Chapter 3

Varieties of restrained domination
number of a graph

In this chapter we study six different domination parameters- Global restrained domination, Restrained double domination, Uniform restrained domination, Connected restrained domination, Total restrained domination and Efficient restrained domination. We obtain certain bounds for the respective domination numbers of the domination parameters mentioned above. We establish the sharpness for certain bounds and characterize graphs attaining some of the bounds. We also derive certain Nordhaus-Gaddum-type results for these domination numbers.

3.1 Introduction

Even though the concept of domination was introduced by Ore [38] in his book Theory of graphs in the year 1962, rapid growth has been made in the area only when Walikar et al. published their monograph [42] on domination.

Since then the vistas of domination was expanded by Graph theorists by defining
hundreds of domination parameters. These parameters are defined by imposing additional conditions on a dominating set.

In this chapter we study six different domination parameters namely Global restrained domination, Restrained double domination, Uniform restrained domination, Connected restrained domination, Total restrained domination and Efficient restrained domination.

Global domination, Double domination, Uniform domination, Connected domination, Total domination and Efficient domination have been introduced by Sampathkumar [39], Frank Harary, Teresa W. Haynes [25], Acharya [1], S.T. Hedetniemi [40] E.J. Cockayne, R.M. Dawes, S.T. Hedetniemi [14] E.J. Cockayne, Hartnell, Hedetniemi and Laskar [16] respectively. We extend the study of these parameters to a restrained dominating set.

### 3.2 Global restrained domination number of a graph

Definition 1.52 is extended as follows:

**Definition 3.2.1.** A subset $S$ of $V$ is called a global restrained dominating set if it is a restrained dominating set of both $G$ and $\bar{G}$. The minimum cardinality of a minimal global restrained dominating set is called global restrained domination number of a graph, denoted by $\gamma_{gr}(G)$.

**Example 3.2.2.**

(i) If $G \cong K_p$ then $\gamma_{gr}(G) = p$.

(ii) If $G \cong K_{3,3}$ then $\gamma_{gr}(G) = 2$. 

47
(iii) If \( G \cong G_1 \) where \( G_1 \) is given in Fig 3.1, then \( \gamma_{gr}(G) = 2 \).

The following are simple observations.

**Proposition 3.2.3.** A restrained dominating set \( S \) of a graph \( G \) is a global restrained dominating set if and only if for every \( v \in V - S \) there exists \( u \in S \) and \( w \in V - S \) such that \( v \) is not adjacent to \( u \) and \( w \).

**Proposition 3.2.4.** \( \gamma_r(G) \leq \gamma_{gr}(G) \), \( \gamma_{gr}(G) = \gamma_{gr}(\bar{G}) \) and so \( \gamma_{gr}(G) \geq \frac{\gamma_r(G) + \gamma_{gr}(\bar{G})}{2} \).

**Proposition 3.2.5.**

\[
\gamma_{gr}(G) \geq \frac{2q - p(p - 5)}{4}.
\]

**Proof.** By Proposition 3.2.3,

\[
q \leq \left( \frac{p}{2} \right) - 2(p - \gamma_{gr}(G))
\]

\[
= \frac{p(p - 1)}{2} - 2p + 2\gamma_{gr}(G)
\]

\[
= \frac{p(p - 5) + 4\gamma_{gr}(G)}{2}
\]

and so \( \gamma_{gr}(G) \geq \frac{2q - p(p - 5)}{4} \). \( \square \)

**Theorem 3.2.6.**

\[
\gamma_{gr}(C_p) = \gamma_r(C_p) \quad \text{for all } p \geq 6.
\]
Proof. By Theorem 1.73, $\gamma_r(C_p) = k + r$ where $p = 3k + r(0 \leq r \leq 2)$. Let $C_p = (v_1, v_2, \ldots, v_p, v_1)$.

If $r = 0$, since $p \geq 6$ we have $S = \{v_1, v_4, \ldots, v_{3k-2}\}$ is a restrained dominating set of $G$ and $\bar{G}$. Since $|S| = k, \gamma_{gr}(C_p) = \gamma_r(C_p)$. Proof is similar if $r = 1$ or $2$. \hfill $\square$

Theorem 3.2.7. Every minimal restrained dominating set in a nontrivial tree $T$ is also a minimal global restrained dominating set if and only if either $\text{diam}(T) \leq 2$ or $\text{diam}(T) \geq 8$.

Proof. Suppose $\text{diam}(T) \leq 2$. If $\text{diam}(T) = 1$, $T \cong K_2$ and if $\text{diam}(T) = 2$ then $T \cong K_{1,p-1}$ where $p-1 \neq 1$. In both the cases $\gamma_r(G) = p$ and so the result is obvious.

Suppose $\text{diam}(T) \geq 8$. Let $u$ and $v$ be the end vertices of a diametrical path. Suppose $A$ is a minimal restrained dominating set in $T$. Since $u$ and $v$ are pendant vertices, $\{u, v\} \subseteq A$ and also $\{u, v\}$ is a dominating set in $\bar{T}$ so that $A$ dominates $\bar{T}$.

Claim. $\langle V - A \rangle$ has no isolated vertices in $\bar{T}$.

If $\langle V - A \rangle$ has three or more isolated vertices in $\bar{T}$, then these vertices induce a complete graph in $T$ which is impossible.

Suppose $\langle V - A \rangle$ has two isolated vertices in $\bar{T}$, say $x_1$ and $x_2$. If $|V - A| = 2$ then every vertex $z \in A$ is adjacent to either $x_1$ or $x_2$ in $\bar{T}$ since otherwise they induce a triangle in $T$. Let $u_1$ and $v_1$ be the supports of $u$ and $v$ in $T$ respectively. Since $d(u, v) \geq 8$, if $u_1 = x_1$ [or equivalently $v_1 = x_2$] then $d(v_1, x_2) \geq 5[d(u_1, x_1) \geq 5]$ and now $A - \{v_1, v'_1\} \setminus \{A - \{u_1, u'_1\}\}$ is a restrained dominating set where $v'_1(u'_1)$ is the vertex adjacent to $v_1(u_1)$ in the $d(v_1, x_2)(d(u_1, x_1))$ path. Hence $u_1 \neq x_1$ and $v_1 \neq x_2$.

Since $d(u, v) \geq 8$, if $u_1 \in N(x_1)(v_1 \in N(x_2))$ then $d(v_1, x_2) \geq 4[d(u_1, x_1) \geq 4]$ and so as above we get a restrained dominating set of smaller cardinality which is a contradiction. Hence $d(u_1, x_1) \geq 2$ and $d(v_1, x_2) \geq 2$. Since $d(u, v) \geq 8$, either
\[ d(u_1, x_1) > 2 \text{ or } d(u_1, x_2) > 2 \]. Without loss of generality suppose \( d(u_1, x_1) > 2 \). Let \( z_2 \in N(u_1) \) in the \( d(u_1, x_1) \) path. Then \( A - \{z_2, u_1\} \) is a restrained dominating set which is a contradiction.

Since \( |V - A| \geq 2, \langle V - A \rangle \) has no isolated vertices in \( \bar{T} \).

Conversely suppose \( 2 < \text{diam} (T) < 8 \). If \( P_p = \langle 1, 2, \ldots, p \rangle \) is a path on \( p \) vertices, then for \( 4 \leq i \leq 8 \), \( P_i \) are paths having \( \text{diam} (P_i) = i - 1 \) and \{1, 4\}, \{1, 2, 5\}, \{1, 2, 3, 6\}, \{1, 2, 3, 6, 7\}, \{1, 2, 3, 6, 7, 8\} are minimal restrained dominating sets of them respectively which are not global. This contradiction shows that either \( \text{diam} (T) \leq 2 \) or \( \text{diam} (T) \geq 8 \).

**Definition 3.2.8.** A restrained dominating set \( S \subset V(G) \) is said to be a connected restrained dominating set when \( \langle S \rangle \) is connected. The minimum cardinality of a minimal connected restrained dominating set is defined to be the connected restrained domination number and is denoted by \( \gamma_{cr}(G) \).

**Proposition 3.2.9.** For any graph \( G \), at least one of the following holds:

(i) \( \gamma_{cr}(G) \leq \gamma_{gr}(G) \)

(ii) \( \gamma_{cr}(\bar{G}) \leq \gamma_{gr}(G) \).

**Proof.** If \( S \) is a minimum restrained global dominating set, then \( \langle S \rangle \) is connected either in \( G \) or in \( G \) and the proof follows.

**Theorem 3.2.10.** For any graph \( G \), \( \gamma_{gr}(G) = \gamma_r(G) \) if and only if there exists a \( \gamma_r \)-set \( S \) satisfying the following two conditions.

(i) \( S \not\subseteq N(u) \forall u \in V - S \).

(ii) \( \Delta((V - S)) < p - 1 - \gamma_r(G) \).
Proof. Let $S$ be a $\gamma_r$-set satisfying conditions (i) and (ii). Condition (i) guarantees that $S$ is a dominating set in $\bar{G}$ and condition (ii) shows that it is a restrained dominating set in $\bar{G}$. Hence $\gamma_{gr}(G) = \gamma_r(G)$.

Conversely since $\gamma_{gr}(G) = \gamma_r(G)$, there exists a $\gamma_r$-set, say $S$ which is also a restrained dominating set of $G$. Clearly $S$ satisfies conditions (i) and (ii). $\square$

Corollary 3.2.11. Let $G$ be a connected graph in which every non-pendant vertex is a support and if $D$ is the set of supports such that $\langle D \rangle \not\cong H + K_1$ for any graph $H$, then $\gamma_{gr}(G) = \gamma_r(G)$.

Proof. The set of all pendant vertices of $G$, $S$ forms a $\gamma_r$-set of $G$ satisfying conditions (i) and (ii) of theorem 3.2.10 and so the result follows. $\square$

We now characterize several classes of graphs $G$ with $\gamma_{gr}(G) = \gamma_r(G)$.

Proposition 3.2.12.

(1) If $G \cong K_{m,n}(3 \leq m \leq n)$ then $\gamma_{gr}(G) = \gamma_r(G)$.

(2) If $G \cong K_{1,n}(n \geq 1)$ then $\gamma_{gr}(G) = \gamma_r(G)$.

Proof. (i) Let $G \cong K_{m,n} = (X, Y)$ with $|X| = m, |Y| = n(3 \leq m \leq n)$. For any $x \in X$ and $y \in Y, S = \{x, y\}$ is a restrained dominating set of $G$ and $\bar{G}$ and so $\gamma_{gr}(G) = \gamma_r(G)$.

(ii) If $G \cong K_{1,n}(n \geq 1)$ then $\gamma_{gr}(G) = \gamma_r(G) = p$. $\square$

Theorem 3.2.13. For any tree $T, \gamma_{gr}(T) = \gamma_r(T)$ if and only if there exists at least one $\gamma_r$-set $S$ such that $\langle V - S \rangle$ is not a star.
Proof. Let $S$ be a $\gamma_r$-set such that $\langle V - S \rangle$ is not a star. If $V - S = \phi$ then by theorem 1.69 $T$ is a star and now $\gamma_{gr}(T) = \gamma_r(T) = p$. Otherwise, as $\langle V - S \rangle$ is not a star, it has no isolated vertices in $T$. Also $T$ has at least 2 distinct supports and so $S$ is a dominating set of $T$. Thus $\gamma_{gr}(T) = \gamma_r(T)$.

Conversely, if $\gamma_{gr}(T) = \gamma_r(T) = p$ then $T$ is a star. Suppose $\gamma_r(T) \leq p - 2$. If $\langle V - S \rangle$ is a star for every $\gamma_r$-set $S$, then the central vertex of the star say $v$ will be an isolated vertex of $\langle V - S \rangle$ in $\bar{T}$ and so no $\gamma_r$-set $S$ is a restrained dominating set of $\bar{T}$ which is a contradiction. Hence there exists at least one $\gamma_r$-set $S$ such that $\langle V - S \rangle$ is not a star. \[\square\]

Corollary 3.2.14. If $T$ is a tree which is not a star, then $\gamma_{gr}(T) = p$ if and only if $\langle V - S \rangle$ is a star for every $\gamma_r$-set $S$ of $T$.

Proof. Suppose $\langle V - S \rangle$ is a star for every $\gamma_r$-set $S$ of $T$. By theorem 3.2.13, $\gamma_{gr}(T) \neq \gamma_r(T)$. Also if $S$ is any $\gamma_r$-set of $T$ then for every $v \in V - S$, $S \cup \{v\}$ fails to be a restrained dominating set of either $T$ or $\bar{T}$ and so $\gamma_{gr}(T) = p$.

Conversely, since $T$ is not a star, $\gamma_{gr}(T) \neq \gamma_r(T)$ and so by theorem 3.2.13, $\langle V - S \rangle$ is a star for every $\gamma_r$ set $S$ of $T$. \[\square\]

Remark 3.2.15. If $G$ is any graph with $\gamma_r(G) = p - 2$ or $p - 3$ then $\gamma_{gr}(G) = p$.

Corollary 3.2.16. For any tree $T$, $\gamma_{gr}(T) = \gamma_r(T)$ if and only if diam $T \neq 3, 4$ and $T$ is non-isomorphic to $P_6, T_1$ or $T_2$ where $T_1$ and $T_2$ are as given in Fig (3.2).

Proof. Let $T$ be a tree with diam $T \neq 3, 4$ and $T \not\cong P_6, T_1$ and $T_2$. By theorem 3.2.7 $\gamma_{gr}(T) = \gamma_r(T)$ for all trees $T$ with either diam $(T) \leq 2$ or diam $(T) \geq 8$. If diam $T = 5$, $\langle V - S \rangle$ is $P_4$ or a bistar for every $\gamma_r$-set $S$. If diam $T = 6$, for every

\[52\]
\( \gamma_r \)-set \( S \) of \( G, \langle V - S \rangle \) is \( P_3 \) or a galaxy. Also if diam \( T = 7, \langle V - S \rangle \) is not a star for all \( \gamma_r \)-sets of \( S \) and so by theorem 3.2.13, \( \gamma_{gr}(T) = \gamma_r(T) \) in all these cases.

Conversely let \( \gamma_{gr}(T) = \gamma_r(T) \). If diam \( (T) = 3, T \) has a unique \( \gamma_r \)-set, and if diam \( (T) = 4, T \) has two \( \gamma_r \)-set \( S_1 \) and \( S_2 \) and \( \langle V - S \rangle = \langle V - S_1 \rangle = \langle V - S_2 \rangle = P_2 \). Thus by theorem 3.2.13, \( \gamma_{gr}(T) \neq \gamma_r(T) \). Similarly if \( T \cong P_6, T_1 \) or \( T_2 \), for every \( \gamma_r \)-set \( S, \langle V - S \rangle \) is a star and hence the result follows.

**Theorem 3.2.17.** Let \( G \) be a connected cubic graph. Then \( \gamma_{gr}(G) = \gamma_r(G) = 2 \) if and only if \( G \cong G_i (1 \leq i \leq 5), \) where \( G_i \) are given in Fig. 3.3.

**Proof.** Suppose \( \gamma_{gr}(G) = \gamma_r(G) = 2 \) and let \( S = \{u, v\} \) be a \( \gamma_r \)-set of \( G \).

**Case 1.** \( \langle S \rangle \cong P_2 \).

As \( G \) is a cubic graph, \(|N(u) \cap (V - S)| = |N(v) \cap (V - S)| = 2 \). If \( u \) and \( v \) have two common neighbors then \( G \cong K_4 \) and if they have one common neighbor then \( p = 5 \), both leading to contradictions and so \( (N(u) \cap (V - S)) \cap (N(v) \cap (V - S)) = \emptyset \). Let \( u_1, u_2 \in N(u) \) and \( v_1, v_2 \in N(v) \).

Since \( G \) is cubic, either both \( (u_1, u_2) \) and \( (v_1, v_2) \) are in \( E(G) \) or both are not in \( E(G) \).
In the former case $G \cong G_1$ and in the latter case $G \cong G_2$.

**Case 2.** $(S) \not\equiv P_2$.

If $u$ and $v$ have 3 common neighbors or one common neighbor then $p = 5$ or 7 accordingly, which is not possible. Hence $|N(u) \cap (V - S)) \cap (N(v) \cap (V - S))| = 0$ or 2. In the latter case $G \cong G_1$. Suppose $u_1, u_2, u_3 \in N(u) \cap (V - S)$ and $v_1, v_2, v_3 \in N(v) \cap (V - S)$ where all six vertices are distinct. Since $G$ is cubic and connected it is enough if we consider the following cases.

1. If $(u_1, u_2), (u_2, u_3), (v_1, v_2), (v_2, v_3)$ lie in $E(G)$ then $G \cong G_3$.

2. If any one of $\{(u_1, u_2), (u_2, u_3)\}$ and any one of $\{(v_1, v_2), (v_2, v_3)\}$ lie in $E(G)$ then $G \cong G_3$ or $G_5$.

3. If none of these lie in $E(G)$ then $G \cong G_4$.
Conversely for every $G_i$ given in Fig 3.3 $\gamma(G) = 2$. Thus $\gamma(G) = 2$ if and only if $G \cong G_i (1 \leq i \leq 5)$. For all these $G_i$, we observe that $\gamma_{gr}(G) = 2$ and hence the theorem follows.

\textbf{Theorem 3.2.18.} For every connected cubic graph $G$, $\gamma_r(G) = 3$ if and only if $\gamma_{gr}(G) = 3$.

\textbf{Proof.} Suppose $\gamma_r(G) = 3$ and let $S = \{u, v, w\}$ be a $\gamma_r$-set of $G$. If an $x \in V - S$ is adjacent to all vertices in $S$ then $x$ is an isolated vertex in $(V - S)$ and so $|N(x) \cap S| \leq 2$ for every $x \in V - S$. If $|V - S| \geq 4$ then $S$ is also a restrained dominating set in $\overline{G}$ and so $\gamma_{gr}(G) = 3$. If $|V - S| = 2$ then $p = 5$ which is impossible and so $|V - S| = 3$.

But now $G \cong \overline{C_6, C_3 \cup C_3}$ according as $(S) \cong C_3, P_3$. In these cases $\gamma_r(G) = 2$.

Since $G$ is a cubic graph, $(S) \not\cong K_2 \cup K_1$ and $3K_1$. Hence $\gamma_{gr}(G) = 3$.

Conversely let $\gamma_{gr}(G) = 3$. Since $\gamma_r(G) \leq \gamma_{gr}(G), \gamma_r(G) \leq 3$. If $\gamma_r(G) = 1$ then $G \cong K_4$ which is not possible. Hence by theorem 3.2.17 it follows that $\gamma_r(G) = 3$.

\textbf{Theorem 3.2.19.} Let $G = (X, Y)$ be a connected bipartite graph with $|X| \leq |Y|$ and $|X| \leq 2$. Then $\gamma_{gr}(G) = \gamma_r(G)$ if and only if $|X| = 1$.

\textbf{Proof.} If $|X| = 1$ then $G \cong K_{1, p-1}$ and $\gamma_{gr}(G) = \gamma_r(G) = p$. Conversely suppose $\gamma_{gr}(G) = \gamma_r(G)$ and $|X| = 2$. Let $X = \{u, v\}$. If $\deg u = \deg v = p - 2$ then $G \cong K_{2, p-2}$ and $\gamma_{gr}(G) \neq \gamma_r(G)$. Otherwise, let $S = \{w \in Y : \exists \deg w = 1\}$. By choice of $u$ and $v$, $S$ is non-empty. So $S \cup \{u\}$ and $S \cup \{v\}$ are the only $\gamma_r$-sets of $G$, but both are not restrained dominating sets of $\overline{G}$ as in $\overline{G}, v$ or $u$ is an isolated vertex of $(V - (S \cup \{u\}))$ or $(V - S \cup \{v\})$ respectively. Hence $|X| = 1$.
Theorem 3.2.20. If $G = (X, Y)$ is a connected bipartite graph with $3 \leq |X| \leq |Y|$ and $\delta(G) \geq 2$ then $\gamma_{r}(G) = \gamma_{r}(G)$.

Proof. Let $S$ be a $\gamma_{r}$-set of $G$. Since $(V - S)$ cannot have isolated vertices, the sets $S \cap X, S \cap Y, (V - S) \cap X$ and $(V - S) \cap Y$ are all nonempty and so $S$ dominates $G$.

Case 1. $|(V - S) \cap X| = 1$ and $|(V - S) \cap Y| = 1$.

Let $x_1 \in (V - S) \cap X$ and $y_1 \in (V - S) \cap Y$. Since $S$ is a dominating set, $\delta(G) \geq 2$ and $3 \leq |X| \leq |Y|$ there exists $x_2, x_3 \in S \cap X$ and $y_2, y_3 \in S \cap Y$ such that $y_2 \in N(x_1)$ and $x_2 \in N(y_1)$.

As $\delta(G) \geq 2, (S \cap X) \cap N(y_1) = \{x_2\} ((S \cap Y) \cap N(x_1) = \{y_2\})$. If $x(y) \in S \cap X (S \cap Y)$ such that $x(y)$ is adjacent to $y_1(x_1)$ then $S - \{x\} (S - \{y\})$ is a $\gamma_{r}$-set of $G$.

Hence there exists an edge $zw$ with $z \in S \cap X - \{x_2\}$ and $w \in S \cap Y - \{y_2\}$. But now $S - \{z, w\}$ is a $\gamma_{r}$-set of $G$, which is a contradiction. Thus case 1 is impossible.

Case 2. $|(V - S) \cap X| = 1$ and $|(V - S) \cap Y| \geq 2$.

Let $(V - S) \cap X = \{u\}$. Since $(V - S)$ cannot have isolated vertices, $(V - S) \cap Y \subseteq N(u)$. Further since $\delta(G) \geq 2$, if there exists 2 vertices $u_1$ and $u_2$ in $(S \cap Y) \cap N(u), S - \{u_2\}$ is a restrained dominating set and so $|(S \cap Y) \cap N(u)| = 1$.

Let $(S \cap Y) \cap N(u) = \{u_1\}$. Since $\delta \geq 2, u_1$ has a neighbor in $S \cap X$ say $y_1$. We claim that $y_1$ must be adjacent to at least one vertex in $(V - S) \cap N(u)$, since otherwise $y_1$ has a neighbor say $x_1$ in $S \cap Y$ which in turn has a neighbor in $S \cap X$ so that $S - \{x_1, y_1\}$ is a restrained dominating set in $G$, which is a contradiction.

We claim that no vertex in $S \cap X$ can have all its neighbors in $S \cap Y$. Suppose $y_2 \in S \cap Y$ with $N(y_2) \subseteq S \cap Y$. Choose $x_2 \in N(y_2) \cap (S \cap Y)$ such that $x_2 \notin N(u)$.
This is possible since \( N(u) \cap (S \cap Y) = \{u_1\} \). Now \( S - \{x_2, y_2\} \) is a restrained dominating set which is a contradiction.

**Claim:** No vertex in \((S \cap X) - \{y_1\}\) can have neighbor in \((S \cap Y) - \{u_1\}\). Let

\[
\begin{align*}
A_1 &= \{x \in ((S \cap X) - \{y_1\}) \text{ such that } N(x) \cap ((S \cap Y) - \{u_1\}) \neq \emptyset\} \text{ and } \\
A_2 &= (S \cap X) - (A_1 \cup \{y_1\}).
\end{align*}
\]

Every vertex in \( A_2 \) has at least 2 neighbors in \( N(u) \cap (V - S) \). Suppose \( A_1 \neq \emptyset \). For every \( x \in A_2 \) choose a neighbor in \( N(u) \cap (V - S) \) and let \( A_3 \) be such a set with minimum cardinality. Clearly \( |A_3| \leq |A_2| \).

Let \( S' = (S \cap Y) \cup A_3 \cup \{u\} \). By choice of \( A_3 \), \( S' \) is a dominating set. By corollary 2.8.13, every vertex in \( A_1 \) has at least a private neighbor in \((V - S) \cap N(u)\) and so \( S' \) is a restrained dominating set which is a contradiction. Hence \( A_1 = \emptyset \). Using the claim and the fact that \( \delta(G) \geq 2 \), we have \( S \cap Y = \{u_1\} \).

Let \( S \cap X = B_1 \cup B_2 \) where

\[
\begin{align*}
B_1 &= \{x \in S \cap X / x \in N(u_1)\} \text{and} \\
B_2 &= \{x \in S \cap X / x \notin N(u_1)\}.
\end{align*}
\]

For every \( x \in B_2 \) choose a neighbor in \( N(u) \cap (V - S) \) and let \( B_3 \) be such a set with minimum cardinality. Then \( |B_3| \leq |B_2| \).

If \( |B_1| \geq 2 \), \( S' = \{u, u_1\} \cup B_3 \) is a restrained dominating set which is a contradiction. Hence \( |B_1| = 1 \).

Let \( P \) be the set of all private neighbors of \( y_1 \) in \( N(u) \). By corollary 2.8.13, \( P \neq \emptyset \). If there exists a vertex in \( N(u) - P \) which is not a private neighbor of any vertex in \( B_2 \) then \( |B_3| < |B_2| \). But now \( B_3 \cup \{u, u_1\} \) is a restrained dominating set and so every vertex in \( N(u) \) is a private neighbor of some vertex in \( S \cap X \).
For every vertex in $S \cap X$, choose exactly one private neighbor in $N(u)$ and let $C_3$ be this set of vertices. $C_3 \cup \{u\}$ is a $\gamma_r$-set of $G$ which is also a restrained dominating set in $\bar{G}$.

Thus in all possible cases we have $\gamma_r(G) = \gamma_{gr}(G)$.

**Case 3.** $|(V - S) \cap X| \geq 2$ and $|(V - S) \cap Y| \geq 2$.

Now $(V - S)$ has no isolated vertices in $\bar{G}$ and so $S$ is also a restrained dominating set of $\bar{G}$. Thus in all possible cases we have $\gamma_r(G) = \gamma_{gr}(G)$.

**Remark 3.2.21.** If $\gamma_{gr}(G) = \gamma_r(G)$ it is not necessary that $\delta(G) \geq 2$.

For the graph given in Fig (3.4), $\gamma_{gr}(G) = \gamma_r(G) = 3$ but $\delta(G) = 1$.

![Fig. 3.4](image)

**Problem 3.2.22** Characterize the connected bipartite graphs with $3 \leq |X| \leq |Y|$ and $\delta(G) = 1$ for which $\gamma_{gr}(G) = \gamma_r(G)$.

### 3.3 Restrained double domination number of a graph

Restrained double domination number of a graph is defined as follows:

**Definition 3.3.1.** A set $S \subseteq V(G)$ is a restrained double dominating set for $G$ if every vertex in $V$ is dominated by at least 2 vertices in $S$ and $(V - S)$ has no isolated
vertices. The minimum cardinality of a minimal restrained double dominating set is the restrained double domination number and is denoted by \( \gamma_2^r(G) \).

Example 3.3.2.

(i) \( \gamma_2^r(K_p) = 2 \) for \( p \neq 3 \) and \( \gamma_2^r(K_3) = 3 \).

(ii) \( \gamma_2^r(K_{1,p-1}) = p \).

(iii) \( \gamma_2^r(G) = 6 \) where \( G \) is Petersen graph.

(iv) \( \gamma_2^r(C_p) = p \forall p \).

(v) \( \gamma_2^r(W_p) = p - 2k \) when \( p = 3k + r \) where \( r = 1, 2, 3 \).

(vi) \( \gamma_2^r(K_{m,n}) = 4 \) when \( 3 \leq m \leq n \).

Remark 3.3.3.

(i) Every graph \( G \) without isolated vertices has a restrained double dominating set, as \( V(G) \) is such a set for \( G \).

(ii) If \( S \) is any restrained double dominating set of \( G \) then \( S \) contains all pendants, supports and vertices of degree 2 and hence by theorem 2.8.8 if \( G \) is a graph with \( \gamma_r(G) = \) number of pendants then \( \gamma_2^r(G) = p \).

(iii) For any graph \( G \) with \( \delta(G) \geq 3 \), \( \gamma_2^r(G) \leq p - 2 \).

(iv) By theorems 1.74 and 1.75 we observe that there is no graph \( G \) with \( \gamma_r(G) = \gamma_2^r(G) = p - 2 \).

(v) For a graph \( G \) without isolated vertices \( \gamma_r(G) \leq \gamma_2^r(G) \) and by theorem 1.69, \( \gamma_r(G) = \gamma_2^r(G) = p \) if and only if \( G \) is a galaxy.
(vi) For every tree $G(\not\cong K_{1,\nu-1})$ and unicyclic graph $G, \gamma_r(G) \neq \gamma_{2r}(G)$.

**Theorem 3.3.4.** Let $G$ be a graph without isolated vertices and $P$ be the set of all pendants and supports of $G$ ($P$ may be empty). Then $2 \leq \gamma_{2r}(G) \leq p$. Lower bound is attained if and only if $G$ has at least two vertices with full degree and $\delta(G) \geq 3$ if $V(G) - P \neq \emptyset$. Upper bound is attained if and only if for every edge $(u,v)$ in $(V(G) - P)$, either $\deg u$ or $\deg v$ equals 2 or $\delta((V(G) - \{u,v\})) = 0$.

**Proof.** Clearly $2 \leq \gamma_{2r}(G) \leq p$. Suppose $\gamma_{2r}(G) = 2$ and let $S = \{u,v\}$ be a $\gamma_{2r}$-set. Then $\deg u = \deg v = p - 1$ and $(V - S)$ has no isolated vertices and so that $\delta(G) \geq 3$. Converse is obvious.

Assume $\gamma_{2r}(G) = p$. If there exists an edge $(u,v)$ in $(V(G) - P)$ with $\deg u \geq 3, \deg v \geq 3$ and $\delta((V(G) - \{u,v\})) > 0$ then $V(G) - \{u,v\}$ is a restrained double dominating set which is a contradiction.

Conversely let $(u,v)$ be an edge in $(V(G) - P)$. If $\deg u = 2$ or $\deg v = 2$, then $V(G) - \{u,v\}$ is not a restrained double dominating set. If $\deg u \geq 3$ and $\deg v \geq 3$ since $\delta(V(G) - \{u,v\}) = 0, V(G) - \{u,v\}$ is not a restrained double dominating set. Hence $\gamma_{2r}(G) = p$.

**Theorem 3.3.5.** Let $G$ be a graph without isolated vertices. Then $\gamma_{2r}(G) \geq \frac{5p - 2g}{4}$ and the bound is attained by $G_1$ where $G_1$ is given in Fig 3.5.

![Fig. 3.5](image-url)
Proof. Let $S$ be a $\gamma_2$-set. Every vertex in $V - S$ is adjacent to at least 2 vertices in $S$ and one vertex in $V - S$. Also every vertex in $S$ must have at least one neighbor in $S$. So

$$q \geq 2|V - S| + \frac{|V - S| + \gamma_2r(G)}{2}$$

$$= \frac{5}{2}|V - S| + \frac{\gamma_2r(G)}{2}$$

$$= \frac{5}{2}(p - \gamma_2r(G)) + \frac{\gamma_2r(G)}{2}.$$ 

Hence $2q \geq 5p - 4\gamma_2r(G)$.

and so $\gamma_2r(G) \geq \frac{5p - 2q}{4}$.

The bound is attained if $G \cong G_1$. \hfill \square

Theorem 3.3.6. If $G$ has no isolated vertices, then $\gamma_2r(G) \geq \frac{2p}{\Delta(G) + 1}$.

Proof. Let $S$ be a $\gamma_2$-set of $G$. Let $s$ be the number of edges with one end in $S$ and the other in $V - S$. Since every vertex in $S$ has at least one neighbor in $S$,

$$s \leq (\Delta(G) - 1)|S| = (\Delta(G) - 1)\gamma_2r(G).$$

Also every vertex in $V - S$ is adjacent to at least 2 vertices in $S$ and so $s \geq 2|V - S| = 2(p - \gamma_2r(G))$.

Thus

$$2p - 2\gamma_2r(G) \leq (\Delta(G) - 1)\gamma_2r(G)$$

$$\Rightarrow 2p \leq (\Delta(G) + 1)\gamma_2r(G)$$

and so $\gamma_2r(G) \geq \frac{2p}{\Delta(G) + 1}$.

When $G \cong mK_2$ and $K_p(p \geq 2)$ the bound is attained, so the bound is sharp. \hfill \square
Theorem 3.3.7. Let $G = (X, Y)$ be a connected bipartite graph with $|X| = m$, $|Y| = n$, $2 \leq m \leq n$. Then $\gamma_{2r}(G) = p$ if and only if either $\Delta(G) \leq 2$ or for every $v \in V(G)$ with $\deg v \geq 3$, whenever $u \in N(v)$ either $\deg u \leq 2$ or $u$ is a support.

Proof. Let $\gamma_{2r}(G) = p$ and $\Delta(G) \geq 3$. If there exists $v \in V(G)$ and a non-support vertex $u \in N(v)$ with $\deg v \geq 3$ and $\deg u \geq 3$ then $V(G) - \{u, v\}$ is a restrained double dominating set which is a contradiction.

Converse follows by Remark 3.3.3 (ii). \qed

Theorem 3.3.8. Let $G$ be a connected graph. Then $\gamma_{2r}(G) = 2$ if and only if $G \cong K_4$ and $\gamma_{2r}(G) = 4$ if and only if $G \cong G_i (1 \leq i \leq 5)$ where $G_i (1 \leq i \leq 5)$ are given in Fig (3.6).

![Fig. 3.6](image)

Proof. Suppose $\gamma_{2r}(G) = 2$. Since $G$ is a cubic graph, by theorem 3.3.4, $G \cong K_4$. Converse is obvious.
Suppose $\gamma_{2r}(G) = 4$. Let $S = \{u, v, w, x\}$ be a $\gamma_{2r}$-set.

**Case 4.** $\langle S \rangle$ is a path.

In $\langle S \rangle$, $\deg v = \deg w = 2$. Let $y$ and $z$ be the neighbors of $v$ and $w$ in $V - S$ respectively. Since $G$ is cubic, $|V - S| = 2$. Hence $u$ and $x$ are adjacent. Obviously $y$ and $z$ are adjacent. If $y$ is adjacent to $u$ and $z$ is adjacent to $x$ then $G \cong G_1$. If $y$ is adjacent to $x$ and $z$ is adjacent to $u$ then $G \cong G_2$.

**Case 5.** $\langle S \rangle$ is not a path.

In this case $\langle S \rangle \cong 2K_2$. Without loss of generality, let $u$ and $v$ be adjacent and $w$ and $x$ be adjacent. Clearly $|V - S| = 4$. If $u$ and $v$ have 2 common neighbors in $V - S$ then $w$ and $x$ also have 2 common neighbors and so $G \cong G_3$.

If $u$ and $v$ have only one common neighbor say $s$, then $w$ and $x$ have only one common neighbor $t$.

Let $w_1 \in N(u) \cap (V - S)$ and $v_1 \in N(v) \cap (V - S)$. Without loss of generality let $w_1 \in N(w)$ and $v_1 \in N(x)$. If $s \in N(w_1)$ and $v_1 \in N(t)$, then $G \cong G_3$. If $s \in N(v_1)$ and $w_1 \in N(t)$, then also $G \cong G_3$. If $s \in N(t)$ and $w_1 \in N(v_1)$, then $G \cong G_4$.

If $u$ and $v$ do not have common neighbors, then $w$ and $x$ also do not have common neighbors.

Let $u_1 \in N(u) \cap (V - S), v_1 \in N(v) \cap (V - S), w_1 \in N(w) \cap (V - S), x_1 \in N(x) \cap (V - S)$. Without loss of generality let $u_1 \in N(w), v_1 \in N(x), w_1 \in N(u)$ and $x_1 \in N(v)$.

If $u_1$ and $v_1$ are adjacent and $w_1$ and $x_1$ are adjacent, then $G \cong G_5$. If $u_1$ and $w_1$ are adjacent and $v_1$ and $x_1$ are adjacent, then $G \cong G_3$. If $u_1$ and $x_1$ are adjacent and $v_1$ and $w_1$ are adjacent then $G \cong G_5$. Converse is obvious. \(\square\)
Proposition 3.3.9. There exists no connected cubic graph $G$, with $\gamma_{2r}(G) = 3$.

Proof. Let $S$ be a $\gamma_{2r}$-set with $|S| = 3$. Clearly $(S) \cong P_3 = (u,v,w)$. Let $y$ be the neighbor of $v$ in $V - S$. Without loss of generality let $y \in N(u)$. Since $y$ has a neighbor in $V - S$, there exists $x \in V - S$ such that $x \in N(u) \cap N(w)$. But then $\deg w = 2$ and so there exists no connected cubic graph with $\gamma_{2r}(G) = 3$. $\Box$

Theorem 3.3.10. If $T$ is a tree such that $\bar{T}$ has no isolated vertices, then $\gamma_{2r}(T) + \gamma_{2r}(\bar{T}) \leq 2p$. Equality holds if and only if $T \cong P_4, P_5$ or $B(2,1)$.

Proof. $\gamma_{2r}(T) \leq p$ and $\gamma_{2r}(\bar{T}) \leq p$ so that $\gamma_{2r}(T) + \gamma_{2r}(\bar{T}) \leq 2p$. Suppose $\gamma_{2r}(T) + \gamma_{2r}(\bar{T}) = 2p$.

We claim that $\text{diam}(T) = 3$ or 4. Clearly $\text{diam}(T) \geq 3$. Suppose $\text{diam}(T) = d \geq 5$ and let $v_1, v_2, \ldots, v_{d+1}$ be the diametrical path in $T$. Now $V(T) - \{v_1, v_{d+1}\}$ is a restrained double dominating set of $\bar{T}$ which is a contradiction and so $\text{diam}(T) = 3$ or 4.

If $\text{diam}(T) = 3$, let $(u, u_1, v_1, v)$ be the diametrical path. If $\deg u_1 = \deg v_1 = 2, T \cong P_4$. If $\deg u_1 \geq 4$ or if $\deg u_1 = 3$ and $\deg v_1 \geq 3$, then we get a restrained double dominating set of $\bar{T}$ with cardinality $p - 2$ and so $T \cong B(2,1)$.

If $\text{diam}(T) = 4$, then $T \cong P_5$, since in all other cases we get a restrained dominating set of $\bar{T}$ with cardinality $p - 2$. $\Box$

Theorem 3.3.11. Let $G$ be a connected unicyclic graph such that $\bar{G}$ has no isolated vertices. Then $\gamma_{2r}(G) + \gamma_{2r}(\bar{G}) \leq 2p$ and equality holds if and only if $G \cong C_4, C_5$ or $G_i(1 \leq i \leq 4)$ where $G_i$ are given in Fig 3.7.

Proof. Clearly $\gamma_{2r}(G) + \gamma_{2r}(\bar{G}) \leq 2p$. Suppose $\gamma_{2r}(G) + \gamma_{2r}(\bar{G}) = 2p$ and let $C_n = (v_1, v_2, \ldots, v_n = v_1)$ be the cycle in $G$. If $n \geq 6, V(G) - \{v_2, v_3\}$ is a restrained
double dominating set of $\bar{G}$ and so $n \leq 5$. If $n = 5$ and if $u \in V(G) - C_n$, then for any $v \in C_n \cap (V - N(u)), V(G) - \{u, v\}$ is a restrained double dominating set of $\bar{G}$ and so $G \cong C_5$. Suppose $n = 4$. If there exists a pendant vertex $u \in V(G) - C_n$ such that $d(u, v_i) \geq 2$ for some $i(1 \leq i \leq 4)$, then $V(G) - \{u, v_{i-1}\}$ is a restrained double dominating set of $\bar{G}$. If $G$ contains 2 pendant vertices say $u$ and $v$, then $V(G) - \{u, v\}$ is a restrained double dominating set of $\bar{G}$. Hence $G \cong C_4$ or $G_1$. If $n = 3$, as above we can show that every vertex not on $C_3$ is at most at distance 2 from a vertex of $C_3$. If there exists 2 vertices at distance 2 from a vertex of $C_3$, we get a similar contradiction. Also every vertex of $C_3$ is deg 2 or 3 since otherwise either $\bar{G}$ has isolated vertices or $\bar{G}$ has a restrained double dominating set of cardinality $p - 2$. Hence $G \cong G_2, G_3$ or $G_4$. Converse is obvious.

\[ \text{Fig. 3.7} \]

**Theorem 3.3.12.** If $T$ is a tree such that $T \not\cong K_{1, p-1}$, then

\[ \gamma_{2r}(T) = \begin{cases} 
5 & \text{if } T \cong P_5, B(r, s) \text{ with } r + s = 3 \\
4 & \text{if } T \cong B(r, s) \text{ with } r + s \neq 3, T_1 \text{ or } T_2 \\
& \text{where } T_1, T_2 \text{ are given in Fig (3.8).} \\
3 & \text{otherwise.}
\end{cases} \]

**Proof.** Let $S$ be the set of all supports of $T$. Since $G \not\cong K_{1, p-1}, |S| \geq 2$. 

65
Case (i). $|S| \geq 4$.

Let $u_1, v_1, w_1, x_1$ be 4 distinct supports with pendants $u \in N(u_1), v \in N(v_1), w \in N(w_1)$ and $x \in N(x_1)$. Without loss of generality we can assume that $x_1$ is nonadjacent to $w_1$. Let $D = \{u, v, w\}$. Clearly $D$ is a double dominating set in $\overline{T}$ and by choice of $x_1, (V - D)$ has no isolated vertices. Hence $D$ is a restrained double dominating set of $T$. Also no set of cardinality 2 can be a restrained double dominating set of $T$ and so $\gamma_{2r}(T) = 3$.

Case (ii). $|S| = 3$.

Let $S = \{u, v, w\}$ where $u, v$ and $w$ are pendant vertices with $u \in N(u_1), v \in N(v_1), w \in N(w_1)$. Suppose $\langle S \rangle \cong P_3$. If $\deg u_1 \geq 3$ or $\deg w_1 \geq 3$ then $D = \{u, v, w\}$ is a minimal restrained double dominating set in $\overline{T}$ and so $\gamma_{2r}(\overline{T}) = 3$. If $\deg u_1 = \deg w_1 = 2$ and $\deg v_1 \geq 3$ then there exists no set with cardinality 3 which is a $\gamma_{2r}(\overline{T})$-set and so if $T \cong T_1$ or $T_2$ then $\gamma_{2r}(\overline{T}) = 4$. If $\langle S \rangle \cong K_2 \cup K_1$ or $3K_1$ then there exists a vertex $x$ which is neither a pendant nor a support and $x$ is non-adjacent to at least one $\{u_1, v_1, w_1\}$. In this case $D = \{u, v, w\}$ is a $\gamma_{2r}$-set in $\overline{T}$ and so $\gamma_{2r}(\overline{T}) = 3$.

Case (iii). $|S| = 2$.

Let $S = \{u_1, v_1\}$ and $u$ and $v$ are pendant vertices with $u \in N(u_1)$ and $v \in N(v_1)$. 
If \( u_1 \) and \( v_1 \) are adjacent then \( T \cong B(r,s) \). It is easy to observe that \( \gamma_{2r}(\bar{T}) = 5 \) if \( r + s = 3 \) and \( \gamma_{2r}(\bar{T}) = 4 \) otherwise.

Suppose \( u_1 \) and \( v_1 \) are nonadjacent. If there exists a vertex \( x \) adjacent to both \( u_1 \) and \( v_1 \), either \( T \cong P_5 \) in which case \( \gamma_{2r}(\bar{T}) = 5 \) or \( \deg u_1(\deg v_1) \geq 3 \) and \( \{u,u_1,v\}\{u,v_1,v\} \) is a \( \gamma_{2r} \)-set in \( \bar{T} \) so that \( \gamma_{2r}(\bar{T}) = 3 \). Otherwise \( \{u,u_1,v\} \) is a \( \gamma_{2r} \)-set and so \( \gamma_{2r}(\bar{T}) = 3 \).

**Corollary 3.3.13.** If \( T \) is a tree such that \( T \not\cong K_{1,p-1} \), \( \gamma_{2r}(T) + \gamma_{2r}(\bar{T}) \leq p + 5 \) and equality holds if and only if \( T \cong P_5 \), \( B(r,s)(r+s = 3) \). If \( T \not\cong P_5 \), \( B(r,s)(r+s = 3) \), then \( \gamma_{2r}(T) + \gamma_{2r}(\bar{T}) \leq p + 4 \) and equality holds if and only if \( T \cong T_1 \), \( T_2 \), \( B(r,s)(r+s \neq 3) \). In all other cases, \( \gamma_{2r}(T) + \gamma_{2r}(\bar{T}) \leq p + 3 \) and this bound is sharp.

**Proof.** Follows from theorem 3.3.12. Sharpness is exhibited by the graph given in Fig (3.9).

![Fig. 3.9](image)

**Theorem 3.3.14.** Let \( G = (X,Y) \) be a connected bipartite graph with \( |X| = m \), \( |Y| = n(2 \leq m \leq n) \) such that \( G \) has no isolated vertices. If \( G \not\cong K_{2,3} \) and \( K_{3,n} \), then \( \gamma_{2r}(\bar{G}) = 3 \) or 4. Furthermore \( \gamma_{2r}(\bar{G}) = 4 \) if and only if for every pair of vertices \( u,v \) in \( X, N(u) \cap N(v) \cap Y \neq \emptyset \) and vice versa.

**Proof.** Clearly \( \gamma_{2r}(\bar{G}) \geq 2 \). If \( \{u,v\} \) is a \( \gamma_{2r} \)-set in \( \bar{G} \), then \( u \) and \( v \) are isolated vertices in \( G \) and so \( \gamma_{2r}(\bar{G}) \geq 3 \). Since \( G \not\cong K_{2,3} \) and \( K_{3,n} \), there exist sets \( S \) with

67
\(|S \cap X| \geq 2\) and \(|S \cap Y| \geq 2\) such that \(S\) is a restrained double dominating set of \(\bar{G}\) and hence \(\gamma_{2r}(G) \leq 4\).

Suppose \(\gamma_{2r}(\bar{G}) = 4\). If there exists \(u, v\) in \(X\) with \(N(u) \cap N(v) \cap Y = \emptyset\), then for any \(w \in Y, \{u, v, w\}\) is a \(\gamma_{2r}\)-set in \(\bar{G}\), which is a contradiction. Conversely no 3 element set is a \(\gamma_{2r}\)-set in \(\bar{G}\) and so \(\gamma_{2r}(\bar{G}) = 4\).

\[\square\]

**Corollary 3.3.15.** Let \(G\) be a connected bipartite graph such that \(\bar{G}\) has no isolated vertices. Then \(\gamma_{2r}(G) + \gamma_{2r}(\bar{G}) \leq p + 4\) and the bound is sharp.

**Proof.** Follows from theorem 3.3.14. Sharpness is exhibited by the graph given in Fig 3.10.

![Fig. 3.10](image)

**Problem 3.3.16.** Characterize the graphs with \(\gamma_{2r}(G) = \gamma_r(G)\).

### 3.4 Uniform restrained domination number of a graph

Arumugam and Paulraj Joseph[4] define uniform domination number \(\gamma_u(G)\) of a graph \(G\) which is extended as follows:

**Definition 3.4.1.** The uniform restrained domination number \(\gamma_{wr}(G)\) of a graph \(G\) is the least positive integer \(k\) such that every \(k\) element restrained subset \(S\) of \(V\) is a dominating set of \(G\).
Example 3.4.2.

(i) \( \gamma_{ur}(C_p) = p - 2 \).

(ii) \( \gamma_{ur}(K_{m,n}) = n \) where \( 2 \leq m \leq n \).

(iii) Let \( G \cong G_1 \) where \( G_1 \) is given in Fig (3.11).

\[ S = \{u, u_1, u_2, u_3, u_4, v, v_1, v_2, v_3, w_1, w_2\} \] is a restrained set but not a dominating set and so \( \gamma_{ur}(G) \geq 13 \). Since \( \delta(G) = 2 \) every pair of adjacent vertices is dominated by the remaining \( p - 2 \) vertices and so \( \gamma_{ur}(G) \leq 13 \). Hence \( \gamma_{ur}(G) = 13 \).

Remark 3.4.3. We observe that if \( G \) is \( D_r \)-complete then \( \gamma_{ur}(G) = \gamma_r(G) \). But the converse fails.

For example if \( G \cong C_4, \gamma_{ur}(C_4) = \gamma_r(C_4) = 2 \) but \( C_4 \) is not \( D_r \)-complete.

Theorem 3.4.4. Let \( G \) be any graph and \( I \) be the set of isolated vertices of \( G \) (I may be empty) and \( \delta' = \delta((V - I)) \). Then

(i) \( \gamma_{ur}(G) = p \) if and only if \( G \cong K_p \) or \( \delta' = 1 \).

(ii) \( \gamma_{ur}(G) = p - \delta' \) if and only if \( \delta' \geq 2 \).
Proof. Assume \( \gamma_{ur}(G) = p \). If \( I = \emptyset \) and \( \delta' \geq 2 \), every restrained set \( S \) with \( |S| = p - 2 \) is also a dominating set so that \( \gamma_{ur}(G) \leq p - 2 \) which is a contradiction so that \( \delta' = 1 \).

If \( I \neq \emptyset \) and \( V(G) - I = \emptyset \) then \( G \cong K_p \).

If \( I \neq \emptyset \) and \( V(G) - I \neq \emptyset \) then \( \langle V - I \rangle \) has no isolated vertices and so by the above argument \( \delta' = 1 \).

Conversely if \( G \cong K_p \) then obviously \( \gamma_{ur}(G) = p \). Suppose \( \delta' = 1 \). Let \( \deg u = 1 \) and \( v \) be the neighbor of \( u \). \( V(G) - \{u, v\} \) is a restrained set which is not a dominating set and so \( \gamma_{ur}(G) = p \). If \( \gamma_{ur}(G) = p - \delta' \) by (i) \( \delta' \geq 2 \).

Conversely let \( \delta' \geq 2 \). Let \( I = \emptyset \). Let \( u \in V \) be \( \exists \deg u = \delta' = \delta(G) \). Consider \( S = V - N[u] \). \( S \) is a restrained set but not a dominating set and so \( \gamma_{ur}(G) > |S| = p - (\delta(G) + 1) \) so that \( \gamma_{ur}(G) \geq p - \delta(G) \). Any restrained set \( S \) with \( |V - S| = \delta(G) \) is obviously a dominating set and so \( \gamma_{ur}(G) \leq p - \delta(G) \).

Hence \( \gamma_{ur}(G) = p - \delta(G) \).

If \( I \neq \emptyset \) and \( V(G) - I = \emptyset \) then \( G \cong K_p \) which is a contradiction. Hence \( V(G) - I \neq \emptyset \).

Now \( \langle V - I \rangle \) has no isolated vertices and so by the above argument \( \gamma_{ur}(G) = p - \delta' \).

\[ \square \]

Corollary 3.4.5. For any tree \( T \),

(i) \( \gamma_{ur}(T) + \chi(T) = p + 2 \)

(ii) \( \gamma_{ur}(T) + \kappa(T) = p + 1 \) and

(iii) \( \gamma_{ur}(T) + \lambda(T) = p + 1 \).
Proof. Proof follows by theorem 1.27.

Corollary 3.4.6. Let $G$ be any unicyclic graph with a cycle $C$ such that $\delta(G) = 1$. Then $\gamma_{wr}(G) + \chi(G) = p + 2$ or $p + 3$ according as $C$ is even or odd. Also $\gamma_{wr}(G) + \kappa(G) = p + 1$ and $\gamma_{wr}(G) + \lambda(G) = p + 1$.

Corollary 3.4.7. For any graph $G$, $\gamma_{wr}(G) = \gamma_r(G) = p$ if and only if $G$ is a galaxy.

Proof. Proof follows from theorems 1.69 and 3.4.4.

Corollary 3.4.8. Let $G$ be any graph and $I$ be the set of isolated vertices of $G$ (I may be empty) and $\delta' = \delta((V - I))$. Then,

$$\gamma_{wr}(G) + \chi(G) \leq \begin{cases} p + \Delta(G) + 1 & \text{if } \delta' = 1 \\ p - \delta' + \Delta(G) + 1 & \text{if } \delta' \geq 2 \end{cases}$$

Equality holds in the former case if and only if $G \cong mK_2 \cup nK_1 (m \geq 1, n \geq 0)$ or $G_1$ and in the latter case $G \cong G_1$. Here $G_1$ is a graph having either $C_p (p \text{ odd})$ or $K_p (p \geq 3)$ as a Component.

Proof. By theorems 3.4.4 and 1.34 it follows that

$$\gamma_{wr}(G) + \chi(G) \leq \begin{cases} p + \Delta(G) + 1 & \text{if } \delta' = 1 \\ p - \delta' + \Delta(G) + 1 & \text{if } \delta' \geq 2 \end{cases}$$

By theorem 1.35 equality holds in the former case if and only if $G \cong mK_2 \cup nK_1 (m \geq 1, n \geq 0)$ or $G_1$ in the latter case $G \cong G_1$. Here $G_1$ is a graph having either $C_p (p \text{ odd})$ or $K_p (p \geq 3)$ as a Component.

Theorem 3.4.9. Let $G$ be a unicyclic graph. $\gamma_{wr}(G) = \gamma_r(G) = p - 2$ if and only if $G \cong C_3, C_4$ or $C_5$. 
Proof. Proof follows from theorems 3.4.4 and 1.75.

**Theorem 3.4.10.** $\gamma_{ur}(G) = 1$ if and only if $G \cong K_p(p \neq 2)$.

**Proof.** $\gamma_{ur}(G) = 1 = p - (p - 1)$. By theorem 3.4.4, if $\delta(G) = p - 1$ then $G \cong K_p(p \neq 2)$. Converse is obvious.

**Theorem 3.4.11.** For any graph $G$ with $p \geq 4, \gamma_{ur}(G) = 2$ if and only if $G \cong K_pUK_1$ or $K_{2n} - X$ where $X$ is a perfect matching or $K_p \setminus Y$ where $Y$ is an independent set of edges with $1 \leq |Y| \leq \lfloor \frac{p}{2} \rfloor$ where $p$ is odd and $1 \leq |Y| \leq \lfloor \frac{p-1}{2} \rfloor$ when $p$ is even.

**Proof.** Suppose $\gamma_{ur}(G) = 2$.

**Case (i)** $G$ has isolated vertices.

Since $\gamma_{ur}(G) = 2$, $G$ can have exactly one isolated vertex and so $G \cong G_1UK_1$ where $\gamma_r(G_1) = 1$.

But by theorem 1.76 , $\gamma_r(G_1) = 1$ if and only if $G_1 \cong K_1 + H$, where $H$ is a graph having no isolated vertices. Also $\gamma_{ur}(G_1) = 1$ and so $H$ is complete. So $G_1 \cong K_p$ and hence $G \cong K_pUK_1$.

**Case (ii)** $G$ has no isolated vertices.

Since $\gamma_{ur}(G) = 2 = p - (p - 2)$, by theorem 3.4.3 $\delta(G) = p - 2$. So $\Delta(G) = p - 2$ or $p - 1$.

If $\delta(G) = \Delta(G) = p - 2$ then $G$ is $p - 2$ regular and so $G \cong K_{2n} - X$ where $X$ is a perfect matching.

If $\delta(G) = p - 2$ and $\Delta(G) = p - 1$ every vertex is not adjacent to at most one vertex and at least one vertex is adjacent to every vertex. So $G \cong K_p - Y$ where $Y$ is an independent set of edges with $1 \leq |Y| \leq \lfloor p/2 \rfloor$ when $p$ is odd and $1 \leq |Y| \leq \lfloor \frac{p-1}{2} \rfloor$ when $p$ is even.
Conversely in all the 3 cases, since every restrained set $SCV$ with $|S| = 2$ is a dominating set $\gamma_{ur}(G) \leq 2$. Hence by theorem 3.4.9, $\gamma_{ur}(G) = 2$.

**Theorem 3.4.12.** In a connected graph $G$, $\gamma_{ur}(G) + \text{diam}(G) \leq 2p - 1$. Equality holds if and only if $G$ is a path.

**Proof.** $\gamma_{ur}(G) \leq p$ and $\text{diam}(G) \leq p - 1$. So $\gamma_{ur}(G) + \text{diam}(G) \leq 2p - 1$.

If $\gamma_{ur}(G) + \text{diam}(G) = 2p - 1$ then $\gamma_{ur}(G) = p$ and $\text{diam}(G) = p - 1$.

Since $\text{diam}(G) = p - 1$ if and only if $G$ is a path, proof follows by theorem 3.4.4.

We now extend the definition of $\gamma_{ur}(G)$ to that of a strict uniform restrained domination number $\gamma_{sur}(G)$ defined as follows.

**Definition 3.4.13.** Strict uniform restrained domination number of a graph $G$ is defined as the least positive integer $k$ such that every $k$ element subset of $V(G)$ is a restrained dominating set and is denoted by $\gamma_{sur}(G)$.

**Example 3.4.14.**

(i) $\gamma_{sur}(K_p) = 1(p \neq 2)$.

(ii) $\gamma_{sur}(K_{m,n}) = m + n$ where $3 \leq m \leq n$.

(iii) $\gamma_{sur}(K_{1,p-1}) = p$.

The following proposition is immediate.

**Proposition 3.4.15.**

1. If either $\delta(G) \leq 1$ or $G \cong C_p(p \geq 4)$ then $\gamma_{sur}(G) = p$.
2. \( \gamma_r(G) \leq \gamma_{sur}(G) \leq \gamma_{sur}(G) \) and equality holds throughout if and only if \( G \) is \( D_r \)-complete.

3. If \( G \) is a connected cubic graph with \( p \neq 4 \), then \( \gamma_{sur}(G) = p \).

Theorem 3.4.16. Let \( G \) be any graph with \( \delta(G) \geq 2 \). Then either \( \gamma_{sur}(G) < \delta(G) \) or \( \gamma_{sur}(G) = p \). Moreover if \( p < 2\delta(G) \) then \( \gamma_{sur}(G) \leq p - \delta(G) \). In particular \( p = 2\delta(G) - 1 \) if and only if \( \gamma_{sur}(G) = \delta(G) - 1 \).

Proof. Suppose \( \gamma_{sur}(G) \neq p \). Let \( u \) be a vertex of degree \( \delta(G) \). Any set containing \( N(u) \) is not a restrained dominating set and so \( \gamma_{sur}(G) < \delta(G) \).

Suppose \( p < 2\delta(G) \) and let \( S \) be any set with \( p - \delta(G) \) vertices.

Then every vertex \( v \) in \( V - S \) has neighbors in both \( S \) and \( V - S \) and so \( S \) is a restrained dominating set. Hence \( \gamma_{sur}(G) \leq p - \delta(G) \).

Let \( \gamma_{sur}(G) = \delta(G) - 1 \). Since \( p < 2\delta(G), \gamma_{sur}(G) \leq p - \delta < \delta \) and so \( \gamma_{sur}(G) \leq p - \delta(G) \leq \delta(G) - 1 \). Since \( \gamma_{sur}(G) = \delta(G) - 1, p = 2\delta(G) - 1 \).

Let \( p = 2\delta(G) - 1 \) and \( p < 2\delta(G) \gamma_{sur}(G) \leq \delta - 1 \). Let \( v \) be a vertex of degree \( \delta(G) \). \( V - N[v] \) is not a dominating set and so \( \gamma_{sur}(G) > |V - N[v]| = \delta(G) - 2 \). Hence \( \gamma_{sur}(G) = \delta(G) - 1 \).

Remark 3.4.17. For the graph \( G \cong G_1 \), where \( G_1 \) is given in Fig (3.12), \( \gamma_{sur}(G) < p - \delta(G) \) but \( p \) is not less than \( 2\delta(G) \).
3.5 Connected restrained domination number of a graph

In this section we assume that $G$ is connected.

**Definition 3.5.1.** A restrained dominating set $S \subseteq V(G)$ is said to be a connected restrained dominating set if $\langle S \rangle$ is connected. The minimum cardinality of all minimal connected restrained dominating sets is called the connected restrained domination number of a graph and is denoted by $\gamma_{cr}(G)$.

**Example 3.5.2.**

(i) $\gamma_{cr}(K_p) = 1 \ (p \neq 2)$.

(ii) $\gamma_{cr}(K_{m,n}) = 2 \ \text{where} \ 2 \leq m \leq n$.

(iii) For any graph $G$ with $\delta(G) > 0$, $\gamma_{cr}(K_p + G) = 1 \ \text{for every} \ p$.

The following proposition is immediate.

**Proposition 3.5.3.** If $G$ is any graph and $H$ is any connected spanning subgraph of $G$, then $\gamma_{cr}(G) \leq \gamma_{cr}(H)$.
Proposition 3.5.4. For a graph $G$, $\gamma_{cr}(G) + \varepsilon_T \geq p + e$ where $\varepsilon_T$ is the maximum number of pendants in any spanning tree of $G$ and $e$ is the number of pendants of $G$.

Proof. Let $S$ be any $\gamma_{cr}$-set of $G$. If $G \cong K_{1,p-1}$, then $\gamma_{cr}(G) + \varepsilon_T = p + p - 1 = p + e$. Suppose $G \not\cong K_{1,p-1}$. If $V - S = \emptyset$, $\gamma_{cr}(G) = p$ and $\varepsilon_T \geq e$ so that $\gamma_{cr}(G) + \varepsilon_T \geq p + e$. Suppose $V - S \neq \emptyset$. Let $|S| = k$ and $|V - S| = p - k$. Remove all the edges in $(V - S)$ and all the edges that have one end in $S$ and another in $V - S$, leaving exactly one such edge for each vertex in $V - S$. Clearly $\varepsilon_T \geq p - k + e$ and so $\gamma_{cr}(G) + \varepsilon_T \geq p + e$.

Theorem 3.5.5. For any tree $T$, $\gamma_{cr}(T) = p$.

Proof. Suppose $\gamma_{cr}(T) < p$ and let $S$ be a $\gamma_{cr}$-set of $T$. Hence there exists $u, v \in V - S$ such that $u$ and $v$ are adjacent. If $|N(u) \cap S| \geq 2$ and if $u_1, u_2 \in N(u) \cap S$, then the $u_1 - u_2$ path in $S$ together with $u$ forms a cycle in $T$ and so $|N(u) \cap S| = 1$ for every $u \in V - S$. But now $u_1 \in N(u) \cap S$ and $v_1 \in N(v) \cap S$, the $u_1 - v_1$ path in $S$ together with edges $u_1 u, uv, vv_1$ forms a cycle and so $V - S = \emptyset$. Hence $\gamma_{cr}(T) = p$.

Corollary 3.5.6. Let $G$ be a unicyclic graph with cycle $C$. Then

$$\gamma_{cr}(G) = \begin{cases} p - 2 & \text{if and only if two adjacent vertices on } C \text{ have deg } 2 \\ p & \text{otherwise.} \end{cases}$$

Corollary 3.5.7. For any tree $T$, $\gamma_{cr}(T) + \chi(T) = p + 2$ and $\gamma_{cr}(T) + \text{diam}(T) = 2p - 1$.

Corollary 3.5.8. If $G$ is any Hamiltonian graph, then $\gamma_{cr}(G) \neq p$.

Proof. Since a Hamiltonian graph contains a Hamiltonian cycle, the proof follows.
Corollary 3.5.9. If $G$ is a graph with $\gamma_r(G) = \text{number of pendant vertices}$, then $\gamma_{cr} = p$.

**Proof.** Follows by theorem 2.8.8.

Theorem 3.5.10. If $G$ is a $r$-regular graph ($r \neq 1$), then $\gamma_{cr}(G) \leq p - r$ and the bound is attained.

**Proof.** If $r = 2$ then $G \cong C_p$ and so $\gamma_{cr}(G) = p - 2 = p - r$. Now $r \geq 3$.

Suppose $\gamma_{cr}(G) > p - r$. Let $\gamma_{cr}(G) = p - r + k$ where $k \geq 1$. Obviously $1 \leq k \leq r$. Let $S$ be a $\gamma_{cr}$-set of $G$. Then $|S| = p - r + k$ and $|V - S| = r - k$. Clearly $r - k = 0$ or $r - k \geq 2$. If $r - k = 0$ then $|S| = p$ so that $\gamma_{cr}(G) = p$. Since $G$ is regular there exists at least one edge which is not a cut edge and so $\gamma_{cr}(G) \leq p - 2$ which is a contradiction and so $r - k \geq 2$.

For any $u \in V - S$, $|N(u) \cap S| \geq k + 1$ and for any $v \in S$, $|N(v) \cap S| \geq k$. If $k \geq 2$, $\deg v \geq 2$ and $\langle S \rangle$ contains a cycle. Now $S - \{v_1, v_2\}$ is a $\gamma_{cr}$-set which is a contradiction. If $k = 1$ and if there exists $v \in S$ with $|N(u) \cap S| = 1$ then $S - \{v\}$ is a $\gamma_{cr}$-set which is a contradiction. If $v \in S$ is with $|N(u) \cap S| \geq 2$ contradiction follows as above. So $\gamma_{cr}(G) \leq p - r$.

Theorem 3.5.11. For a graph $G$, $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor \leq \gamma_{cr}(G) \leq q + 1$ and the lower bound is attained if and only if $\Delta(G) = p - 1$ and $\delta(G) \geq 2$. Upper bound is attained if and only if $G$ is a tree.

**Proof.** Since $\gamma_c(G) \leq \gamma_{cr}(G)$, first part of the theorem follows from theorem 1.56 and, if $\gamma_{cr}(G) = q + 1$, we have $q \leq p - 1$ and so $q = p - 1$. By theorem 1.19, $G$ is a tree. Converse follows from theorem 3.5.5.

Theorem 3.5.11 can be improved as follows.
Theorem 3.5.12. For a graph $G$, \( \left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor \leq \gamma_{cr}(G) \leq q + 1 \) and the lower bound is sharp.

**Proof.** Bounds follow by theorems 1.57 and 3.5.11. Sharpness of the lower bound is exhibited by $G_1$, where $G_1$ is given in Fig. 3.13.

![Fig. 3.13](image)

Theorem 3.5.13. If $G$ is a graph with $\gamma_{cr}(G) = p$, then $G$ is either of the following.

(i) $G$ is a tree

(ii) $G$ is a cyclic irregular graph with $\delta(G) = 1$ and no cycle in $G$ can contain 2 adjacent vertices of degree 2.

**Proof.** Suppose $G$ is not a tree and $G$ is regular with degree $r$. If $r = 1$, $G \cong K_2$ and if $r \geq 2$, by theorem 3.5.10, $\gamma_{cr}(G) \neq p$. Hence $G$ is a cyclic irregular graph.

Suppose $\delta(G) \geq 2$. Since $G$ has at least one edge which is not a cut edge, $\gamma_{cr}(G) \leq p - 2$. If $G$ contains 2 adjacent vertices of degree 2 then again $\gamma_{cr}(G) \leq p - 2$. These contradictions show that $G$ is a cyclic irregular graph with $\delta(G) = 1$ and no cycle in $G$ can contain 2 adjacent vertices of degree 2. 

78
We now establish some Nordhaus-Gaddum-type inequalities on $\gamma_{cr}(G)$.

**Theorem 3.5.14.** Let $G$ be a graph of order $p \geq 4$ such that $\bar{G}$ is connected. Then
\[ \gamma_{cr}(G) + \gamma_{cr}(\bar{G}) \leq \frac{p(p-1)}{2} + 2 \]
and the bound is attained if and only if $G \cong P_4$.

**Proof.** By theorem 3.5.11,
\[ \gamma_{cr}(G) + \gamma_{cr}(\bar{G}) \leq q(G) + q(\bar{G}) + 2 = \frac{p(p-1)}{2} + 2. \]
By theorem 3.5.11, equality holds if and only if both $G$ and $\bar{G}$ are trees and so $G \cong P_4$. □

**Theorem 3.5.15.** If $G$ is a graph and $\bar{G}$ is connected with $\text{diam}(G) \geq 5$, then
\[ \gamma_{cr}(G) + \gamma_{cr}(\bar{G}) \leq p + 2 \]
and the bound is sharp.

**Proof.** Since $\text{diam}(G) \geq 5$, there exists 2 vertices $u$ and $v$ such that $d(u, v) = \text{diam}(G) \geq 5$. Let $P = \{u, v_1 \ldots v_{n-1}, v\}$ be a diametrical path in $G$. Every vertex in $V - \{v_1, v_{n-1}\}$ is adjacent to at least one of $\{v_1, v_{n-1}\}$ and at least one of $\{u, v\}$ in $\bar{G}$ so that $\{v_1, v_{n-1}\}$ is a connected restrained dominating set in $\bar{G}$. Thus $\gamma_{cr}(\bar{G}) \leq 2$. Hence $\gamma_{cr}(G) + \gamma_{cr}(\bar{G}) \leq p + 2$. If $G \cong C_n \circ K_1 (n \geq 4)$, $\gamma_{cr}(G) = p, \gamma_{cr}(\bar{G}) = 2$ and so the bound is sharp. □

**Theorem 3.5.16.** Let $G$ be a regular graph such that $G \not\cong K_2$ and $\bar{G}$ is connected. Then
\[ \gamma_{cr}(G) + \gamma_{cr}(\bar{G}) \leq p + 1 \]
and the bound is sharp. Moreover if equality holds then $\gamma_{cr}(G) \geq 3$.

**Proof.** By theorem 3.5.10,
\[ \gamma_{cr}(G) + \gamma_{cr}(\bar{G}) \leq p - \Delta(G) + p - \Delta(\bar{G}). \]
\[ = 2p - (\Delta(G) + \Delta(\bar{G})) \]
\[ \leq 2p - (p - 1) \]
\[ = p + 1 \]
If $G \cong C_5$, $\gamma_{cr}(G) + \gamma_{cr}(\bar{G}) = 3 + 3 = 6 = p + 1$ and so the bound is sharp. Suppose $\gamma_{cr}(G) + \gamma_{cr}(\bar{G}) = p + 1$. If $\gamma_{cr}(G) = 1$ then $\bar{G}$ is not connected and if $\gamma_{cr}(G) = 2$ then $\gamma_{cr}(\bar{G}) = p + 1 - 2 = p - 1$ which is impossible. Hence $\gamma_{cr}(G) \geq 3$. \hfill \Box

**Theorem 3.5.17.** Let $T$ be any tree such that $T$ is connected. Then

$$\gamma_{cr}(T) + \gamma_{cr}(\bar{T}) = \begin{cases} p + 4 & \text{if and only if } T \cong P_4 \\ p + 3 & \text{if and only if } T \cong T_1 \text{ where } T_1 \text{ is given in Fig.3.14} \\ p + 2 & \text{otherwise.} \end{cases}$$

**Proof.** By theorem 3.5.5, $\gamma_{cr}(T) = p$. If diam $T \geq 5$, by theorem 3.5.15, $\gamma_{cr}(\bar{T}) = 2$. Since $T$ is connected, $T \not\cong K_{1,p-1}$ and so diam $(T) \geq 3$. If diam $(T) = 3$ then $\gamma_{cr}(\bar{T}) = 2, 3$ and 4 according as $G \cong T_2, T_1$ and $P_4$, where $T_1$ and $T_2$ are given in Fig. 3.14.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.14.png}
\caption{Fig. 3.14}
\end{figure}

Suppose diam $T = 4$ and let $(v_1, v_2, v_3, v_4, v_5)$ be the diametrical path. $\{v_2, v_5\}$ is a $\gamma_{cr}$-set of $T$ and so $\gamma_{cr}(\bar{T}) = 2$. Thus we have,

$$\gamma_{cr}(T) + \gamma_{cr}(\bar{T}) = \begin{cases} p + 4 & \text{only if } T \cong P_4 \\ p + 3 & \text{only if } T \cong T_1 \\ p + 2 & \text{otherwise.} \end{cases}$$

Converse is obvious \hfill \Box

**Theorem 3.5.18.** For a graph $G$, $\gamma_{r}(G) = \gamma_{cr}(G) = p$ if and only if $G$ is a star.
Proof. Follows by theorems 1.69 and 3.5.5

Remark 3.5.19.

1. $\gamma_c(K_p) = \gamma_r(K_p) = \gamma_{cr}(K_p) = 1$.

2. $\gamma_c(K_{m,n}) = \gamma_r(K_{m,n}) = \gamma_{cr}(K_{m,n}) = 2 \forall m, n \geq 2$.

3. By theorems 1.69, 1.58 and 3.5.5 there is no non-trivial tree $T$ with $\gamma_c(T) = \gamma_r(T) = \gamma_{cr}(T)$.

Theorem 3.5.20. Let $G$ be a unicyclic graph. Then $\gamma_c(G) = \gamma_r(G) = \gamma_{cr}(G)$ if and only if $G \cong C_3, C_4$ or $C_5$.

Proof. By theorem 1.75, $\gamma_r(G) = p - 2$ if and only if $G \cong C_3, C_4, C_5$ or $H_1$ where $H_1$ is the graph obtained from $C_3$ by adding any number of pendants to at most 2 vertices. Hence by corollary 3.5.6, $\gamma_r(G) = \gamma_{cr}(G) = p - 2$ if and only if $G \cong C_3, C_4, C_5$ or $H_2$ where $H_2$ is the graph obtained from $C_3$ by adding any number of pendants to exactly one vertex. But $\gamma_c(H_2) = 1$ and so the theorem follows.

Theorem 3.5.21. For $n \geq 3$, $(n - 1)! \leq \gamma_{cr}(S_n) \leq 2(n - 1)!$.

Proof. Define

$$A = \{\alpha \in V(S_n)/\alpha(1) = 1\}.$$ 

and

$$B_i = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = 1\} \text{ for } i = 2, 3, \ldots, n.$$ 

As proved in theorem 1.59, $X = A \cup (\cup_{i=2}^n B_i)$ is a connected dominating set for $S_n$.

When $n = 3$, every vertex in $V - X$ is adjacent to only one vertex in $A$ and no vertex in $B_2 \cup B_3$. But $S_n$ is $(n - 1)$ regular and so $\langle V - X \rangle$ has no isolated vertices.
When \( n \geq 4 \), every vertex in \( V - X \) is adjacent to exactly one vertex in \( A \) and a vertex in \( \bigcup_{i=2}^{n-2} B_i \) and so \( \langle V - A \rangle \) has no isolated vertices. Thus \( X \) is a connected restrained dominating set and so \( \gamma_{cr}(S_n) \leq |X| = 2(n - 1)! \)

Since \( \gamma_{cr}(S_n) \geq \gamma_r(S_n) \), by theorem 2.2.2 \( \gamma_{cr}(S_n) \geq (n - 1)! \) for \( n \geq 3 \). Hence \( (n - 1)! \leq \gamma_{cr}(S_n) \leq 2(n - 1)! \) \( \forall n \geq 3 \). \( \Box \)

### 3.6 Total restrained domination number of a graph

In this section we assume that \( G \) has no isolated vertices.

**Definition 3.6.1.** A restrained dominating set \( S \subseteq G \) is said to be a total restrained dominating set if \( \langle S \rangle \) has no isolated vertices. Minimum cardinality of such a minimal total restrained dominating set is called total restrained domination number and is denoted by \( \gamma_{tr}(G) \).

**Example 3.6.2.**

1. For \( p \geq 4 \), \( \gamma_{tr}(K_p) = 2 \).
2. For \( p \geq 4 \), \( \gamma_{tr}(W_p) = 2 \).
3. For \( p \geq 2 \),

\[
\gamma_{tr}(P_p) = \begin{cases} 
\lfloor p/2 \rfloor + 1 & \text{if } p \equiv 2 \text{ or } 3 \text{ (mod 4)} \\
\lfloor p/2 \rfloor + 2 & \text{otherwise.}
\end{cases}
\]

4. For \( p \geq 3 \),

\[
\gamma_{tr}(C_p) = \begin{cases} 
\lfloor p/2 \rfloor + 1 & \text{if } p \equiv 2 \text{ or } 3 \text{ (mod 4)} \\
\lfloor p/2 \rfloor & \text{otherwise.}
\end{cases}
\]

**Theorem 3.6.3.** Let \( G \) be a graph and let \( S \) be the set of pendant vertices and supports of \( G \) (\( S \) may be empty). Then \( |S| \leq \gamma_{tr}(G) \leq p \). Lower bound is sharp and
upper bound is attained if and only if for every edge $uv \in E(G)$, either $u$ or $v$ is a support.

Proof. By definition, every total restrained dominating set contains $S$ and so $|S| \leq \gamma_{tr}(G)$. Sharpness of the bound is exhibited by the bistar $B(r, s)$.

Suppose $\gamma_{tr}(G) = p$ and there exists an edge $uv \in E(G)$ such that neither $u$ nor $v$ is a support. If there exists a vertex $\omega$ with $\deg(\omega) = 2$ and $\omega \in N(u) \cap N(v)$, then either $V(G) - \{u, \omega\}$ or $V(G) - \{v, \omega\}$ is a total restrained dominating set of $G$. Otherwise $V(G) - \{u, v\}$ is a total restrained dominating set of $G$. Hence $\gamma_{tr}(G) < p - 2$, which is a contradiction.

Converse is obvious. \hfill \Box

**Corollary 3.6.4.** Let $T$ be a tree. Then $\gamma_{tr}(T) + \chi(T) = p + 2$ if and only if for every edge $uv \in E(G)$, either $u$ or $v$ is a support.

Proof. Since $\chi(T) = 2$, proof follows from theorem 3.6.3. \hfill \Box

**Corollary 3.6.5.** If $G$ is any connected graph with $\gamma_r(G) =$ number of pendent vertices, then $\gamma_{tr}(G) = p$.

Proof. Follows from theorems 2.8.8 and 3.6.3. \hfill \Box

**Theorem 3.6.6.** For any graph $G$ with $\delta(G) \geq 3$, $\gamma_{tr}(G) \leq p - \delta(G) + 1$ and the bound is sharp.

Proof. Let $u \in V(G)$ with $\deg u = \delta(G) \geq 3$ and let $S = (V(G) - N[u]) \cup \{v, \omega\}$ where $v, \omega \in N(u)$. By choice of $u$ and $\delta(G)$, $S$ is a total restrained dominating set of $G$ and so $\gamma_{tr}(G) \leq |S| = p - \delta(G) + 1$. Bound is attained by $G \cong K_4$. \hfill \Box
**Theorem 3.6.7.** Let $G$ be a connected graph with 2 vertices $u$ and $v$ such that $\deg u = \deg v = \Delta(G)$, $N(u) = N(v)$ and not every $x \in N(u)$ is a support. Let $D$ be the set of all isolated vertices in $\langle V - N[u] \rangle$ or $\langle V - (N[u] \cup \{v\}) \rangle$ according as $v \in N(u)$ or not. Then

$$\gamma_{tr}(G) \leq \begin{cases} 
 p - \Delta(G) & \text{if } uv \notin E(G) \text{ and } D = \emptyset \\
 p - \Delta(G) + 1 & \text{if } uv \in E(G) \text{ and } D = \emptyset. \\
 p - \Delta(G) + |D| - 1 & \text{if } uv \notin E(G) \text{ and } D \neq \emptyset \\
p - \Delta(G) + |D| & \text{if } uv \in E(G) \text{ and } D \neq \emptyset.
\end{cases}$$

Moreover, all these bounds are sharp.

**Proof.** Case (i). $uv \notin E(G)$.

Let $D$ be the set of all isolated vertices in $\langle V - (N[u] \cup \{v\}) \rangle$. If $D = \emptyset$ then every vertex in $V - (N[u] \cup \{v\})$ has a neighbor in $V - (N[u] \cup \{v\})$ itself and so for $\omega \in N(u)$, $(V - N[u]) \cup \{\omega\}$ is a total restrained dominating set so that $\gamma_{tr}(G) \leq p - (\Delta(G) + 1) + 1 = p - \Delta(G)$. The bound is attained by $G_1$ given in Fig. 3.15.

![Fig. 3.15](image)

Suppose $D \neq \emptyset$. By assumption, there exists at least one vertex in $N(u)$ which is
not a support. Let $Y$ be the set of minimum number of vertices in $N(u)$ such that every vertex in $D$ has a neighbor in $Y$. Clearly $|Y| \leq |D|$ and so $(V - N[u]) \cup Y$ is a total restrained dominating set of $G$ so that $\gamma_{tr}(G) \leq p - \Delta(G) + |D| - 1$. The bound is attained by $G_2$ given in Fig. 3.16.

![Fig. 3.16](image)

**Case (ii).** $uv \in E(G)$.

Let $D$ be the set of isolated vertices in $(V - N[u])$. If $D = \emptyset$, then $(V - N[u]) \cup \{v, w\}$ where $w \in N(u)$ is a total restrained dominating set and so $\gamma_{tr}(G) \leq p - \Delta(G) + 1$. The bound is attained by $G_3$ given in Fig. 3.17.

![Fig. 3.17](image)

Suppose $D \neq \emptyset$. Now $(V(G) - N[u]) \cup Y \cup \{v\}$ is a total restrained dominating set...
of $G$ where $Y$ is chosen as in case (i). Now $\gamma_{tr}(G) \leq p - \Delta(G) + |D|$. The bound is attained by $G_4$ given in Fig. 3.18.

![Fig. 3.18](image)

**Definition 3.6.8.** A caterpillar is a tree $T$ with the property that the removal of its pendant vertices results in a path known as spine. The code $C$ of a caterpillar with spine $v_1, v_2, \ldots, v_s$ is the sequence of non-negative integers $t_1, t_2, \ldots, t_s$ where $t_i$ is the number of pendant vertices adjacent to $v_i$ in $T$. The substrings of consecutive zeros in $C$ are called zero strings of $C$ and are labelled from 1 to $k$.

**Theorem 3.6.9.** Let $G$ be a caterpillar. If $Z_i (1 \leq i \leq m)$ are the zero strings of $G$ then

$$\gamma_{tr}(G) = p - \sum_{i=1}^{m} s_i$$

where

$$s_i = \begin{cases} 0 & \text{if } |Z_i| = 1 \\ \left\lfloor \frac{|Z_i|}{2} \right\rfloor + 1 & \text{if } |Z_i| \geq 2 \text{ and } |Z_i| \equiv 2 \text{ or } 3 \text{(mod 4)} \\ \left\lfloor \frac{|Z_i|}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

**Proof.** Let $S$ be a $\gamma_{tr}$-set of $G$.

Let $Z_i$ be one of the substrings of $G$, such that $Z_i = (v_j, \ldots, v_n)$ ($j \geq 2, n \leq s - 1$). Let $v_{j-1}$ and $v_{n+1}$ be 2 consecutive supports such that $u_{j-1}$ and $u_{n+1}$
are their pendants respectively. (choose one if there are more than one). Then \(\{u_{j-1}, v_{j-1}, v_j \ldots v_n, v_{n+1}, u_{n+1}\}\) is a path say \(P_i\). Let \(s_i = |(V - S) \cap Z_i|\).

If \(|Z_i| \equiv 0(\mod 4)\) say \(4k(k \geq 0)\) then \(|V(P_i)| = 4k + 4 \equiv 0(\mod 4)\) and so by example 3 of 3.6.2, \(\gamma_{tr}(P_i) = 2k + 4\). Hence \(s_i = 4k + 4 - (2k + 4) = 2k = \left\lfloor \frac{|Z_i|}{2} \right\rfloor\).

If \(|Z_i| = 1\) then clearly \(s_i = 0\).

If \(|Z_i| \equiv 1(\mod 4)\) say \(4k + 1(k \geq 1)\) then \(|V(P_i)| = 4k + 5 \equiv 1(\mod 4)\). So

\[
s_i = 4k + 5 - (2k + 5) = 2k = \left\lfloor \frac{|Z_i|}{2} \right\rfloor.
\]

Similarly when \(|Z_i| = 2\) or \(3(\mod 4)\) we find

\[
s_i = \left\lfloor \frac{|Z_i|}{2} \right\rfloor + 1.
\]

Hence the result follows. \(\Box\)

The following are immediate.

**Proposition 3.6.10.** For any connected graph \(G\),

\[
\gamma_{tr}(G) + \varepsilon_T \geq \begin{cases} p + e & \text{if } \delta(G) = 1 \\ p & \text{if } \delta(G) \geq 2 \end{cases}
\]

where \(\varepsilon_T\) is the maximum number of pendants in a spanning tree and \(e\) is the number of pendant vertices in \(G\).

**Proposition 3.6.11.** If \(G\) is a graph with \(\Delta(G) = p - 1\) and \(\delta(G) \geq 3\) then \(\gamma_{tr}(G) = 2\).

**Proposition 3.6.12.** If \(G\) is a \(D_r\)-complete graph, then \(\gamma_{tr}(G) = \gamma_r(G)\) if and only if \(G \cong K_{2n} \setminus X(n \geq 3)\) where \(X\) is a perfect matching or \(G\) is a galaxy.

87
Fig. 3.19
Theorem 3.6.13. Let \( G \) be a connected cubic graph. Then (1) \( \gamma_{tr}(G) = \gamma_r(G) = 2 \) if and only if \( G \cong G_1, G_2 \). (2) \( \gamma_{tr}(G) = \gamma_r(G) = 3 \) if and only if \( G \cong G_3, G_4, G_5 \) where \( G_i(1 \leq i \leq 5) \) are given in Fig. 3.19

Proof. (1) Suppose \( \gamma_{tr}(G) = \gamma_r(G) = 2 \) and let \( S = \{u, v\} \) be a \( \gamma_{tr} \)-set. Hence \( u \) and \( v \) are adjacent. Let \( u_1, u_2 \in N(u) \) and \( v_1, v_2 \in N(v) \). Since \( G \) is a cubic graph with \( \gamma_r(G) = 2 \), \( u_1, u_2, v_1, v_2 \) are all distinct. Hence \( G \cong G_1, G_2 \). Converse is obvious. (2) Suppose \( \gamma_{tr}(G) = \gamma_r(G) = 3 \) and let \( S = \{u, v, w\} \) be a \( \gamma_{tr} \)-set. Then \( S \cong P_3 \). Let \( u_1, u_2 \in N(u), v_1 \in N(v) \) and \( w_1, w_2 \in N(w) \). If \( w_1 = u_1 \) and \( w_2 = u_2 \) then \( \{v, v_1\} \) is a \( \gamma_{tr} \)-set and if \( w_1 = u_1 \) and \( w_2 = v_1 \) then \( \{w, u_1\} \) is a \( \gamma_{tr} \)-set. Hence \( |V(G)| = 8 \) and we have the following 3 possibilities.

(i) Both \( u_1u_2 \) and \( w_1w_2 \in E(G) \)

(ii) Exactly one of \( u_1u_2 \) and \( w_1w_2 \in E(G) \)

(iii) Neither of \( u_1u_2, w_1w_2 \in E(G) \).

Accordingly \( G \cong G_3, G_4, G_5 \). Converse is obvious

Theorem 3.6.14. If \( G = (X, Y) \) is any bipartite graph with \( 3 \leq |X| \leq |Y| \) which is not complete then \( \gamma_{tr}(\tilde{G}) = 2 \).

Proof. Since \( G \) is not complete, there exist \( u \in X \) and \( v \in Y \) such that \( u \) and \( v \) are non-adjacent in \( G \). Now \( \{u, v\} \) is a \( \gamma_{tr} \)-set in \( \tilde{G} \) and so \( \gamma_{tr}(\tilde{G}) = 2 \).

Corollary 3.6.15. For any tree \( T \supset \tilde{T} \) has no isolated vertices,

(i) \( \gamma_{tr}(\tilde{T}) = 2 \) if and only if \( T \ncong B(r, s) \) where at least one of \( \{r, s\} \) equals 1.
(ii) $\gamma_{tr}(\bar{T}) = 3$ if and only if $T \cong B(r,s)$ where $r = 1$ and $s > 1$ or vice versa.

(iii) $\gamma_{tr}(\bar{T}) = 4$ if and only if $T \cong B(r,s)$ where $r = s = 1$.

**Proof.** If $\text{diam}(T) \geq 5$, by theorem 3.6.14, $\gamma_{tr}(\bar{T}) = 2$. If $\text{diam} T = 4$ let $v_1, v_2, v_3, v_4, v_5$ be the diametrical path. Then $\{v_2, v_5\}$ is a $\gamma_{tr}$-set of $\bar{T}$ and so $\gamma_{tr}(\bar{T}) = 2$. If $\text{diam} (T) = 3$ then $T \cong B(r,s)$. If both $r$ and $s > 1$ then $\gamma_{tr}(\bar{T}) = 2$. If exactly one of $\{r, s\}$ equals 1 then $\gamma_{tr}(\bar{T}) = 3$. If both $r$ and $s$ are equal to 1, then $\gamma_{tr}(\bar{T}) = 4$. If $\text{diam}(T) \leq 2$ then $\bar{T}$ has isolated vertices. Hence the result follows. \[ \square \]

**Corollary 3.6.16.** (i) If $G \cong K_{m,n}(2 \leq m \leq n)$ and $G \not\cong K_{2,3}$ then $\gamma_{tr}(G) + \gamma_{tr}(\bar{G}) = 6$.

(ii) If $G = (X,Y)$ is any bipartite graph with $3 \leq |X| \leq |Y|$ then $\gamma_{tr}(G) + \gamma_{tr}(\bar{G}) \leq p + 2$ and the bound is sharp.

**Proof.** (i) $\gamma_{tr}(K_{m,n}) = 2$ and if $G \not\cong K_{2,3}$, $\gamma_{tr}(K_{m,n}) = \gamma_{tr}(K_m \cup K_n) = 4$ and so $\gamma_{tr}(G) + \gamma_{tr}(\bar{G}) = 6$.

(ii) If $G$ is a complete bipartite graph, result follows from (i). Suppose $G$ is not complete, by theorem 3.6.14, $\gamma_{tr}(\bar{G}) = 2$ and so $\gamma_{tr}(G) + \gamma_{tr}(\bar{G}) \leq p + 2$. Bound is attained by $G \cong G_1$ where $G_1$ is given in Fig. 3.20 \[ \square \]

Fig. 3.20

90
Theorem 3.6.17. For \( n \geq 4 \), \( \gamma_{tr}(S_n) = \frac{n!}{(n-1)!} \) if \( n \) is even and \( \frac{n!}{(n-1)!} \leq \gamma_{tr}(S_n) \leq \frac{(n-1)!}{(n-2)!} \) if \( n \) is odd.

Proof. By theorem 1.61, \( \gamma_{tr}(S_n) \geq \gamma_t(S_n) \geq \frac{n!}{(n-1)!} \).

Case (i) \( n = 2m \)

Let

\[ A_i = \{ \alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = i + 1 \} \text{ if } i \text{ is odd} \]
\[ A_i = \{ \alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = i - 1 \} \text{ if } i \text{ is even} \]

Claim: \( A = \bigcup_{i=1}^{2m} A_i \) is a total restrained dominating set of \( S_n \).

For odd \( i \), each vertex of \( A_i \) has a unique adjacent vertex in \( A_{i+1} \) and so \( \langle A \rangle \) has no isolated vertices. Let \( \alpha \in V - A \) and \( \alpha(2) = i \). Let \( \alpha' \) be obtained from \( \alpha \) by interchanging \( \alpha(1) \) and \( i + 1 \) if \( i \) is odd and \( \alpha(1) \) and \( i - 1 \) if \( i \) is even. Then \( \alpha' \in A \) and adjacent to \( \alpha \). So \( A \) dominates \( S_n \). Since every vertex in \( V - A \) is adjacent to a unique vertex in \( A \) and \( S_n \) is \( n - 1 \) regular, \( \langle V - A \rangle \) has no isolated vertices. Thus \( A \) is a total restrained dominating set of \( S_n \). So \( \gamma_{tr}(S_n) \leq |A| = \frac{n!}{(n-1)!} \). Hence \( \gamma_{tr}(S_n) = \frac{n!}{(n-1)!} \).

Case (ii).

Let us define \( A_i \) and \( B_i \) for \( i = 1, 2 \ldots 2m \).

\[ A_i = \{ \alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = i + 1 \} \text{ if } i \text{ is odd.} \]
\[ A_i = \{ \alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = i - 1 \} \text{ if } i \text{ is even.} \]
\[ B_i = \{ \alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = 2m + 1, \alpha(3) = i + 1 \} \text{ if } i \text{ is odd.} \]
\[ B_i = \{ \alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = 2m + 1, \alpha(3) = i - 1 \} \text{ if } i \text{ is even.} \]
Let $A = \bigcup_{i=1}^{2m} (A_i \cup B_i)$. As discussed in case (1), $\langle A \rangle$ has no isolated vertices. Also $\bigcup_{i=1}^{2m} A_i$ dominates all the vertices with $\alpha(2) = k(k = 1, 2 \ldots 2m)$ and $\bigcup_{i=1}^{2m} B_i$ dominates all the vertices with $\alpha(2) = 2m + 1$. So $A$ dominates $S_n$. Now every vertex $\alpha$ in $V - A$ with $\alpha(1) = j(j = 1, 2 \ldots 2m)$ is adjacent to a vertex in $\bigcup_{i=1}^{2m} A_i$ and every vertex $\alpha \in V - A$ has $\alpha(1) = 2m + 1$ is adjacent to a vertex in $\bigcup_{i=1}^{2m} A_i$ and a vertex in $\bigcup_{i=1}^{2m} B_i$. Since $n \geq 4$ and $S_n$ is $n - 1$ regular, $\langle V - A \rangle$ has no isolated vertices and so $A$ is a total restrained dominating set of $S_n$. So $\gamma_{tr}(S_n) \leq |A| = \frac{(n-1)^2(n-1)}{n-2}$. Hence the result follows. 

**Problem 3.6.18.** Characterize connected cubic graphs for which $\gamma_{tr}(G) = \gamma_r(G)$

### 3.7 Restrained efficient domination number of a graph

We now extend definition 1.62 as follows:

**Definition 3.7.1.** A set $S$ of vertices in a graph $G = (V, E)$ is defined to be restrained efficient dominating set if every vertex $u$ in $V - S$ is adjacent to exactly one vertex in $S$ and $\langle V - S \rangle$ has no isolates. The minimum cardinality of a minimal restrained efficient dominating set is called restrained efficient domination number of a graph, denoted by $\gamma_{re}(G)$.

**Example 3.7.2.**

(i) $\gamma_{re}(K_p) = 1(p \neq 2)$

(ii) $\gamma_{re}(K_{m,n}) = 2(m, n \geq 2)$

(iii) $\gamma_{re}(C_p) = k + r$ where $p = 3k + r(0 \leq r \leq 2)$.
Theorem 3.7.3. If $G$ is any connected graph with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ then $\gamma_{er}(G) < p$.

Proof. Case (i). $\delta(G) = \Delta(G) = 2$.

For any two adjacent vertices $u$ and $v$, $V(G) - \{u,v\}$ is a restrained efficient dominating set of $G$ and $\gamma_{er}(G) < p$.

Case (ii). $\delta(G) = \Delta(G) = 3$.

For any chordless cycle $C$, $V - C$ is a restrained efficient dominating set of $G$ and so $\gamma_{er}(G) < p$.

Case (iii). $\delta(G) = 2$ and $\Delta(G) = 3$.

Let $S_1$ and $S_2$ be the set of all vertices of $G$ with degrees 2 and 3 respectively. If $\langle S_1 \rangle$ has an edge or if $\delta(\langle S_2 \rangle) \geq 2$ theorem follows by cases (i) and (ii). Suppose $\langle S_1 \rangle$ is independent and $\delta(S_2) \leq 1$. Hence there exists a vertex $u \in S_2$ such that $u$ is adjacent to at least two vertices, say $v$ and $w$ where $v$ and $w \in S_1$ and so $V(G) - \{u,v,w\}$ is a restrained efficient dominating set. Hence $\gamma_{er}(G) < p$. Hence the result. $\square$

Theorem 3.7.4. If $G = (V_1, V_2, \ldots V_r)$ is a complete $r$-partite graph with $|V_i| \geq 2(1 \leq i \leq r)$ and $r \geq 3$, then $\gamma_{er}(G) = p$.

Proof. Let $D$ be a $\gamma_{er}$-set of $G$. Clearly $|D| \geq 2$. Without loss of generality, let $u, v \in D$ where $u \in V_1$ and $v \in V_2$. But $D$ is an efficient dominating set and so $\bigcup_{i=3}^{r} V_i \subseteq D$ which in turn gives $\bigcup_{i=1}^{r} V_i \subseteq D$ so that $\gamma_{er}(G) = p$. $\square$

Theorem 3.7.5. If $G = G_1 + G_2$ where $G_1 = nK_1(n \geq 2)$ and $G_2$ is any graph with $1 \leq \delta(G_2) \leq \Delta(G_2) < p - 1$ then $\gamma_{er}(G) = p$.

Proof. Let $D$ be a $\gamma_{er}$-set of $G$. If either $|D \cap V(G_1)| \geq 2$ or $|D \cap V(G_2)| \geq 2$, clearly $D = V(G)$. If either $|D \cap V(G_1)| = 0$ or $|D \cap V(G_2)| = 0$, $D$ is not a $\gamma_{er}$-set.
and so $|D \cap V(G_1)| = |D \cap V(G_2)| = 1$. Let $u \in V(G_2) \cap D$. Since $\delta_{G_2}(u) \geq 1$, $u$ has a neighbor $v$ in $V(G_2)$ and so $v \in D$. Now by above argument $D = V(G)$ and so $\gamma_{er}(G) = p$.

We now investigate graphs $G$ with $\gamma_r(G) = \gamma_{er}(G)$.

(i) $\gamma_r(P_p) = \gamma_{er}(P_p) = p - 2 \lceil \frac{p-1}{3} \rceil$.

(ii) $\gamma_r(C_p) = \gamma_{er}(C_p) = k + r$ where $p = 3k + r (0 \leq r \leq 2)$.

(iii) $\gamma_r(K_{m,n}) = \gamma_{er}(K_{m,n}) = 2(m, n \geq 2)$.

(iv) $\gamma_r(K_p) = \gamma_{er}(K_p) = 1$ if $p \neq 2$ and $\gamma_r(K_2) = \gamma_{er}(K_2) = 2$.

(v) $\gamma_r(K_{1,p-1}) = \gamma_{er}(K_{1,p-1}) = p$.

(vi) $\gamma_r(W_p) = \gamma_{er}(W_p) = 1$.

\[ \Box \]

**Theorem 3.7.6.** For any tree $T$, $\gamma_r(T) = \gamma_{er}(T) = p - 2$ if and only if $T \cong P_4, P_5, P_6$ or $T$ is obtained by adding any number of pendant vertices to the supports of $P_6$.

**Proof.** Follows from theorem 1.74.

**Theorem 3.7.7.** Let $G$ be a unicyclic graph. Then $\gamma_r(G) = \gamma_{er}(G) = p - 2$ if and only if $G \cong C_4, C_5, C_3$, or $G_1$ where $G_1$ is obtained from $C_3$ by adding any number of pendant vertices to exactly one vertex of $C_3$.

**Proof.** Follows from theorem 1.75.

**Theorem 3.7.8.** Let $G$ be a connected cubic graph. Then $\gamma_r(G) = \gamma_{er}(G) = 2$ if and only if $G \cong G_i (1 \leq i \leq 5)$ where $G_i$ are given in Fig 3.21 and $\gamma_r(G) = \gamma_{er}(G) = 1$ if and only if $G \cong K_4$. 

94
Proof. Let $S$ be a $\gamma_{er}$-set. Suppose $\gamma_r(G) = \gamma_{er}(G) = 2$.

Case (i). $\Delta(\langle S \rangle) = 0$.

Then $S$ is independent, $S = \{u, v\}$ and let $N(u) = \{u_1, u_2, u_3\}$ and $N(v) = \{v_1, v_2, v_3\}$. Then $\deg_{(V-S)} u_i = \deg_{(V-S)} v_i = 2 (1 \leq i \leq 3)$ and so $(V-S)$ is a cycle.

It is easy to observe that $G \cong G_1, G_2$ or $G_3$ according as $|E(\langle N(v) \rangle)| = |E(\langle N(u) \rangle)|$ = 2, 1 or 0 respectively.

Case (ii). $\Delta(\langle S \rangle) = 1$.

Now $u$ and $v$ are adjacent and let $u_1, u_2 \in N(u) \cap (V-S)$ and $v_1, v_2 \in N(v) \cap (V-S)$. Then $(V-S)$ is a cycle and $G \cong G_4$ or $G_5$ according as $|E(\langle N(u) \cap (V-S) \rangle)| = |E(\langle N(v) \cap (V-S) \rangle)| = 1$ or 0. Converse is obvious.

Suppose $\gamma_r(G) = \gamma_{er}(G) = 1$ and let $S = \{u\}$ be a $\gamma_{er}$-set. Then $N(u) =$
\{u_1, u_2, u_3\} \text{ and } \langle N(u) \rangle \cong K_3 \text{ so that } G \cong K_4. \text{ Converse is obvious.} \quad \square

**Definition 3.7.9.** Let \( V_k = \{1, 2, \ldots, 2k + 1\} \) for \( k \geq 3 \). Let \( G_k \) be the circulant on \( V_k \) in which vertex \( i \) is adjacent to vertices \( i + 1, i + k, i + k + 1, i + 2k \) where \( 1 \leq i \leq 2k + 1 \).

**Theorem 3.7.10.** \( \gamma_{er}(G_k) = p \) if and only if \( 2k + 1 \not\equiv 0 \text{ (mod 5)} \).

**Proof.** Let \( D \) be a minimum restrained efficient dominating set in \( G_k \) if \( 2k + 1 \not\equiv 0 \text{ (mod 5)} \).

Suppose \( |D| < 2k + 1 \).

**Claim:** \( D \) is an independent set.

Since \( |D| < 2k + 1 \), \( V - D \neq \emptyset \). Let \( i \in V - D \) and by definition \( N(i) = \{i + 1, i + k, i + k + 1, i + 2k\} \) for \( 1 \leq i \leq 2k + 1 \).

**Case (i).** \( i + 1 \in D \).

Now \( N(i) - \{i + 1\} \subset V - D \). We claim that \( N(i + 1) \subset V - D \). By definition, \( N(i + 1) = \{i + 2, i, i + k + 1, i + k + 2\} \). \( i + k + 1 \in N(i + 1) \cap (N(i) - \{i + 1\}) \) and so \( i + k + 1 \in V - D \). Also \( i + k + 1 \in N(i + 1) \cap N(i + k + 2) \) and \( D \) is \( \gamma_{er} \)-set so that \( i + k + 2 \in V - D \). In a similar way \( i + 2 \in N(i + 1) \cap N(i + k + 2) \) and so \( i + 2 \in V - D \). Thus \( N(i + 1) \subset V - D \).

**Case (ii).** \( i + k \in D \).

As in case (i) \( N(i) - \{i + k\} \subset V - D \). Also \( N(i + k) = \{i + k + 1, i + k - 1, i + 2k, i\} \). Since \( i + 2k \) and \( i + k + 1 \) lie in \( N(i) - \{i + k\} \), \( i + 2k \in V - D \) and \( i + k + 1 \in V - D \). Further \( i + 2k \in N(i + k - 1) \cap N(i + k) \) and so \( i + k - 1 \in V - D \). Thus \( N(i + k) \subset V - D \).

**Case (iii).** \( i + k + 1 \in D \).

\( N(i + k + 1) = \{i + k + 2, i + k, i + 2k + 1 = i, i + 1\} \). Clearly \( i + 1 \) and \( i + k \) lie in
By case (i) \( i + k + 2 \in V - D \) and so \( N(i + k + 1) \subset V - D \).

Case (iv). \( i + 2k \in D \).

\[ N(i + 2k) = \{ i, i + 2k - 1, i + k - 1, i + k \}. \]
Clearly, \( i + k \in V - D, i + k \in N(i + 2k) \cap N(i + k - 1) \) and so \( i + k - 1 \in V - D \). Also \( i + k - 1 \in N(i + 2k) \cap N(i + 2k - 1) \) and hence \( i + 2k - 1 \in V - D \). Thus \( N(i + 2k) \subset V - D \).

By cases (i) to (iv) every vertex in \( D \) which dominates vertices in \( V - D \) has all its neighbors in \( V - D \). If \( E_1 \subseteq D \) is the set of all vertices which do not dominate any vertex in \( V - D \) then \( \langle E_1 \rangle \) is disconnected. So \( E_1 = \emptyset \) and \( D \) is an independent set.

Since \( G_k \) is 4 regular graph and \( D \) is independent we have \( |V - D| = 4|D| \) and so \( 2k + 1 = 5|D| \) which is a contradiction. Hence \( \gamma_{er}(G_k) = p \).

Conversely suppose \( \gamma_{er}(G_k) = p \) and \( 2k + 1 \equiv 0 \pmod{5} \).

Now by definition of \( G_k \) (by symmetry), we find \( D = \{ i, i + 3, \ldots, i + 3(m - 1) \} \forall i \in 1 \leq i \leq 2k + 1 \) is a restrained dominating set of \( G_k \) and the neighbors of every vertex in \( D \) are given by \( i + 1, i - 1, i + k, i + k + 1; i + 1 + 5, i - 1 + 5, i + k + 5, i + k + 1 + 5; \ldots i + 1 + 5(m - 1), i - 1 + 5(m - 1), i + k + (5(m - 1), i + k + 1 + 5(m - 1) \) respectively. Obviously \( D \) is a \( \gamma_{er} \) set and \( |D| \neq p \), which is a contradiction and so \( 2k + 1 \not\equiv 0 \pmod{5} \).

\[ \boxed{\text{Theorem 3.7.11. } \gamma_{er}(S_n) = (n - 1)! \quad \forall n \geq 3.} \]

\[ \text{Proof. } S = \{ \alpha \in V(S_n) | \alpha(1) = 1 \} \text{ is an efficient restrained dominating set of } S_n \text{ and so } \gamma_{er}(S_n) \leq |S| = (n - 1)!. \text{ Hence by theorem 2.2.2 it follows that } \gamma_{er}(S_n) = (n - 1)!. \]