $(v_i, u_j)$. We call $V_i$ as the set of vertices of $G \circ H$ that corresponds to the vertex $v_i$ of $G$. Then we may write $V(G \circ H) =$
Clearly, for each edge $v_i v_j \in E(G)$ the subgraph of $G \circ K_m$ induced by $V_i \cup V_j$ is $K_{V_i, V_j} \setminus \{\alpha_i(V_i, V_j)\}$, where $\alpha_k(V_i, V_j) = \{u_1^i u_k^j, u_2^i u_{k+1}^j, u_3^i u_{k+2}^j, \ldots, u_m^i u_{k-1}^j\}$. Clearly $\alpha_k(V_i, V_j), 1 \leq k \leq m$, is a 1-factor of $K_{V_i, V_j}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ give a 1-factorization of $K_{V_i, V_j}$.

Let $K_{r(n)}$ be the complete $r$-partite graph with each partite set containing $n$ vertices. By 6.1.2. and by 1.70, it follows that unless $n$ is odd and $r$ is even we have $(r-1)n+1 \leq \gamma_2(K_{r(n)}) \leq (r-1)n+2$.

We give the relation $K_r \circ K_n = K_{r(n)} \setminus \bigcup_{v, v' \in E(K_r)} \{\alpha_1(V_i, V_j)\}$.

6.1.12. Theorem: For $r \geq 3$, $(K_r \circ K_{2n+1})^{(2)}$ is decomposed into $2n(r - 1)$ hamiltonian paths and $n$ vertex disjoint cycles.

Proof: Let $V(K_r) = \{v_0, v_1, v_2, \ldots, v_{r-1}\}$ and let $V(K_{2n+1}) = \{u_0, u_1, \ldots, u_{2n}\}$.

Throughout this theorem whenever we calculate modulo $s$, we always use $1, 2, \ldots, s$ as residues instead of $0, 1, 2, \ldots, s-1$. Let $C = (u_1, u_2, \ldots, u_{2n+1}, u_1)$ be a cycle.

Define for $1 \leq i \leq r-1$,

$$H_i = \begin{cases} (v_o v_i v_{i+1} v_{i+2} v_{i+3} \ldots v_{i+k-1} v_{i+k} v_o) & \text{when } r = 2k \\ (v_o v_i v_{i+1} v_{i-1} v_{i+2} v_{i+3} \ldots v_{i+k+1} v_{i+k} v_o) & \text{when } r = 2k +1 \end{cases}$$

where indices are taken modulo $r - 1$ except the index 0. Clearly $\{H_1, H_2, \ldots, H_{r-1}\}$ is edge disjoint union of hamiltonian cycles covering all the edges of $K_r^{(2)}$ exactly once.
Define for $1 \leq i \leq n$, $C_i = (u_0 u_i u_{i+1} u_{i-1} u_{i+2} u_{i-2} \ldots u_{n+i-1} u_{n+i+1} u_{n+i} u_0)$

where the subscripts of $u$'s except the index 0 are taken modulo $2n$. First we define the permutations $\rho_i$ ($1 \leq i \leq r-1$) and $\beta_j$ ($1 \leq j \leq n$) as follows:

\[
\rho_i = \begin{cases} 
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & r-1 & r \\
0 & i & i+1 & i-1 & i+2 & i-2 & i+3 & i+1 & i+k & i+k 
\end{cases} \quad \text{when } r = 2k,
\]

\[
\rho_i = \begin{cases} 
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & r-1 & r \\
0 & i & i+1 & i-1 & i+2 & i-2 & i+3 & i+k & i+k 
\end{cases} \quad \text{when } r = 2k + 1
\]

where the indices are taken modulo $r-1$ except the index 0

and $\beta_j = \begin{cases} 
1 & 2 & 3 & 4 & 5 & 6 & \ldots & 2n-1 & 2n & 2n+1 \\
0 & j & j+1 & j-1 & j+2 & j-2 & \ldots & n+j-1 & n+j+1 & n+j 
\end{cases} \quad \text{where the indices are taken modulo } 2n \text{ except the index 0}.

Now $H_i \circ C$ ($1 \leq i \leq r-1$) can be decomposed into hamiltonian cycles $H_i'$ and $H_i''$ and the construction is as follows:

When $r = 2k$ and $1 \leq i \leq r-1$

\[
H_i' = \bigcup_{j=1}^{k-1} \{ \alpha_2(V_{\rho_i(2j)}, V_{\rho_i(2j+1)}) \cup \alpha_{2n+1}(V_{\rho_i(2j+1)}, V_{\rho_i(2j+2)}) \}
\cup \{ \alpha_2(V_{\rho_i(1)}, V_{\rho_i(2)}) \cup \alpha_{2n+1}(V_{\rho_i(1)}, V_{\rho_i(2k)}) \}
\]

\[
H_i'' = \bigcup_{j=1}^{k-1} \{ \alpha_{2n+1}(V_{\rho_i(2j)}, V_{\rho_i(2j+1)}) \cup \alpha_2(V_{\rho_i(2j+1)}, V_{\rho_i(2j+2)}) \}
\cup \{ \alpha_{2n+1}(V_{\rho_i(1)}, V_{\rho_i(2)}) \cup \alpha_2(V_{\rho_i(1)}, V_{\rho_i(2k)}) \}.
\]
When \( r = 2k+1 \) and \( 1 \leq i \leq r-1 \),

\[
H_i' = \bigcup_{j=1}^{k-1} \{ \alpha_{2n+1}(V_{p_i(2j)}, V_{p_i(2j+1)}) \cup \alpha_2(V_{p_i(2j+1)}, V_{p_i(2j+2)}) \\
\cup \{ \alpha_{2n+1}(V_{p_i(2k)}, V_{p_i(2k+1)}) \cup \alpha_2(V_{p_i(1)}, V_{p_i(2)}) \} \cup \\
\alpha_{2n+1}(V_{p_i(1)}, V_{p_i(2k+1)}) \}
\]

\[
H_i'' = \bigcup_{j=1}^{k-1} \{ \alpha_2(V_{p_i(2j)}, V_{p_i(2j+1)}) \cup \alpha_{2n+1}(V_{p_i(2j+1)}, V_{p_i(2j+2)}) \} \\
\cup \{ \alpha_2(V_{p_i(2k)}, V_{p_i(2k+1)}) \cup \alpha_{2n+1}(V_{p_i(1)}, V_{p_i(2)}) \} \cup \\
\alpha_2(V_{p_i(1)}, V_{p_i(2k+1)}) \}
\]

Clearly \( H_i' \) and \( H_i'' \) (\( 1 \leq i \leq r-1 \)) are edge-disjoint hamiltonian cycles covering all the edges of \((K_r \circ C)^{(2)}\) exactly once. Let \( \beta_j(C) = (u_{\beta_j(1)}, u_{\beta_j(2)}, \ldots, u_{\beta_j(2n+1)}) \). So we have \( \beta_j(C) = C_j \). Similarly, we decompose \( H_i \circ \beta_j(C), 1 \leq i \leq r-1 \) and \( 1 \leq j \leq n \) into hamiltonian cycles \( H_{ij}' \) and \( H_{ij}'' \). As \( K_r \circ (C_1 \cup C_2 \cup \ldots \cup C_n) = (K_r \circ C_1) \cup (K_r \circ C_2) \cup \ldots \cup (K_r \circ C_n) \), \( (K_r \circ C_{2n+1})^{(2)} \) is decomposed into \( 2n(r-1) \) hamiltonian cycles (See Fig. 9 for \((K_8 \circ K_7)^{(2)}\)). For each fixed \( j(1 \leq j \leq n) \), remove an edge from each of the cycles \( H_{ij}' \) and \( H_{ij}'' \) (\( 1 \leq i \leq r-1 \)) to form matchings \((u_{\beta_j(0_{2j-1})}^i, u_{\beta_j(0_{2j})}^{i+1})\) and \((u_{\beta_j(0_{2j})}^i, u_{\beta_j(0_{2j-1})}^{i+1})\) (\( 1 \leq i \leq r-1 \)) respectively where \( l_k \in \{1,2,3, \ldots, 2n+1\} \) \( (1 \leq k \leq 2n) \) and the superscripts are taken modulo \( r-1 \). Here we choose \( l_i \) \( (1 \leq i \leq 2n) \) in such a way that the numbers in \( \{\beta_j(0_{2j-1}), \beta_j(0_{2j}) : 1 \leq j \leq n\} \) are distinct and \((\beta_j(l_{2j-1}), \beta_j(l_{2j}))\) is an edge of \( C_j (1 \leq j \leq n) \).
\[ \text{For each } j \ (1 \leq j \leq n), \text{ union of the two matchings form a cycle } C_j' \text{ of length } 2r-2. \]

We are using the above theorem in the following example.

\textbf{6.1.13. Example : } \gamma_2(K_r \circ K_7) = 6r - 5, \text{ when } r \text{ is even.}

\textbf{Proof : } Let \( r = 2k \). By 6.1.4. \( \gamma_2(K_r \circ K_7) \geq 6r - 6+1 = 6r - 5 \). By 6.1.12, \((K_r \circ K_7)^{(2)}\) is decomposed into \(6(r-1)\) hamiltonian paths and 3 vertex disjoint cycles. Here we choose \( l_1 = 7, l_2 = 1, l_3 = 2, l_4 = 3, l_5 = 5 \) and \( l_6 = 6 \). Now, \( \beta_1(7) = 4, \beta_1(1) = 0, \beta_2(2) = 2, \beta_2(3) = 3, \beta_3(5) = 5 \) and \( \beta_3(6) = 1 \). Clearly \((4,0), (2,3)\) and \((5,1)\) are the edges of \( C_1, C_2 \) and \( C_3 \) respectively. So we have 3 cycles as in 6.1.12.

\[ C_1' = (u_4^1 u_2^0 u_4^3 u_0^4 \ldots u_4^{r-1} u_0^1 u_4^2 u_0^3 \ldots u_0^{r-1} u_4^1) \]
\[ C_2' = (u_2^1 u_3^2 u_2^3 u_3^4 \ldots u_2^{r-1} u_3^1 u_2^2 \ldots u_3^{r-1} u_2^1) \text{ and} \]
\[ C_3' = (u_5^1 u_1^2 u_5^3 u_1^4 \ldots u_5^{r-1} u_1^1 u_5^2 \ldots u_1^{r-1} u_5^1) \]

Delete \((u_4^1 u_0^2), (u_2^k u_3^{k+1})\) and \((u_1^{r-1} u_5^1)\) from \( C_1', C_2', \text{ and } C_3' \) respectively. Choose \((u_4^1 u_5^0 u_3^{k+1})\) instead of \((u_4^1 u_0^2)\) from \( H_{1,1}' \). Select a path segment \((u_2^k u_0^0 u_5^1)\) instead of \((u_2^k u_3^{k+1})\) from \( H_{k,2}' \). Choose \((u_1^{r-1} u_6^0)\) instead of \((u_1^{r-1} u_5^1)\) from \( H_{r-1,3}'' \). Then we will get new paths.

\[ P_1 = (u_0^2 u_4^3 u_0^4 \ldots u_0^{r-1} u_4^1 u_5^0 u_3^{k+1}) \]
\[ P_2 = (u_3^{k+1} u_2^{k+2} \ldots u_3^{k-1} u_2^k u_0^0 u_5^1) \text{ and} \]
\[ H_{ij}' (1 \leq i \leq 7, 1 \leq j \leq 3) \]

\[ H_{ij}'' (1 \leq i \leq 7, 1 \leq j \leq 3) \]

\[ K_8 \circ K_7 \]

Fig. 9

\[ P_3 = (u_{5^1}u_1u_{5^3}u_{1^4}...u_{5^{r-1}}u_{1^1}u_{5^2}...u_{1^{r-1}}u_6^0) \]

\[ P = P_1 \cup P_2 \cup P_3 \] is clearly a path. So we have \( \gamma_2(K_7 \circ K_7) \leq 6r - 5. \)

Hence \( \gamma_2(K_r \circ K_7) = 6r - 5. \)

6.1.14. Remark: Let \( P_n(G) \) be the minimum number of paths required to cover all the edges of \( G \) exactly once. Clearly \( \gamma_2(G) \leq 2P_n(G) \). By 1.39 and by 6.1.4, we have \( \gamma_2(K_{2n}) = 2P_n(K_{2n}) \). It is not true in general that \( P_n(G) < \gamma_2(G) \).

Let \( G = C_m \times K_2 \). By 6.1.10 and by 1.38, \( \gamma_2(G) < P_n(G) \) if \( m \geq 5 \) and \( \gamma_2(G) = P_n(G) \) if \( m = 4 \). But for a tree \( T \), \( P_n(T) \leq n-1 < n = \gamma_2(T) \).
6.2. PATH COVERS OF DIGRAPHS

In [3] path decomposition of a digraph is studied. In [4], path number of tournaments is studied. In this section we determine path number of certain standard digraphs. Throughout this chapter D stands for a weakly connected digraph without loops or multiple arcs. By a path in D we mean a directed path. In [3], it is defined that \( \text{dg}(v) = \text{od}(v) - \text{id}(v) \). If more than one digraph is under consideration we write \( \text{dg}_D(v) \) relative to D. The quantity \( x(v) = \max(0, \text{dg}(v)) \) called as the excess at \( v \) and the excess of the digraph \( D \) is defined to be \( x(D) = \sum_{v \in V(D)} x(v) \). It was also shown in [4] that for any digraph \( D, x(D) \leq \text{Pn}(D) \). Digraphs for which equality holds will be called consistent.

6.2.1. Definition : Let \( D = (V,E) \) be a digraph. Let \( \Psi \) be a collection of directed paths in \( D \) satisfying the following conditions.

(i) Every path in \( \Psi \) has atleast length one.

(ii) Every arc of \( D \) is in exactly one path of \( \Psi \)

Then \( \Psi \) is called a path cover of \( D \). Let \( G \) denote the set of all path covers of \( D \). \( E(D) \) is trivially a path cover of \( D \) and hence \( G \) is non–empty. Let \( \text{Pn}(D) = \min_{\Psi \in G} |\Psi| \). \( \text{Pn}(D) \) is called the path number of \( D \). When there is no possibility of confusion, we write \( \text{Pn} \) instead of \( \text{Pn}(D) \). A path cover \( \Psi \) of \( D \) with \( |\Psi| = \text{Pn} \) is called a minimum path cover of \( D \).
6.2.2. Example

Consider the digraph D given in the figure.

![Digraph D](image)

Let $P_1 = (2 3 7 5 6 1)$, $P_2 = (4 5 7 1 2)$, $P_3 = (1 7 3 4 8 9 6)$ and $P_4 = (9 8)$.

Then $\Psi = \{ P_1, P_2, P_3, P_4 \}$ is a minimum path cover of D and $P_n(D) = 4$.

Let $\Psi$ be a path cover of a directed graph D. Let $i(v, \Psi)$ denote the number of paths in $\Psi$ having $v$ as an internal vertex and let $i_\Psi = \max_{v \in V(D)} i(v, \Psi)$.

6.2.3. Theorem: $P_n(D) = \frac{1}{2} \sum_{v \in V(D)} |dg(v)| + \sum_{v \in V(D)} \min(id(v), od(v)) - i$, where $i = \max i_\Psi$ and the maximum is taken over all path covers $\Psi$ of D.

Proof: $2|\Psi| = \text{The number of external vertices of directed paths in } \Psi$

\[
= \sum_{v \in V(D)} |dg(v)| + \sum_{v \in V(D)} 2 \left( \min(id(v), od(v)) - i(v, \Psi) \right)
\]

\[
|\Psi| = \frac{1}{2} \sum_{v \in V(D)} |dg(v)| + \sum_{v \in V(D)} \min(id(v), od(v)) - \sum_{v \in V(D)} i(v, \Psi)
\]
\[ = \frac{1}{2} \sum_{v \in V(D)} |\text{dg}(v)| + \sum_{v \in V(D)} \min(\text{id}(v), \text{od}(v)) - i \]

Hence \( Pn(D) = \frac{1}{2} \sum_{v \in V(D)} |\text{dg}(v)| + \sum_{v \in V(D)} \min(\text{id}(v), \text{od}(v)) - i \) 

6.2.4. **Corollary**: For any digraph \( D \), \( Pn(D) \geq \frac{1}{2} \sum_{v \in V(D)} |\text{dg}(v)| \). Moreover the following statements are equivalent.

(i) \( Pn(D) = \frac{1}{2} \sum_{v \in V(D)} |\text{dg}(v)| \)

(ii) There exists a path cover \( \Psi \) of \( D \) for which \( i(v, \Psi) = \min(\text{id}(v), \text{od}(v)) \) for every vertex \( v \) of \( D \).

6.2.5. **Corollary** [8]: For a directed tree \( T \), \( Pn(T) = \frac{1}{2} \sum_{v \in V(T)} |\text{dg}(v)| \).

**Proof**: Clearly \( T \) has a path cover \( \Psi \) with \( i(v, \Psi) = \min(\text{id}(v), \text{od}(v)) \) for every vertex \( v \) in \( T \) and hence \( Pn(T) = \frac{1}{2} \sum_{v \in V(T)} |\text{dg}(v)| \).

6.2.6. **Corollary**: If \( G \) is a 3–regular graph then there exists an orientation \( D \) of \( G \) such that \( Pn(D) = \frac{1}{2} \sum_{v \in V(D)} |\text{dg}(v)| \).

**Proof**: By 1.38 and 1.42, there exists a path cover \( \Psi \) of \( G \) in which every vertex is an internal vertex of exactly one path in \( \Psi \) and \( Pn(G) = \frac{p}{2} \). Let \( D \) be the digraph obtained by orienting the edges of \( G \) in such a way that each path in \( \Psi \) is a directed path. Then \( \text{id}(v) = 1, \text{od}(v) = 2 \) or \( \text{id}(v) = 2, \text{od}(v) = 1 \) for

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every vertex \( v \) of \( D \). So we have, \( i(v, \Psi) = \min(\text{id}(v), \text{od}(v)) \) for every vertex \( v \) of \( D \). Hence by 6.2.4., \( P_n(D) = \frac{1}{2} \sum_{v \in V(D)} |d_{g_D}(v)| \).

6.2.7. Corollary: Let \( D = (V, E) \) be an isograph. Then \( P_n(D) = q - i \).

Proof: Since \( \text{od}(v) = \text{id}(v) \) for every \( v \in V(D) \) (see 1.78) and also by 6.2.3 we have, \( P_n(D) = \sum_{v \in V(D)} \text{od}(v) - i = q - i \).

6.2.8. Remark: \( P_n(D) = \frac{1}{2} \sum |d_{g(u)}| + \sum \min(\text{id}(u), \text{od}(u)) - i \)

\[ = \frac{1}{2} \sum [\max(d_{g(u)}, 0) - \min(d_{g(u)}, 0)] + \sum \min(\text{id}(u), \text{od}(u)) - i \]

\[ = \frac{1}{2} x(D) - \frac{1}{2} \sum \min(d_{g(u)}, 0) + \sum \min(\text{id}(u), \text{od}(u)) - i \]

\[ = \frac{1}{2} x(D) + \sum [\min(\text{id}(u), \text{od}(u)) - \min(d_{g(u)}, 0)] + \frac{1}{2} \sum \min(d_{g(u)}, 0) - i \]

\( P_n(D) = \frac{x(D)}{2} + \sum \text{id}(u) + \frac{1}{2} \sum \min(d_{g(u)}, 0) - i \)

\[ = \frac{x(D)}{2} + q + \frac{1}{2} \sum \min(d_{g(u)}, 0) - i \]

6.2.9. Theorem: If \( D \) is consistent then \( P_n(D) = 2(q - i) + \sum \min(d_{g(u)}, 0) \).

Proof: If \( D \) is consistent then \( P_n(D) = x(D) \). The result follows from the above remark.

The following two lemmas are useful in theorem 6.2.13.
6.2.10. **Lemma:** Let $D$ be a digraph and $P_n(D) = \frac{1}{2} \sum_{u \in V(D)} |dg(u)|$. Let $vw$ be an arc of $D$. Let $D'$ be a digraph obtained by replacing $vw$ by a directed path $P$ joining $v$ and $w$. Then $P_n(D) = P_n(D')$.

**Proof:** Let $\Psi$ be a minimum path cover for $D$. Let $Q$ be a path of $\Psi$ containing the arc $vw$. Let $P_1$ be a path obtained from $Q$ by replacing the arc $vw$ by the directed path $P$. Then $\{ \Psi - \{Q\} \} \cup \{ P_1 \}$ is a path cover for $D'$.

Hence $P_n(D') \leq P_n(D)$. Clearly $P_n(D') \geq \frac{1}{2} \sum_{u \in V(D')} |dg_{D'}(u)| = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)| = P_n(D)$.

6.2.11. **Lemma:** Let $D$ be a digraph. Let $vw \notin E(D)$ with $dg_D(v) \geq 0$, $dg_D(w) \leq 0$ and $P_n(D) = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)|$. Let $D'$ be a graph obtained from $D$ by adding the arc $vw$. Then $P_n(D') = \frac{1}{2} \sum_{u \in V(D')} |dg_{D'}(u)| = P_n(D) + 1$.

**Proof:** Let $\Psi$ be a minimum path cover for $D$. Let $\Psi_1 = \Psi \cup \{ vw \}$ be a path cover for $D'$. Hence $P_n(D') \leq |\Psi| + 1 = P_n(D) + 1$. As $P_n(D') \geq \frac{1}{2} \sum_{u \in V(D')} |dg_{D'}(u)|$ and $|dg_D(v)| + |dg_D'(w)| = |dg_D(v)| + |dg_D(w)| + 2$. We have $P_n(D') \geq \frac{1}{2} \sum_{u \in V(D')} |dg_{D'}(u)| = \frac{1}{2} \left[ \sum_{u \in V(D)} |dg_D(u)| + 2 \right] = P_n(D) + 1$. Hence the result.
6.2.12. **Theorem**: Let $D$ be a digraph. Let $(vw)$ be an arc of $D$ with $\text{dg}_D(v) \leq 0$, $\text{dg}_D(w) \geq 0$. Let $H$ be the digraph obtained from $D$ by reversing the arc $(vw)$. Then $D$ is consistent if and only if $H$ is consistent and $P_n(H) = P_n(D) + 2$.

**Proof**: Let $H$ be consistent and $P_n(H) = 2 + P_n(D)$. We have $\text{dg}_H(v) = \text{dg}_D(v) - 2$, $\text{dg}_H(w) = \text{dg}_D(w) + 2$ and $\text{dg}_H(u) = \text{dg}_D(u)$ for all other vertices. As $\text{dg}_D(v) \leq 0$, $\text{dg}_H(v) \leq 0$, $\text{dg}_D(w) \geq 0$ and $\text{dg}_H(w) \geq 0$ we have, $x(H) = x(D) + 2$. As $H$ is consistent, $P_n(D) = P_n(H) - 2 = x(H) - 2 = x(D)$. Hence $D$ is consistent. The converse follows from 1.81.

6.2.13. **Theorem**: Let $D$ be a unicyclic graph with unique directed cycle $C$ where the directed cycle is obtained from a cycle by orienting the cycle clockwise. Let $m$ denote the number of vertices of either out degree $> 1$ or in degree $> 1$ on $C$. Then

$$P_n(D) = \begin{cases} 
2 & \text{if } m = 0 \\
\frac{1}{2} \sum_{u \in V(D)} |\text{dg}_D(u)| + 1 & \text{if } m = 1 \\
\frac{1}{2} \sum_{u \in V(D)} |\text{dg}_D(u)| & \text{if } m \geq 2
\end{cases}$$

**Proof**: Case (i): If $m = 0$ then $D = C$ is a directed cycle and $P_n(D) = 2$.

Case (ii): Let $m = 1$. Let $v$ be the vertex on $C$ for which $\text{id}(v) > 1$ or $\text{od}(v) > 1$. In any minimum path cover $\Psi$ of $D$ atleast one vertex on $C$ other than $v$ is not internal in any path of $\Psi$ and hence $\sum_{v \in V(D)} (\min(\text{id}(v), \text{od}(v)) - i) \geq 1$. Hence

$$P_n(D) \geq \frac{1}{2} \sum_{u \in V(D)} |\text{dg}_D(u)| + 1.$$ 

Let $(wv)$ be a path on $C$. 

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Subcase a: Let $\text{dg}_D(v) > 0$. Let $D_1 = D - vs$ be a tree. Since $\text{dg}_D(v) > 0$, $\text{dg}_D(s) = 0$, we have $|\text{dg}_D(v)| = |\text{dg}_{D_1}(v)| + 1$ and $|\text{dg}_D(s)| = |\text{dg}_{D_1}(s)| - 1$. Let $\Psi'$ be a minimum path cover for $D_1$. Then $\Psi = \Psi' \cup \{(vs)\}$ is a path cover for $D$. Hence $P_n(D) \leq |\Psi| = P_n(D_1) + 1$. Since $D_1$ is a tree we have, $P_n(D_1) = \frac{1}{2} \sum_{u \in V(D_1)} |\text{dg}_{D_1}(u)|$. 

Now, $P_n(D) \leq P_n(D_1) + 1 = \frac{1}{2} \sum_{u \in V(D_1)} |\text{dg}_{D_1}(u)| + 1 = \frac{1}{2} \sum_{u \in V(D)} |\text{dg}_D(u)| + 1$. Hence $P_n(D) = \frac{1}{2} \sum_{u \in V(D)} |\text{dg}_D(u)| + 1$.

Subcase b: Let $\text{dg}_D(v) < 0$. Let $D_1 = D - wv$ be a tree. Since $\text{dg}_D(v) < 0$, $|\text{dg}_D(v)| = |\text{dg}_{D_1}(v)| + 1$ and $|\text{dg}_D(w)| = |\text{dg}_{D_1}(w)| - 1$. Let $\Psi'$ be a minimum path cover for $D_1$. Then $\Psi = \Psi' \cup \{(wv)\}$ is a path cover for $D$. Hence $P_n(D) \leq |\Psi| = P_n(D_1) + 1$. Since $D_1$ is a tree, $P_n(D_1) = \frac{1}{2} \sum_{u \in V(D_1)} |\text{dg}_{D_1}(u)|$. Hence $P_n(D) \leq \frac{1}{2} \sum_{u \in V(D_1)} |\text{dg}_{D_1}(u)| + 1 = \frac{1}{2} \sum_{u \in V(D)} |\text{dg}_D(u)| + 1$.

Subcase c: Let $\text{dg}_D(v) = 0$. Clearly, $\text{od}_D(v) \geq 2$ and $\text{id}_D(v) \geq 2$. Let $D_1 = D - wv$. Note that $|\text{dg}_D(v)| = |\text{dg}_{D_1}(v)| - 1$ and $|\text{dg}_D(w)| = |\text{dg}_{D_1}(w)| - 1$. Let $P = C - wv$. Let $\Psi'$ be a minimum path cover for $D_1$. Then either $P \in \Psi'$ or there exists a path $P_1 \in \Psi'$ contains $P$.

Let $P_1 \in \Psi'$. Since $\text{dg}_{D_1}(v) > 0$ there exists a path $Q \in \Psi''$ which starts at $v$ and contains no arc of $C$. Define $Q' = Q \cup \{(wv)\}$. Now $\Psi = \{\Psi'' - Q\} \cup \{Q'\}$ is a path cover for $D$. Hence $P_n(D) \leq |\Psi| = |\Psi'| = P_n(D_1) =$ 151
\[
\frac{1}{2} \sum_{u \in V(D)} |dg_{D_1}(u)| = \frac{1}{2} \left( \sum_{u \in V(D)} |dg_D(u)| + 2 \right) = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)| + 1. \text{ Hence } P_n(D) = \\
\frac{1}{2} \sum_{u \in V(D)} |dg_D(u)| + 1.
\]

Let \( P \in \Psi' \). There exists a path \( P_2 \in \Psi' \) contains \( u \) as an internal vertex and containing no arc of \( C \), as \( od_{D_1}(v) \geq 2 \) and \( id_{D_1}(v) \geq 1 \). Divide \( P_2 \) into two paths \( P_2' \) and \( P_2'' \) such that \( P_2' \) ends at \( v \) and \( P_2'' \) starts at \( v \). Let \( P_3 = P_2' \cup P \) and \( P_4 = P_2'' \cup \{(uv)\} \). Now \( \Psi' = \Psi - \{ P, P_2 \} \cup \{ P_3, P_4 \} \). \( P_n(D) \leq |\Psi| = |\Psi'| = \frac{1}{2} \sum_{u \in V(D_1)} |dg_{D_1}(u)| = \frac{1}{2} \left( \sum_{u \in V(D)} |dg_D(u)| + 2 \right) = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)| + 1.
\]

Case iii: Let \( m \geq 2 \). Let \( v, w \in V(C) \) such that every vertex on the directed path from \( v \) to \( w \) of \( C \) has in degree one and out degree one and \( id(v) > 1 \) or \( od(v) > 1 \) and \( id(w) > 1 \) or \( od(w) > 1 \). With out loss of generality assume that \( vw \) is an arc of \( D \), as we are going to prove that \( P_n(D) = \sum_{u \in V(D)} |dg_D(u)| \) and by 6.2.10, \( P_n(D) \) is not altered if we replace \( vw \) by a directed path joining \( v \) and \( w \). Clearly \( P_n(D) \geq \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)| \). Let \( D_1 = D - vw \). By 6.2.5., \( P_n(D_1) = \\
\frac{1}{2} \sum_{u \in V(D_1)} |dg_{D_1}(u)| \).

Subcase a: Let \( dg_D(v) > 0 \) and \( dg_D(w) < 0 \). Then \( dg_{D_1}(v) \geq 0 \) and \( dg_{D_1}(w) \leq 0 \). By 6.2.11, \( P_n(D) = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)| \).
Subcase b: Let $dg_D(v) \leq 0$ and $dg_D(w) \geq 0$. There exists a minimum path cover $\Psi$ of $D_1$ such that there is a path $P_1$ which ends at $v$ and there is a path $P_2$ which starts at $w$ and both contain no arc of $C$. Let $P = P_1 \cup (vw) \cup P_2$. Clearly $\Psi_1 = \{\Psi - \{P_1, P_2\}\} \cup \{P\}$ is a path cover of $D$. Moreover, $|dg_D(v)| = |dg_{D_1}(v)| - 1$ and $|dg_D(w)| = |dg_{D_1}(w)| - 1$. Now $P_n(D) \leq |\Psi_1| = |\Psi| - 1 = P_n(D_1) - 1 = \frac{1}{2} \sum_{u \in V(D_1)} |dg_{D_1}(u)| - 1 = \frac{1}{2} (\sum_{u \in V(D_1)} |dg_{D_1}(u)| - 2) = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)|$.

Hence $P_n(D) = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)|$.

Subcase c: Let $dg_D(v) \leq 0$ and $dg_D(w) < 0$. It is possible to find a minimum path cover $\Psi'$ of $D_1$ such that $\Psi'$ contains a path $Q$ which ends at $v$ and does not contain any arc of $C$. Let $\Psi = \{\Psi' - Q\} \cup \{Q \cup (vw)\}$. Clearly $\Psi$ is a path cover for $D$. Also $|dg_D(v)| = |dg_{D_1}(v)| - 1$ and $|dg_D(w)| = |dg_{D_1}(w)| + 1$.

$P_n(D) \leq |\Psi| = |\Psi'| = \frac{1}{2} \sum_{u \in V(D_1)} |dg_{D_1}(u)| = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)|$. Hence $P_n(D) = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)|$.

Subcase d: Let $dg_D(v) > 0$ and $dg_D(w) \geq 0$. Then $dg_{D_1}(v) \geq 0$ and $dg_{D_1}(w) > 0$. There exists a minimum path cover $\Psi'$ of $D_1$ such that $\Psi'$ contains a path $Q$ which starts at $w$ and contains no arc of $C$. Let $P = (vw) \cup Q$. Clearly
\[ \Psi = \{ \Psi' - \{ Q \} \} \cup \{ P \} \] is a path cover for D. Note that \(|dg_D(v)| = |dg_{D_1}(v)| + 1\) and \(|dg_D(w)| = |dg_{D_1}(w)| - 1\). \(Pn(D) \leq |\Psi| = |\Psi'| = \frac{1}{2} \sum_{u \in V(D)} |dg_{D_1}(u)| = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)|.\) Hence \(Pn(D) = \frac{1}{2} \sum_{u \in V(D)} |dg_D(u)|.\)

6.2.14. Theorem: Let D be an anti-symmetric digraph. Then \(Pn(D) = |E(D)|\) if and only if \(id(v) = 0\) or \(od(v) = 0\) for all vertices \(v\) of D.

**Proof:** Let \(id(v) = 0\) or \(od(v) = 0\) for all vertices \(v\) of D. \(Pn(D) \geq \frac{1}{2} \sum |dg(v)| = \frac{1}{2} \sum (od(v) + id(v)) = q(D).\) Always \(Pn(D) \leq q.\) Hence \(Pn(D) = q.\) Conversely, let \(Pn(D) = q.\) Suppose there is a vertex \(v\) such that \(id(v) \geq 1\) and \(od(v) \geq 1\). Then there exists two different vertices \(u\) and \(w\) of D such that \(uv, vw \in E(D)\). Let \(P = (u \ v \ w).\) Now \(\Psi = \{ P \} \cup \{ E(G) - E(P) \}\) is a path cover of D and \(|\Psi| < q\) which is a contradiction.

6.2.15. Theorem: For any graph G, there exists an orientation D of G such that \(Pn(D) = Pn(G)\).

**Proof:** Let \(\Psi\) be a minimum path cover of G. Let D be the digraph obtained by orienting the edges of G in such a way that each path in \(\Psi\) is a directed path. Then \(Pn(D) = Pn(G).\)
If $P = (u_1, u_2, \ldots, u_n)$ then we define the directed paths $\bar{P} = (u_1, u_2, \ldots, u_n)$ and $\bar{P} = (u_n, \ldots, u_2, u_1)$.

**6.2.16. Theorem** Let $G$ be a $(p,q)$ graph. Let $D$ be the symmetric digraph obtained from $G$ by replacing each edge of $G$ by a symmetric pair of arcs. Then $P_n(D) \leq 2P_n(G)$

**Proof**: Let $P_n(G) = m$. Let $\Psi = \{P_1, P_2, P_3, \ldots, P_m\}$ be a minimum path cover of $G$. We orient the paths $P_i$ $(1 \leq i \leq m)$ in two different directions so that $\bar{P}_i, \bar{P}_i$ $(1 \leq i \leq m)$ are directed paths in $D$. Now $\bar{\Psi} = \{\bar{P}_i, \bar{P}_i : 1 \leq i \leq m\}$ is a path cover for $D$. Hence $P_n(D) \leq 2P_n(G)$.

**6.2.17 Corollary**: Let $D$ be a complete symmetric digraph on $2n$ vertices with $n \geq 2$. Then $P_n(D) = 2n$.

**Proof**: By 6.2.16 and by 1.39 it follows that $P_n(D) \leq 2n$. Also $P_n(D) \geq \frac{|E(D)|}{2n-1}$

$$= \frac{2n(2n-1)}{2n-1} = 2n.$$ Therefore, $P_n(D) = 2n$.