Chapter 6

Restrained domsaturation number of a graph

The restrained domsaturation number $d_{Sr}(G)$ of a graph $G = (V, E)$ is the least positive integer $k$ such that every vertex of $G$ lies in a restrained dominating set of cardinality $k$. In this chapter we obtain certain bounds for $d_{Sr}(G)$ and characterize the graphs with large restrained domsaturation numbers. We derive certain Nordhaus-Gaddum-type results on $d_{Sr}(G)$, characterize graphs with equal restrained domination number and restrained domsaturation number and obtain the relationship between $d_{Sr}(G)$ and certain other graph theoretic parameters.

6.1 Introduction

Acharya [1] introduced the concept of domsaturation number of a graph. The least positive integer $k$ such that every vertex of $G$ lies in a dominating set of cardinality $k$ is called the domsaturation number of $G$ and is denoted by $ds(G)$. A detailed study of this parameter was already done by Arumugam and Kala [5,6] and about 5 papers[5,6] have been published. Motivated by this concept, we introduce the concept of restrained
6.2 Some bounds on restrained domsaturation number of a graph

Definition 6.2.1. The restrained domsaturation number $d_{sr}(G)$ of a graph $G$ is the least positive integer $k$ such that every vertex of $G$ lies in a restrained dominating set of cardinality $k$.

Example 6.2.2.

(i) If $G \cong K_p$ then $d_{sr}(G) = 1$.

(ii) If $G \cong K_{m,n}$, $2 \leq m \leq n$ then $d_{sr}(G) = 2$.

(iii) If $G \cong G_1$ where $G_1$ is given in Fig. 6.1 then $d_{sr}(G) = 4$.

Theorem 6.2.3. If $P_p$ is any path with $p = 3k + r (k \in Z, k \geq 1, 0 \leq r \leq 2)$ then $d_{sr}(P_p) = k + r + 2$.

Proof. Case (i) $r = 0$
By theorem (1.71), \( \gamma_r(P_{3k}) = k + 2 \). Let \( V(P_{3k}) = \{v_1, v_2, \ldots, v_{3k-1}, v_{3k}\} \)

\[
A = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k-1}, v_{3k}\}
\]
\[
B = \{v_1, v_2, v_5, \ldots, v_{3k-4}, v_{3k-1}, v_{3k}\}
\]
\[
C = \{v_1, v_2, v_3, v_6, \ldots, v_{3k-3}, v_{3k}\}
\]

are \( \gamma_r \)-sets of cardinality \( k + 2 \) and \( A \cup B \cup C = V(P_{3k}) \). Hence \( ds_r(P_{3k}) = k + 2 \).

**Case (ii) \( r = 1 \).**

By theorem 1.71, \( \gamma_r(P_{3k+1}) = k + 1 \).

Let \( V(P_{3k+1}) = \{v_1, v_2, \ldots, v_{3k}, v_{3k+1}\} \) \( A_1 = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\} \) is a \( \gamma_r \)-set. \( B_1 = \{v_1, v_2, v_5, v_8, \ldots, v_{3k-4}, v_{3k-1}, v_{3k}, v_{3k+1}\} \) and \( C_1 = \{v_1, v_2, v_3, v_6, \ldots, v_{3k-3}, v_{3k}, v_{3k+1}\} \) are minimal restrained dominating sets of cardinality \( k + 3 \). Also \( B_1 \) and \( C_1 \) are the only minimal set containing \( v_2 \) and also \( A_1 \cup B_1 \cup C_1 = V(P_{3k+1}) \). Hence \( ds_r(P_{3k+1}) = k + 3 \).

**Case (iii) \( r = 2 \).**

By theorem 1.71, \( \gamma_r(P_{3k+2}) = k + 2 \). \( A_2 = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}, v_{3k+2}\} \) and \( B_2 = \{v_1, v_2, v_5, v_8, \ldots, v_{3k-1}, v_{3k+2}\} \) are \( \gamma_r \)-sets, and \( C_2 = \{v_1, v_2, v_3, v_6, \ldots, v_{3k-3}, v_{3k}, v_{3k+1}, v_{3k+2}\} \) is the only minimal restrained dominating sets containing \( v_3 \) and is of cardinality \( k + 4 \). Also \( A_2 \cup B_2 \cup C_2 = V(P_{3k+2}) \) and so \( ds_r(P_{3k+2}) = k + 4 \).

**Theorem 6.2.4.** If \( C_p \) is any cycle with \( p = 3k + r(k \in \mathbb{Z}, k > 1, 0 \leq r \leq 2) \) then \( ds_r(C_p) = k + r \).

**Proof.** Similar to theorem 6.2.3.

**Theorem 6.2.5.** If \( G \) is any connected graph with \( \delta(G) = 1 \) such that \( \gamma_r(G) \) is equal to the number of pendant vertices, then \( \gamma_r(G) + 1 \leq ds_r(G) \leq p \). Furthermore,
\( ds_r(G) = \gamma_r(G) + 1 \) if and only if every support of \( G \) is adjacent to at least two other supports and \( ds_r(G) = p \) if and only if \( G \cong G_1 \) where \( G_1 \) is given in Fig. 6.2.

![Fig. 6.2](image)

**Proof.** Let \( S \) be the set of all pendant vertices of \( G \) and let \( |S| = \lambda \). Since \( \gamma_r(G) = \lambda \), \( S \) is the only \( \gamma_r \)-set of \( G \) and so \( G \not\cong K_{1,p-1} \). Hence by theorem 2.8.8, every non-pendant vertex is a support. For every \( v \notin S \), \( S \cup \{v\} \cup D \) is a minimal restrained dominating set containing \( v \) where \( D \) is the minimum number of supports needed so that \( S \cup \{v\} \cup D \) is a minimal restrained dominating set containing \( v \). Since \( 0 \leq |D| \leq p - \lambda - 1 \), we have \( \gamma_r(G) + 1 \leq ds_r(G) \leq \gamma_r(G) + 1 + p - \lambda - 1 \) so that \( \gamma_r(G) + 1 \leq ds_r(G) \leq p \).

Suppose \( ds_r(G) = \gamma_r(G) + 1 \). If there exists a support \( u \) adjacent to exactly one support \( v \), then the minimal restrained dominating set containing \( v \), say \( A \) should also contain \( u \) since otherwise, \( u \) has no neighbor in \( V - A \). Then \( ds_r(G) > \gamma_r(G) + 1 \) which is a contradiction and so every support of \( G \) is adjacent to at least 2 supports.

Conversely, for every support \( w \) in \( G \), \( S \cup \{w\} \) is a minimal restrained dominating set containing \( w \) and so \( ds_r(G) = \gamma_r(G) + 1 \).

Suppose \( ds_r(G) = p \). By theorem 2.8.8, there exists a support \( u \in V(G) \) such that the only restrained dominating set containing \( v \) is \( V(G) \). As \( G \not\cong K_{1,p-1} \), \( G \) has at least one another support, say \( w \).

**Claim 1.** For any support \( w \neq v \), \( d(v, w) = 1 \).
If not, there exists at least one support \( w' \) such that \( d(v, w') = k > 1 \). If \((v, w_1, w_2, \ldots, w')\) is a \( d(v, w') \) path, then \( V(G) - \{w_1, w_2\} \) is a restrained dominating set containing \( v \), which is a contradiction.

**Claim 2.** \( v \) is unique.

Suppose there exists another support \( v' \) such that \( V(G) \) is the only restrained dominating set containing \( v' \). Since \( G \) is connected, there exists a vertex \( x \) with \( d(x, v') \leq 2 \) and a vertex \( y \) with \( d(y, v) \leq 2 \) such that \( x \) and \( y \) are adjacent. In all the cases we get a contradiction as in claim 1, so that \( G \cong G_1 \). Converse is obvious. \( \Box \)

**Theorem 6.2.6.** For any graph \( G \), \( \gamma_r(G) \leq ds_r(G) \leq \min\{\gamma_r(G) + \Delta(G), p\} \) and these bounds are sharp.

**Proof.** Lower bound is obvious. Suppose \( ds_r(G) = \gamma_r(G) + \Delta(G) + k \) where \( k \geq 1 \). Then there exists a vertex \( v \in V(G) \) such that the cardinality of the minimal restrained dominating set \( A \) containing \( v \) is \( \gamma_r(G) + \Delta(G) + k \). If \( S \) is any \( \gamma_r \)-set, then \( v \not\in S \). Since \( |A| = \gamma_r(G) + \Delta(G) + k \) and \( S \cap N(v) \neq \emptyset \), \( \langle V - S \cup \{v\} \rangle \) has \( \Delta(G) + k - 1 \) isolated vertices and so \( |N(v)| \geq \Delta(G) + k \), which is a contradiction. Thus \( ds_r(G) \leq \gamma_r(G) + \Delta(G) \). But always \( ds_r(G) \leq p \) and so \( ds_r(G) \leq \min\{\gamma_r(G) + \Delta(G), p\} \).

![Fig. 6.3](image)

156
If $G \cong C_6$, $d_{sr}(G) = \gamma_r(G) = 2$ and so the lower bound is sharp. If $G \cong G_1$, where $G_1$ is graph given in Fig. 6.3(a), $d_{sr}(G) = 9 = \min\{\gamma_r(G) + \Delta(G) = 11, p = 9\}$. If $G \cong G_2$, where $G_2$ is given in Fig. 6.3(b), $d_{sr}(G) = 12 = \min\{\gamma_r(G) + \Delta(G) = 12, p = 14\}$. These exhibit the sharpness of the upper bound.

**Theorem 6.2.7.** If $k$ is any integer such that $1 \leq k \leq \gamma_r(G)$, then there exists a connected graph $G$ with $d_{sr}(G) = \gamma_r(G) + k$.

**Proof.** Let $G$ be any connected graph such that $\gamma_r(G)$ equals the number of pendant vertices of $G$. Then by theorem 2.8.8, every non pendant vertex is a support. Since $d_{sr}(G) = \gamma_r(G) + k$ there exists at least one vertex $v \in V(G)$ such that the minimal restrained dominating set containing $v$ is of cardinality $\gamma_r(G) + k$. So $v$ is adjacent to $k - 1$ supports whose neighbors are only pendant vertices. So, the graph $G$ can contain any number of supports, satisfying the following two conditions.

(i) Every support is adjacent to at most $k - 1$ supports whose neighbors are only pendant vertices.

(ii) There exists at least one vertex adjacent exactly to $k - 1$ supports whose neighbors are only pendant vertices. 

**Example 6.2.8.** As an illustration of the above theorem, consider the graph $G$ given in Fig. 6.4. We observe that $d_{sr}(G) = \gamma_r(G) + 5$.

**Theorem 6.2.9.** There exists a graph $G$ for which $d_{sr}(G) - ds(G)$ can be made arbitrarily large.

**Proof.** Let $P_{p-k} = \{u_1, u_2, \ldots, u_{p-k}\}$ be a path on $p - k$ vertices, where $1 \leq k \leq p - 1$. Let $S = \{v, v_1, v_2, \ldots, v_{k-1}\}$ and join the vertex $v$ to each of the vertices
in $P_{p-k}$ and to each vertex in $S - \{v\}$. The resulting graph $G$ is of order $p$ and $\gamma(G) = 1$. Also $\{v, u_i\}$ for $1 \leq i \leq p-k$ and $\{v, v_j\}$ for $1 \leq j \leq k-1$ are minimal dominating sets containing $u_i$ and $v_j$ respectively and so $ds(G) = 2$.

Case 1. $k = p - 1$.

Now $V(P_{p-k}) = \{u_1\}$ and so $S \cup \{u_1\}$ is a restrained dominating set containing $u_1$. Hence $\gamma_r(G) = k + 1$ and $ds_r(G) = k + 1$ so that $ds_r(G) - ds(G) = k + 1 - 2 = k - 1$.

In all the other cases $S$ is the minimum restrained dominating set of $G$ and so $\gamma_r(G) = k$.

Case 2. $k = p - 2$.

Now $V(P_{p-k}) = \{u_1, u_2\}$ and $S \cup \{u_1, u_2\}$ is a minimal restrained dominating set of cardinality $k + 2$ containing $u_1$ as well as $u_2$. Then $ds_r(G) = k + 2$ and $ds_r(G) - ds(G) = k + 2 - 2 = k$.

Case 3. $k = p - 3$.

Let $V(P_{p-k}) = \{u_1, u_2, u_3\}$. Since $S \cup \{u_1, u_2, u_3\}$ is the only minimal restrained dominating set containing $u_2$, $ds_r(G) = k + 3$ and so $ds_r(G) - ds(G) = k + 3 - 2 = k + 1$.

Case 4. $k \leq p - 4$. 

158
Every vertex \( u_i (1 \leq i \leq p-k) \) of \( P_{p-k} \) other than \( u_2 \) and \( u_{p-k-1} \) lies in a restrained dominating set of cardinality \( k + 1 \) since \( S \cup \{u_i\} \) is one such set. \( S \cup \{u_2\} \) and \( S \cup \{u_{p-k-1}\} \) are not restrained dominating sets as \( u_1 \) and \( u_{p-k} \) are isolated vertices in \( (V - (S \cup \{u_2\})) \) and \( (V - S \cup \{u_{p-k-1}\}) \) respectively. Hence \( d_{sr}(G) = k + 2 \) and so \( d_{sr}(G) - ds(G) = k \).

Hence \( d_{sr}(G) - ds(G) = k - 1 \) or \( k \) or \( k + 1 \) where \( k \) can be chosen arbitrarily large.

\[ \square \]

### 6.3 Characterization of graphs with extreme restrained domsaturation numbers

**Theorem 6.3.1.** Let \( G \) be a connected graph. Then \( d_{sr}(G) = p \) if and only if \( G \cong G_i (1 \leq i \leq 2) \) where \( G_i (1 \leq i \leq 2) \) are given in Fig. 6.5.

\[ \text{Fig. 6.5} \]

**Proof.** If \( d_{sr}(G) = p \) then there exists at least one vertex \( v \in V(G) \) such that the only minimal restrained dominating set containing \( v \) is \( V(G) \).

**Case (i).** \( v \) is a pendant vertex.

In this case, we have \( \gamma_r(G) = p \) by choice of \( v \). Hence by Theorem 1.69 \( G \cong G_1 \).

**Case (ii).** \( v \) is a non-pendant vertex.
Let $N(v) = \{v_1, v_2, \ldots, v_k\}(k \geq 2)$. If there exists an edge $(v_i, v_j) \in \langle N(v) \rangle (1 \leq i, j \leq k)$, then $V(G) - \{v_i, v_j\}$ is a restrained dominating set containing $v$ and so $\langle N(v) \rangle$ is independent.

We now claim that every vertex in $V(G) - N[v]$ is a pendant vertex. Suppose there exists $u \in V(G) - N[v]$ such that $d(u) \geq 2$. Since $G$ is connected, there exists a $u - v$ path $P$ with length at least 2. Let $w \in N(u) \cap P$. Then $V(G) - \{u, w\}$ is a restrained dominating set containing $v$ and hence $G \cong G_2$. Converse is obvious. □

The following corollary is immediate.

**Corollary 6.3.2.** Let $G$ be any graph. Then $ds_r(G) = p$ if and only if every component of $G$ is isomorphic to any one of the graphs in Fig (6.5).

**Theorem 6.3.3.** For any connected graph $G$, $ds_r(G) = p - 2$ if and only if $G \cong G_i(1 \leq i \leq 10)$ where $G_i(1 \leq i \leq 10)$ are given in Fig (6.6). Also diam$(G) \leq 7$.

**Proof.** Suppose $ds_r(G) = p - 2$. Then there exists at least one vertex $v$ such that the minimal restrained dominating set containing $v$ is of cardinality $p - 2$. For sake of brevity we denote a restrained dominating set containing $v$ as a $RD - v$ set.

Let $N(v) = \{v_1, v_2, \ldots, v_m\}$. If $\langle N(v) \rangle$ has two edges $v_i v_j$ and $v_k v_l (1 \leq i, j, k, l \leq m)$ then $V(G) - \{v_i, v_j, v_k\}$ and $V(G) - \{v_i, v_j, v_k, v_l\}$ are $RD - v$ sets according as $v_j = v_k$ or $v_j \neq v_k$. Hence $\langle N(v) \rangle$ has at most one edge.

**Case (i).** $\langle N(v) \rangle$ has exactly one edge.

Let $e = v_i v_j \in E(\langle N(v) \rangle)$. If there exists $w \in N(v_i) \cap N(v_j)$ such that $w \neq v$ then $d(w) = 2$ since otherwise $V(G) - \{v_i, v_j, w\}$ is a $RD - v$ set. If there exists another $w' \in N(v_i) \cap N(v_j)$ such that $w' \neq v$ then $V(G) - \{v_i, w, w'\}$ is a $RD - v$ set and
Fig. 6.6
so $w$ is unique. For every $x \in N(v_k) - \{v\}(1 \leq k \leq m, x \neq w)$ if $d(x) \geq 2$ then $V(G) - \{v_i, v_j, x\}$ and $V(G) - \{v_i, v_j, v_k, x\}$ are $RD - v$ sets according as $k \in \{i, j\}$ or otherwise. So $d(x) = 1$. Hence $G \cong G_1$. If there is no such $w$ then $G \cong G_2$.

**Case (ii).** $\langle N(v) \rangle$ is independent.

By Theorem 6.3.1 it follows that $G \not\cong K_{1,p-1}$ and there exists an index $i(1 \leq i \leq m)$ such that $v_i$ has non-pendant neighbors other than $v$. We now claim that there exists at most 2 indices $i$ and $j$. Suppose there exists $v_i, v_j$ and $v_k$ having non-pendant neighbors $w_i, w_j$ and $w_k$.

Suppose $w_i'$ is another non-pendant neighbor of $v_i$. If $w_i'$ is adjacent to $w_i$ then $V(G) - \{v_i, w_i, w_j\}$ and $V(G) - \{v_i, w_i, v_j, w_j\}$ are $RD - v$ sets according as $w_j$ is adjacent to a vertex in $\{v_i, w_i\}$ or $w_j$ is adjacent to a vertex in $V(G) - \{v_i, w_i, w_i'\}$. If $w_i'$ is not adjacent to $w_i$ then $V(G) - \{v_i, w_i, w_i'\}$ is a $RD - v$ set. Hence $w_i, w_j$ and $w_k$ are unique non-pendant neighbors of $v_i, v_j$ and $v_k$.

If $w_i = w_j = w_k$ then $V(G) - \{v_i, v_j, w_i\}$ is a $RD - v$ set. Suppose exactly two of them are equal say $w_i = w_j$. If $w_k$ is adjacent to $w_i$ then $V(G) - \{v_i, v_j, w_i\}$ is a $RD - v$ set. Otherwise $V(G) - \{v_i, w_i, v_k, w_k\}$ is a $RD - v$ set.

Suppose $w_i, w_j, w_k$ are distinct. If $w_i$ is adjacent to any vertex in $\{w_j, w_k\}$ say $w_j$ then $V(G) - \{v_j, w_i, v_k, w_k\}$ and $V(G) - \{v_i, w_i, v_k, w_k\}$ are $RD - v$ sets according as $w_k$ is adjacent to $w_i$ or to a vertex in $\{w_j, V(G) - \{w_i, w_j\}\}$. If not $V(G) - \{v_i, w_i, v_j, w_j, v_k, w_k\}$ is a $RD - v$ set. Hence there exists at most 2 indices $i$ and $j$ such that $v_i$ and $v_j$ have non-pendant neighbors.

**Subcase 1:** There exists exactly two vertices $v_i$ and $v_j(i \neq j; 1 \leq i, j \leq m)$ having non-pendant neighbors $w_i$ and $w_j$.
As above \( v_i \) and \( v_j \) can each have exactly one non-pendant neighbor.

If \( w_i \) and \( w_j \) are distinct and adjacent, then at least one of \( w_i \) and \( w_j \) have degree 2 since otherwise \( V(G) - \{v_i, w_i, v_j, w_j\} \) is a \( RD - v \) set. If \( d(w_i) = d(w_j) = 2 \) then \( G \cong G_3 \). If \( d(w_i) = 2 \) and \( d(w_j) \geq 3 \) then \( x \in N(w_j) - \{v_j\} \) has degree 1 since otherwise \( V(G) - \{w_i, w_j, x\} \) is a \( RD - v \) set. In this case \( G \cong G_4 \). If \( w_i \) and \( w_j \) are non adjacent then \( V(G) - \{v_i, w_i, v_j, w_j\} \) is a \( RD - v \) set. If \( w_i = w_j \) then \( d(w_i) = 2 \) since otherwise \( V(G) - \{v_i, w_i, v_j\} \) is a \( RD - v \) set. Hence \( G \cong G_5 \).

Subcase 2: There exists exactly one vertex \( v_i \) having non-pendant neighbors.

As proved earlier \( v_i \) can have at most 2 non-pendant neighbors. So if \( \langle N(v_i) \rangle \) is not independent then there exists exactly one edge \( \langle w_i, x_i \rangle \) in \( \langle N(v_i) - \{v\} \rangle \). If \( d(w_i) = d(x_i) = 2 \) then \( G \cong G_6 \).

If \( d(w_i) \geq 3 \) and \( d(x_i) \geq 3 \) then \( V(G) - \{v_i, w_i, x_i\} \) is a \( RD - v \) set and so without loss of generality let \( d(w_i) \geq 3 \) and \( d(x_i) = 2 \). If \( w_i \) has non-pendant neighbor \( y_i \) other than \( v_i \) and \( x_i \) then \( V(G) - \{v_i, w_i, y_i\} \) is a \( RD - v \) set and so every neighbor of \( w_i \) is a pendant vertex. In this case \( G \cong G_7 \).

Suppose \( \langle N(v_i) \rangle \) is independent. By Theorem 6.3.1 at least one vertex say \( u_i \) in \( \langle N(v) - \{v\} \rangle \) is a non-pendant vertex. \( u_i \) is unique since if there exists another \( u_i \) then \( V(G) - \{v_i, u_i, w_i\} \) is a \( RD - v \) set. Also \( \langle N(u_i) \rangle \) is independent since if there exists an edge \( e = ab \) in \( \langle N(u_i) \rangle \) then \( V(G) - \{v_i, u_i, a\} \) is a \( RD - v \) set. If every neighbor of \( v_i \) other than \( v_i \) is a pendant vertex then \( G \cong G_8 \).

If \( v_i \) has a non-pendant neighbor say \( v_i'' \) then \( v_i'' \) is the only neighbor of \( v_i \) other than \( v_i \) since otherwise \( V(G) - \{v_i, v_i', v_i''\} \) is a \( RD - v \) set. As above \( \langle N(v_i'') \rangle \) is independent. If every neighbor of \( v_i'' \) other than \( v_i \) is a pendant vertex then \( G \cong G_9 \).

Otherwise, proceeding as above \( v_i'' \) has exactly one non-pendant neighbor \( v_i''' \) which
is the only neighbor of $v''_i$ other than $v'_i$. Also $\langle N(v''_i) \rangle$ is independent. If every neighbor of $v''_i$ other than $v''_i$ is a pendant vertex then $G \cong G_{10}$. Otherwise if $v''''_i$ is a non-pendant neighbor of $v''_i$ then $V(G) - \{v_i, v'_i, v''_i, v''''_i\}$ is a RD - $v$ set and so $\text{diam } G \leq 7$. \hfill $\square$

The following corollary is immediate.

**Corollary 6.3.4.** Let $G$ be any graph. Then $ds_r(G) = p - 2$ if and only if exactly one component of $G$ is isomorphic to any one graph in fig (6.6) and every other component is isomorphic to any graph in fig (6.5).

### 6.4 Some Nordhaus-Gaddum-type results on $ds_r(G)$

**Theorem 6.4.1.** Let $G \cong C_p$ where $p = 3k + r (r \in \mathbb{Z}, 0 \leq r \leq 3)$.

Then

$$ds_r(G) + ds_r(\tilde{G}) = \begin{cases} 4 & \text{if } p = 3 \\ 6 & \text{if } p = 4, 5 \\ k + r + 2 & \text{if } p \geq 6 \end{cases}$$

**Proof.** By theorem 6.2.4, $ds_r(C_p) = k + r$. If $p = 3$ then $ds_r(G) = 1$ and $ds_r(\tilde{G}) = 3$ so that $ds_r(C_p) + ds_r(\tilde{C}_p) = 4$. If $p = 4$, $ds_r(C_p) = 2$ and $ds_r(\tilde{C}_p) = 4$ so that $ds_r(C_p) + ds_r(\tilde{C}_p) = 6$. If $p = 5$, then $ds_r(C_p) = 3$, $ds_r(\tilde{C}_p) = 3$, and so $ds_r(C_p) + ds_r(\tilde{C}_p) = 6$. If $p \geq 6$, for any $u \in V(G)$, $\{u, v\}$ is a $\gamma_r$-set for $\tilde{C}_p$ where $v \in N(u)$ in $G$. Hence $ds_r(\tilde{C}_p) = 2$, so that $ds_r(C_p) + ds_r(\tilde{C}_p) = k + r + 2$. \hfill $\square$

**Theorem 6.4.2.** For any connected graph $G$ with at least two pendant vertices, $4 \leq ds_r(G) + ds_r(\tilde{G}) \leq p + 4$. Lower bound is attained if and only if $G \cong K_2$ and graphs
attaining upper bounds are characterized as follows.

\[
\text{attaining upper bounds are characterized as follows.}
\]

\[
\begin{align*}
ds_r(G) + ds_r(\bar{G}) &= \begin{cases} 
p + 4 & \text{if and only if } G \cong P_4 \\
p + 3 & \text{if and only if } G \cong K_{1,2} \text{ or } G_1 \\
p + 2 & \text{if and only if } G \cong K_{1,p+1}(p \neq 3), G_2 \text{ or } G_3 \quad (G_3 \neq P_4, K_{1,2} \text{ and } G_1) 
\end{cases}
\end{align*}
\]

where \(G_1\), \(G_2\) and \(G_3\) are given in Fig (6.7).

\[
\begin{align*}
\text{Fig. 6.7}
\end{align*}
\]

**Proof.** Lower bound follows by theorem 1.78. To establish the upper bound it is enough to prove that \(ds_r(\bar{G}) \leq 4\). Let \(P = \{u_1, u_2, \ldots, u_m\}\) be the set of pendant vertices of \(G\) and \(S = \{v_i | 1 \leq i \leq m\}\) be the set of corresponding supports (not necessarily distinct). If \(m \geq 3\) and there exists an index \(i\) such that \((V(G) - \{u_i, v_i\})\) has two distinct supports then \(A = \{u_i, v_i\}\) is a restrained dominating set of \(\bar{G}\). If \(w\) is the unique support in \((V(G) - \{u_i, v_i\})\) then \(A = \{u_i, v_i, w\}\) is a restrained dominating set of \(\bar{G}\). Otherwise \(v_i\) is the only support of \(G\) and \(A = \{u_i, v_i\}\) is a restrained dominating set of \(\bar{G}\). In all cases \(A \cup \{x\}\) is a restrained dominating set of \(\bar{G}\), containing \(x\), where \(x \in V(G) - (P \cup S)\). Also for every \(i\), \(\{u_i, v_i\}\) is a restrained dominating set of \(\bar{G}\). Hence \(ds_r(\bar{G}) \leq 4\).

Suppose \(m = 2\). Let the two pendant vertices be \(u\) and \(v\) with supports \(u_1\) and \(v_1\) respectively.
Case (i). $u_1 = v_1$

Let $D = V(G) \setminus \{u, v, u_1\}$. If $D = \phi$, then $\{u, v, u_1\}$ is a restrained dominating set of $\tilde{G}$. If $D \neq \phi$, then $\{u, v_1\}, \{v, v_1\}$ and $\{u, v_1, x\} (x \in V(G) \setminus \{x, v, u_1\})$ are restrained dominating sets of $\tilde{G}$.

Case (ii). $u_1 \neq v_1$

If $(u_1, v_1) \notin E(G)$, then $\{v, u_1\}, \{u, v_1\}$ and $\{u, v_1, x\} (x \in V(G) \setminus \{u, v, u_1, v_1\})$ are restrained dominating sets of $\tilde{G}$. Suppose $(u_1, v_1) \in E(G)$ and let $B = V(G) \setminus \{u, v, u_1, v_1\}$. If $B = \phi$, then $\{u, v, u_1, v_1\}$ is a restrained dominating set of $\tilde{G}$. Suppose $B \neq \phi$. If $|B| \geq 2$, then $\{u, u_1, v_1, x\} (\{v, u_1, v_1, x\}, x \in B)$ are restrained dominating sets of $\tilde{G}$. If $|B| = 1$ and $B = \{w\}$ then $\{u_1, v_1, w\}, \{u, u_1, v_1\}, \{u_1, v_1, v\}$ are restrained dominating sets of $\tilde{G}$. Thus $d_{sr}(\tilde{G}) \leq 4$ and so $4 \leq d_{sr}(G) + d_{sr}(\tilde{G}) \leq p+4$.

If $d_{sr}(G) + d_{sr}(\tilde{G}) = 4$, by hypothesis $d_{sr}(G) = d_{sr}(\tilde{G}) = 2$ since $G$ contains at least 2 pendant vertices. Hence $G \cong K_2$. Converse is obvious.

Suppose $d_{sr}(G) + d_{sr}(\tilde{G}) = p+4$. By above argument, $d_{sr}(G) = p$ and $d_{sr}(\tilde{G}) = 4$ and the conclusion follows from theorem (6.3.1). If $d_{sr}(G) + d_{sr}(\tilde{G}) = p + 3$, then $d_{sr}(G) = p$ and $d_{sr}(\tilde{G}) = 3$ and so by theorem (6.3.1) we have $G \cong K_{1,2}$ or $G_1$.

If $d_{sr}(G) + d_{sr}(\tilde{G}) = p + 2$ then either $d_{sr}(G) = p$ and $d_{sr}(\tilde{G}) = 2$ or $d_{sr}(G) = p - 2$ and $d_{sr}(\tilde{G}) = 4$. By theorems (6.3.1) and (6.3.3) it is easy to verify that $G \cong K_{1, p-1}(p \neq 3), G_2$ or $G_3$ ($G_3 \notin P_4, K_{1,2}$ and $G_1$).

**Theorem 6.4.3.** For a connected graph $G$ with $p \geq 2$ and $\text{diam} G \neq 2$, $4 \leq d_{sr}(G) + d_{sr}(\tilde{G}) \leq p + 4$ and the bounds are sharp.

**Proof.** By theorem 1.78, $\gamma_r(G) + \gamma_r(\tilde{G}) \geq 4$ and hence $d_{sr}(G) + d_{sr}(\tilde{G}) \geq 4$. By theorem 6.3.1, $d_{sr}(G) = p$ if and only if $G \cong G_1$ or $G_2$ where $G_1$ and $G_2$ are
given in Fig. 6.8.

![Fig. 6.8](image)

As diam $G \neq 2$, $G \not\cong G_1$. If $G \cong G_2$, then $ds_r(\tilde{G}) \leq 4$. Suppose $G \not\cong G_2$. Then $ds_r(G) \leq p - 2$. Let $u$ and $v$ be any two vertices with $d(u, v) = \text{diam} G$ and let $P = \{u, v_1, v_2, \ldots, v_{n-1}, v\}$ be a path of length $d(u, v)$.

Let $N = N(v_1) \cap N(v_2)$.

Case (i). $N = \emptyset$.

In this case, $\{v_1, v_2\}$ is a restrained dominating set of $\tilde{G}$. For every $x \in V(G) \setminus \{u, v, v_1, v_2\}$, $\{v_1, v_2, x\}$ is a restrained dominating set of $\tilde{G}$ containing $x$. If $V(G) \setminus \{u, v, v_1, v_2\} = \emptyset$, $G \cong P_4$ and so $ds_r(\tilde{G}) = 4$.

Since $V(G) \setminus \{u, v, v_1, v_2\} \neq \emptyset$, either $\{u, v_2, v\}$ or $\{u, v_1, v\}$ is a restrained dominating set of $\tilde{G}$ containing $u$ and $v$. Thus in this case $ds_r(\tilde{G}) \leq 4$ so that $ds_r(G) + ds_r(\tilde{G}) \leq p + 4$.

Case (ii) $N \neq \emptyset$

Now, $V(G) \setminus N - \{v_1\}(V(G) \setminus N - \{v_2\})$ is a restrained dominating set in $G$ containing $v_2(v_1)$.

Let $x \in N$.

If $|N| = 1$, $V(G) \setminus \{v_1, v_2\}$ is a restrained dominating set in $G$ containing $x$. If
$|N| > 1$, $(V(G) \setminus (N \cup \{v_1\}) \cup \{x\})$ is a restrained dominating set in $G$ containing $x$. So

$$d_{sr}(G) \leq |(V(G) \setminus (N \cup \{v_1\}) \cup \{x\})|$$

$$= p - |N| - 1 + 1 = p - |N|.$$  

Also $N \cup \{v_1, v_2\}$ and $N \cup \{u, v\}$ are restrained dominating sets in $\bar{G}$. For $x \notin N \cup \{u, v, v_1, v_2\}$, $N \cup \{v_1, v_2, x\}$ is a restrained dominating set in $\bar{G}$ containing $x$. So $d_{sr}(\bar{G}) \leq |N| + 3$ so that

$$d_{sr}(G) + d_{sr}(\bar{G}) \leq p - |N| + |N| + 3 = p + 3.$$

Combining cases (i) and (ii) we have

$$d_{sr}(G) + d_{sr}(\bar{G}) \leq p + 4.$$

Sharpness of the lower bound is exhibited by $G \cong K_3$ and that of upper bound is exhibited by $G \cong P_4$.

6.5 Graphs with equal restrained domination and restrained domsaturation number

**Theorem 6.5.1.** Let $G$ be any graph. Then

(i) $d_{sr}(G) = \gamma_r(G) = 1$ if and only if $G \cong K_p (p \neq 2)$.

(ii) $d_{sr}(G) = \gamma_r(G) = p$ if and only if $G \cong$ galaxy.

**Proof.** If $d_{sr}(G) = \gamma_r(G) = 1$ then $\deg v = p - 1$ for every $v \in V(G)$ and so $G \cong K_p$. Converse is obvious.

For any graph $G$ it follows from Theorem 1.69 and the fact that $\gamma_r(G) \leq d_{sr}(G)$, $d_{sr}(G) = \gamma_r(G) = p$ if and only if $G \cong$ galaxy. □
Theorem 6.5.2. Let $G = (V_1, V_2)$ be any connected bipartite graph with $|V_1| \geq |V_2|$. Then $d_{sr}(G) = \gamma_r(G) = 2$ if and only if $G \cong K_2$, $K_{m,n}(n \neq 1)$, $2K_1$ or $K_{m,n} \setminus X (n \geq 3)$ where $X$ is any set of independent edges in $G$.

Proof. If $d_{sr}(G) = \gamma_r(G) = 2$ then every vertex in $V_1$ is non adjacent to at most one vertex in $V_2$ and vice versa. Hence the theorem follows.

Theorem 6.5.3. Let $G$ be a connected cubic graph. Then $d_{sr}(G) = \gamma_r(G) = 2$ if and only if $G$ is isomorphic to one of the graphs $G_1, G_2, G_3$ or $G_4$ given in Fig. 6.9.

![Fig. 6.9](image)

Proof. If $G$ is a cubic graph with $d_{sr}(G) = \gamma_r(G) = 2$, then $5 \leq p \leq 8$. As $G$ is cubic, $p = 6$ or $8$. If $p = 6$ then $G \cong G_1$ or $G_2$. Suppose $p = 8$.

Claim: $G$ is 2-connected.

Suppose $e$ is a cut edge of $G$. As $\gamma_r(G) = 2$, the components of $G \setminus e$ are complete graphs, either $K_4$ or $K_3$. Since $\Delta(G) = 3$ they cannot be $K_4$ and so $p = 6$ which is a contradiction. Hence $G$ is 2-connected and so by theorem 1.29, $G$ is Hamiltonian.

Let $C = (v_1, v_2, \ldots, v_8, v_1)$ be a hamiltonian cycle in $G$. Since $d_{sr}(G) = \gamma_r(G) = 2$, for every $v \in V(G)$ there exists $u \in V(G) - N[v]$ such that $u$ is adjacent to all the vertices in $V(G) - N[v]^*$.

Case 1. $G$ contains a triangle.
Without loss of generality let \( v_1 \) be adjacent to \( v_3 \). By \((*)\), \( v_2 \) cannot be adjacent to \( v_5, v_6 \) or \( v_7 \). Hence \( v_2 \) can be adjacent only to \( v_4 \) or \( v_8 \). If \( v_2 \) is adjacent to \( v_4 \), by symmetry we have \( v_5 \) is adjacent to \( v_7 \), \( v_6 \) is adjacent to \( v_8 \) and so \( G \cong G_3 \).

**Case 2.** \( G \) contains no triangles.

Now \( v_1 \) cannot be adjacent to \( v_3 \) or \( v_7 \). By \((*)\), \( v_1 \) is not adjacent to \( v_5 \) and so \( v_1 \) is adjacent to either \( v_4 \) or \( v_6 \). Without loss of generality, let \( v_1 \) be adjacent to \( v_6 \).

Also \( v_2 \) cannot be adjacent to \( v_4 \) or \( v_8 \). If \( v_2 \) is adjacent to \( v_7 \) then \( v_3 \) is adjacent to \( v_8 \) and so \( d(v_4) = 2 \) which is a contradiction. So \( v_2 \) is adjacent to \( v_5 \). By \((*)\), \( v_3 \) cannot be adjacent to \( v_7 \) and so \( v_3 \) is adjacent to \( v_8 \). Then \( v_4 \) is adjacent to \( v_7 \) and \( G \cong G_4 \). Converse is obvious.

**Theorem 6.5.4.** Let \( T \) be a tree of order \( p \geq 3 \). Then \( ds_r(G) = \gamma_r(G) = p - 2 \) if and only if \( T \) is obtained from \( P_6 \) by adding zero or more number of pendant vertices to the supports.

**Proof.** Follows from theorem 1.74. \( \square \)

**Theorem 6.5.5.** Let \( G \) be a connected graph of order \( p \) containing a cycle. Then \( ds_r(G) = \gamma_r(G) = p - 2 \) if and only if \( G \) is \( C_4, C_5 \) or \( G \) can be obtained from \( C_3 \) by adding zero or more number of pendant vertices to at most 2-vertices of the cycle.

**Proof.** Follows from theorem 1.75. \( \square \)

**Theorem 6.5.6.** If \( G \) is a domatically full graph that is \( k \)-regular \( (k \geq 2) \) then \( ds_r(G) = \gamma_r(G) \).

**Proof.** Let \( \{D_1, D_2, \ldots, D_{k+1}\} \) be a domatic partition of \( G \) so that for any \( i(1 \leq i \leq k + 1), D_1 \cup D_2 \cup \cdots \cup D_{i-1} \cup D_{i+1} \cup \cdots \cup D_{k+1} = V - D_i \). Since \( k \geq 2 \), every
u ∈ V - Di has a neighbor in V - Di and so Di is a restrained dominating set for every i(1 ≤ i ≤ k + 1). Moreover any set Di either contains a vertex or exactly one of its neighbors. So each Di is independent. Since every vertex in Di is adjacent to exactly one vertex in each Dj(j ≠ 1, 1 ≤ j ≤ k + 1), |Di| = γr for every i. Thus $ds_r(G) = \gamma_r(G)$. \hfill \Box

**Theorem 6.5.7.** For any tree T, $ds_r(T) = \gamma_r(T) = 2$ if and only if T is not isomorphic to $K_{1,2}$ or $B(r,s)$ where at least one of r or s equals 1.

**Proof.** Let T be a tree not isomorphic to $K_{1,2}$ or $B(r,s)$ where at least one of r and s equals 1. If $T \cong K_{1,p-1}(p \neq 3)$ then $\gamma_r(T) = 2 = ds_r(T)$. Suppose $T \not\cong K_{1,p-1}$. Then there exists at least 2 pendant vertices u and v with distinct supports $u_1$ and $v_1$ respectively and by choice of T, $d_T(u_1) \leq p - 3$ and $d_T(v_1) \leq p - 3$.

**Case 1.** $d_T(u_1) = p - 3$ and $d_T(v_1) = p - 3$.

If $u_1$ and $v_1$ are adjacent then $T \cong T_1$ where $T_1$ is given in Fig.6.10.

![Fig. 6.10](image)

{$u, v_1$, $\{v_2, v_1\}$, $\{u, u_1\}$, $\{u_2, u_1\}$} are all minimum restrained dominating sets of $\bar{T}$ and so $\gamma_r(\bar{T}) = ds_r(\bar{T}) = 2$.

If $u_1$ and $v_1$ are non-adjacent then $T \cong P_5$ and clearly $\gamma_r(\bar{T}) = ds_r(\bar{T}) = 2$.

**Case 2.** $d_T(u_1) = p - 3$ and $d_T(v_1) \neq p - 3$.  

171
If $u_1$ and $v_1$ are adjacent, then $T \cong T_2$ given in Fig. 6.11.

Since $dT(v_1) \neq p - 3$, $dT(u_1) \geq 4$. For every $u' \in N(u_1), \{u_1, u'\}$ is a $\gamma_r$-set of $\bar{T}$ and for every $v' \in N(v_1), \{v_1, v'\}$ is a $\gamma_r$-set of $\bar{T}$ and so $\gamma_r(\bar{T}) = ds_r(\bar{T}) = 2$.

If $u_1$ and $v_1$ are non-adjacent then $T \cong T_3$ given in Fig. 6.12.

As above $dT(u_1) \geq 3$. For every $u' \in N(u_1), \{u', u_1\}$ is a $\gamma_r$-set of $\bar{T}$. Also $\{u_1, v\}$ and $\{v_1, u\}$ are $\gamma_r$-sets of $\bar{T}$ and so $\gamma_r(\bar{T}) = ds_r(\bar{T}) = 2$.

**Case 3.** $dT(u_1) \neq p - 3$ and $dT(v_1) = p - 3$.

This is symmetrical to case 2.

**Case 4.** $dT(u_1) \neq p - 3$ and $dT(v_1) \neq p - 3$.

If $u_1$ and $v_1$ are adjacent then $dT(u_1) \geq 4$ and $dT(v_1) \geq 4$ and for every $u' \in N(u_1), \{u_1, u'\}$ is a $\gamma_r$-set of $\bar{T}$ and for every $v' \in N(v_1), \{v_1, v'\}$ is a $\gamma_r$-set of $\bar{T}$ so that $ds_r(\bar{T}) = \gamma_r(\bar{T}) = 2$.

Suppose $u_1$ and $v_1$ are non-adjacent. Then $dT(u_1) \geq 3$ and $dT(v_1) \geq 3$. For
every \( x \in V(T) \) with \( d(u_1, x) \neq 2, \{ x, u_1 \} \) is a \( \gamma_r \)-set of \( \hat{T} \) containing \( x \) and if \( d(u_1, x) = 2, \{ x, u \} \) is a \( \gamma_r \)-set of \( \hat{T} \) containing \( x \). The \( \gamma_r \)-sets containing neighbors of \( u_1 \) and \( v_1 \) are as above. Thus \( \gamma_r(\hat{T}) = ds_r(\hat{T}) = 2 \).

Conversely suppose that \( ds_r(\hat{T}) = \gamma_r(\hat{T}) = 2 \).

If \( T \cong K_{1,2} \) then \( \gamma_r(\hat{T}) = 3 \). Suppose \( T \cong B(r, s) \) where \( r = s = 1 \). Then \( T \cong P_4 \) and \( ds_r(P_4) = ds_r(P_4) = 4 \). If \( T \cong B(r, s) \) with exactly one of \( \{ r, s \} \) having value 1, then there is no \( \gamma_r \)-set of \( \hat{T} \) of cardinality 2 containing \( u \). These contradictions exhibit that \( T \) is not isomorphic to \( K_{1,2} \) or \( B(r, s) \) where at least one of \( r \) and \( s \) equals 1. \( \square \)

### 6.6 Relationship of restrained domsaturation number with certain other graph theoretic parameters

**Theorem 6.6.1** Let \( G \) be any graph and let \( k(G) \) be the connectivity of \( G \). Then 
\[
\text{ds}_r(G) + k(G) \leq p + \Delta(G)
\]
and equality holds if and only if \( G \cong K_2 \).

**Proof.** Since \( \text{ds}_r(G) \leq p \) and \( k(G) \leq \Delta(G) \) we have \( \text{ds}_r(G) + k(G) \leq p + \Delta(G) \). If equality holds then \( \text{ds}_r(G) = p \) and \( k(G) = \Delta(G) \) and the result follows from theorems 6.3.1 and 1.27. \( \square \)

**Theorem 6.6.2** For any connected graph \( G \), \( \text{ds}_r(G) + \text{diam}(G) \leq 2p - 1 \). Further

(i) \( \text{ds}_r(G) + \text{diam}(G) = 2p - 1 \) if and only if \( G \cong P_p(p \leq 5) \)

(ii) \( \text{ds}_r(G) + \text{diam}(G) = 2p - 2 \) if and only if \( G \cong K_{1,3}, G_1 \) or \( G_2 \) where \( G_1 \) and \( G_2 \) are given in Fig. 6.13.
(iii) $d_s(G) + \text{diam}(G) = 2p - 3$ if any only if $G \cong K_{1,4}, P_p (6 \leq p \leq 8)$ or $G_i (1 \leq i \leq 7)$ where $G_i$ are given in Fig.6.14.

**Proof.** Since $G$ is connected, $\text{diam}(G) \leq p - 1$. Always $d_s(G) \leq p$ and so $d_s(G) + \text{diam}(G) \leq 2p - 1$. Suppose $d_s(G) + \text{diam}(G) = 2p - 1$. Then $d_s(G) = p$ and $\text{diam}(G) = p - 1$. Since $d_s(G) = p$, by theorem (6.3.1) we observe that $\text{diam}(G) \leq 4$ and so $p \leq 5$. For any graph on $p$ vertices ($3 \leq p \leq 5$) other than $P_p$ we have $d_s(G) + \text{diam}(G) \neq 2p - 1$ and so $G \cong P_p (p \leq 5)$. Converse is obvious.

Suppose $d_s(G) + \text{diam}(G) = 2p - 2$. Since it is not possible that $d_s(G) = p - 1$ we have $d_s(G) = p$ and $\text{diam}(G) = p - 2$. By theorem (6.3.1), we observe that
diam$(G) \leq 4$ and so $p \leq 6$. Among all the graphs with $p \leq 6$, we observe that $K_{1,3}, G_1$ and $G_2$ alone satisfy the given condition. Converse is obvious.

Suppose $d_{sr}(G) + \text{diam}(G) = 2p - 3$. Then either $d_{sr}(G) = p$ and $\text{diam}(G) = p - 3$ (or) $d_{sr}(G) = p - 2$ and $\text{diam}(G) = p - 1$. If $d_{sr}(G) = p$, then by theorem (6.3.1) $p \leq 7$ and among all such graphs we observe that only $K_{1,4}$ and $G_i(1 \leq i \leq 7)$ satisfy the given condition. If $d_{sr}(G) = p - 2$ then by theorem (6.3.3) $p \leq 8$ and among all graph on $p \leq 8$ vertices, we observe that only $P_6, P_7, P_8$, satisfy the given condition.

Converse is obvious. \hfill \Box

**Theorem 6.6.3** Let $G$ be any connected graph and $\chi(G)$ be the chromatic number of $G$. Then $d_{sr}(G) + \chi(G) \leq p + \Delta(G) + 1$ and equality holds if and only if $G \cong K_2$.

**Proof.** Since $d_{sr}(G) \leq p$ we have $d_{sr}(G) + \chi(G) \leq p + \Delta(G) + 1$. If $d_{sr}(G) = p$ then by theorems 6.3.1 and 1.36, $G \cong K_2$. \hfill \Box

**Theorem 6.6.4** Let $G$ be any connected bipartite graph. Then $d_{sr}(G) + \chi(G) = p + 2$ if and only if $G \cong G_1$ or $G_2$ where $G_1$ and $G_2$ are given in Fig. 6.15.

![Fig. 6.15](chart.png)

**Proof.** If $d_{sr}(G) + \chi(G) = p + 2$ then $d_{sr}(G) = p$ and the result follows from theorems 6.3.1 and 1.37. \hfill \Box
Theorem 6.6.5 Let $G$ be any connected bipartite graph. Then $d_{sr}(G) + \chi(G) = p$ if and only if $G \cong G_1, G_2, G_3$, or $G_4$ given in Fig. 6.16.

![Fig. 6.16](image)

Proof. If $d_{sr}(G) + \chi(G) = p$ then $d_{sr}(G) = p - 2$ and the theorem follows from theorem 6.3.3. Converse is obvious.

Corollary 6.6.6 For any tree $G$, $d_{sr}(G) + \chi(G) = p$ if and only if $G \cong G_2, G_3$ or $G_4$ where $G_2, G_3, G_4$ are given in Fig. 6.16.

Proof. Follows from theorem 6.6.5.

Problem 6.6.7 (i) Characterize all connected graphs for which $d_{sr}(G) = \gamma_r(G)$.

(ii) Characterize graphs with $d_{sr}(G) = p - 3$. 

176