Chapter 5

Restrained domination in subdivision graphs

In this chapter we study the restrained domination in subdivision graphs. We obtain certain bounds for $\gamma_r(S(G))$, establish the sharpness for these bounds and derive a Nordhaus-Gaddum type result. Moreover we define $Sd_{\gamma_r}(G)$, the restrained subdivision number of a graph $G$ and characterize several classes of graphs having $Sd_{\gamma_r}(G) = 1$.

5.1 Introduction

Arumugam and Paulraj [9] obtained several results concerning domination parameters in subdivision graphs. Arumugam [8] considered another type of graph modification and defined domination subdivision number $Sd_{\gamma}(G)$ of a graph $G$ to be minimum number of edges that must be subdivided where each edge can be subdivided at most once, in order to increase the domination number. Bhatacharya and Vijayakumar [3], Favaron et. al. [37] and Haynes et. al. [29] obtained several results on $Sd_{\gamma}(G)$. In
this chapter we make a study of restrained domination in subdivision graphs. We introduce the concept of restrained subdivision number $\gamma_r(Sd.)$ and initiate a study of this parameter.

5.2 Restrained domination in subdivision graphs

Definition 5.2.1. If $G$ is a $(p, q)$ graph, the subdivision graph $S(G)$ is a $(p+q, 2q)$ graph which is obtained by subdividing each edge of $G$ exactly once.

Example 5.2.2.

1. If $G \cong P_p$ then $S(G)$ is $P_{2p-1}$.
2. If $G \cong C_p$ then $S(G)$ is $C_{2p}$.
3. If $G \cong \text{star}$ then $S(G)$ is a spider.

Theorem 5.2.3.

(i) $\gamma_r(S(C_p)) = 2p - 2\lfloor \frac{2p}{3} \rfloor$.

(ii) $\gamma_r(S(K_p)) = \begin{cases} p + 1 & \text{if } p = 2 \\ \frac{5k^2 - k}{2} & \text{if } p = 3k \\ \frac{5k^2 + k + 2}{2} & \text{if } p = 3k + 1 \\ \frac{5k^2 + 5k + 4}{2} & \text{if } p = 3k + 2 \end{cases}$

(iii) $\gamma_r(S(K_{m,n})) = m + n$ if $m \leq n$ and $m \geq 2$.

(iv) $\gamma_r(S(W_p)) = p$.

Proof. (i) $\gamma_r(S(C_p)) = \gamma_r(C_{2p}) = 2p - 2\lfloor \frac{2p}{3} \rfloor$.

(ii) When $p = 2$, $S(K_p) = P_3$ and so $\gamma_r(S(K_p)) = p + 1$. 

133
Let $p \geq 3$. Then $K_p$ is a Hamiltonian graph, containing a cycle of length $p$. So $S(K_p)$ contains a cycle of length $2p$. Let $C_{2p} = (1, u_1, 2, u_2, \ldots, p, u_p)$, where $1, 2, \ldots, p \in K_p$ and $u_1, u_2, \ldots, u_p \in V(S(K_p)) - V(K_p)$. Let $D$ be a minimum restrained dominating set of $C_{2p}$ and we aim at constructing a restrained dominating set $S$ of $S(K_p)$ containing $D$. Let $D_0 = D \cap V(K_p)$ and $D_1 = (V - D) \cap V(K_p)$.

By theorem 1.72,

$$\gamma_r(C_{2p}) = 2p - 2 \left\lfloor \frac{2p}{3} \right\rfloor$$

and so

$$|D| = 2p - 2 \left\lfloor \frac{2p}{3} \right\rfloor,$$

which is even. Without loss of generality let $1 \in D$. We observe that

$$|D_0| = \frac{|D|}{2} = p - \left\lfloor \frac{2p}{3} \right\rfloor.$$

\[ \square \]

Case (i). Let $p = 3k$.

Then $D_0 = \{1, 4, 7, \ldots, 3k - 2\}$ and $D_1 = \{2, 3; 5, 6; 8, 9, \ldots, 3k - 1, 3k\}$. Since $1, 4 \in D \subset S$, the vertex of $V(S(K_p))$ which subdivides the edge joining 1 and 4 must be in $S$. So $\binom{|D_0|}{2}$ vertices lie in $S$. Also every vertex which subdivides every edge joining 2 and any vertex in $D_1$ except 3 also must lie in $S$. Similarly for 3. Now

$$|D_1| = p - |D_0| = p - \left( p - \left\lfloor \frac{2p}{3} \right\rfloor \right) = \left\lfloor \frac{2p}{3} \right\rfloor.$$

So

$$2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2 \right) + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 4 \right) + \cdots + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2(k - 1) \right)$$

vertices lie in $S$. 

134
Thus

\[ 2p - 2 \left\lfloor \frac{2p}{3} \right\rfloor + \left( p - \left\lfloor \frac{2p}{3} \right\rfloor \right) + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2 \right) \]
\[ + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 4 \right) + \cdots + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2(k - 1) \right) \]

vertices lie in \( S \).

From the construction of \( S \), \( V - S \) has no isolated vertices, \( S \) dominates \( S(K_p) \).

\( S - \{ v \} \) is not a restrained dominating set of \( S(K_p) \forall v \in S \).

Thus

\[ \gamma_r(S(K_p)) = 2p - 2 \left\lfloor \frac{2p}{3} \right\rfloor + \left( p - \left\lfloor \frac{2p}{3} \right\rfloor \right) + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2 \right) \]
\[ + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 4 \right) + \cdots + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2(k - 1) \right) = \frac{5k^2 - k}{2} \]

Case (ii). \( p = 3k + 1 \).

Since in this case \( u_{3k+1} \), which is the subdivision vertex of the edge joining 1 and

\( 3k + 1 \) already lies in \( D \) so we have,

\[ \gamma_r(S(K_p)) = 2p - 2 \left\lfloor \frac{2p}{3} \right\rfloor + \left( p - \left\lfloor \frac{2p}{3} \right\rfloor \right) - 1 + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2 \right) + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 4 \right) \]
\[ + \cdots + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2(k - 1) \right) = \frac{5k^2 + k + 2}{2} \]

Case (iii). \( p = 3k + 2 \).

In this case \(|D_1| = \left\lfloor \frac{2p}{3} \right\rfloor = 2k + 1 \) and so

\[ \gamma_r(S(K_p)) = 2p - 2 \left\lfloor \frac{2p}{3} \right\rfloor + \left( p - \left\lfloor \frac{2p}{3} \right\rfloor \right) + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2 \right) + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 4 \right) \]
\[ + \cdots + 2 \left( \left\lfloor \frac{2p}{3} \right\rfloor - 2k \right) = \frac{5k^2 + 5k + 4}{2} \]

(iii) Case (i). Let \((X, Y)\) be a bipartition of \( K_{m,n} \) with

\[ X = \{ u_1, u_2, \ldots, u_m \} \text{ and } Y = \{ v_1, v_2, \ldots, v_n \} \]
Let \( w_j \) be the vertex subdividing the edge \( e_{ij} = u_i, v_j \) where \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). If \( A = \{w_i | 1 \leq i \leq m\} \), then \( D = Y \cup A \) is a restrained dominating set of \( S(K_{m,n}) \). For any \( v \in Y \) \( D \setminus \{v\} \) is not a dominating set since either \( v \) turns to be an isolated vertex or some vertex in \( V(S(K_{m,n}) \setminus V(K_{m,n}) \) loses its domination. Similarly for any \( w_{i1} \in A, D \setminus \{w_{i1}\} \) is not a dominating set as the corresponding vertex \( u_i \) is not dominated. Hence \( D \) is a minimal dominating set. Clearly it is a minimum dominating set and so \( \gamma_r(S(K_{m,n})) = |D| = |Y| + |A| = m + n \).

**Case (v).** \( \gamma_r(S(W_p)) = p \).

Let \( u \) be the center of \( W_p \) and \( u_i \ 1 \leq i \leq p - 1 \) be the vertices in \( N(u) \). Let \( u_i(1 \leq i \leq p - 1) \) be the vertices subdividing edges \( uu_i \) and \( w_i \) be the vertices subdividing edges \( uu_i+1(u_i = u_1) \). It is easy to observe that \( D = \{u\} \cup \{w_i\}(1 \leq i \leq p - 1) \) is a \( \gamma_r \)-set of \( S(W_p) \) and so \( \gamma_r(S(W_p)) = p \).

**Theorem 5.2.4.** If \( G \) is a connected \((p, q)\) graph such that \( G \not\cong K_2, 2 \leq \gamma_r(S(G)) \leq p + q - \Delta(G) \). Lower bound is attained if and only if \( G \cong C_3 \) and upper bound is attained by \( G \cong K_{1,p-1}(p \neq 2) \).

**Proof.** If \( \gamma_r(S(G)) = 1 \) then by theorem 1.76, \( S(G) \cong K_1 + H \) where \( H \) is a graph without isolated vertices. But there is no \( G \) for which \( S(G) \cong K_1 + H \) and so \( \gamma_r(S(G)) \geq 2 \). Let \( u \in V(G) \) with \( \deg_G u = \Delta(G) \). Then \( \deg_{S(G)} u = \Delta(G) \). Let \( u_1, u_2, \ldots, u_{\Delta(G)} \) be the vertices in \( V(G) \cap N(u) \) and \( v_1, v_2, \ldots, v_{\Delta(G)} \) be the newly added vertices of \( S(G) \) on these \( \Delta_G \) edges. Any set \( S \subseteq V(S(G)) \) such that \( V - S = \{u, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{\Delta(G)}\} \) is a restrained dominating set of \( S(G) \) and so \( \gamma_r(S(G)) \leq p + q - \Delta(G) \).

Let \( \gamma_r(S(G)) = 2 \). Then \( G \) can contain at most one pendant vertex since if \( G \) contains 2 pendant vertices then by theorem 2.8.8, \( S(G) \cong P_4 \) but there is no such
\[G\]. So \(G \not\cong\) any tree and hence \(G\) contains a cycle. If \(G\) contains \(C_n, n \geq 4\) then 
\(\gamma_r(S(G)) > 2\) and so \(n = 3\). Suppose \(G\) has exactly one pendant vertex then since 
\(S(C_3) \cong C_6 \gamma_r(S(G)) \geq 3\) which is a contradiction. So \(G\) has no pendant vertex.
Thus \(G \cong C_3\). Converse is obvious. Upper bound is attained
by \(G \cong K_{1,p-1}(p \neq 2)\).

\begin{remark}
If \(G \cong K_2, \gamma_r(S(G)) = 3 > p + q - \Delta(G)\).
\end{remark}

\begin{corollary}
Let \(G\) be any \((p, q)\) graph having components \(G_1, G_2, \ldots, G_k\) then
\(\gamma_r(S(G)) \leq p + q - (\Delta_1 + \Delta_2 + \cdots + \Delta_k)\) where \(D_i\) is the maximum degree of \(G_i\)
\(\forall i = 1, 2, \ldots, k\). Also the bound is sharp.
\end{corollary}

\begin{proof}
By theorem 5.2.4, for each \(i\), we have \(\gamma_r(S(G_i)) \leq p_i + q_i - \Delta_i\) and hence
the result follows. Bound is attained by a galaxy, none of whose components is \(K_{1,1}\).
\end{proof}

\begin{corollary}
If \(G\) is any connected \((p, q)\)-graph, \(\gamma_r(G) + \gamma_r(S(G)) \leq 2p + q - \Delta(G)\) and the bound is sharp.
\end{corollary}

\begin{proof}
Since \(\gamma_r(G) \leq p\), the result follows from theorem 5.2.4. The bound is attained
if \(G \cong\) star other than \(K_2\).
\end{proof}

\begin{corollary}
If \(G\) is any \((p, q)\)-graph having \(k\) components \(G_i(1 \leq i \leq k)\) then
\(\gamma_r(G) + \gamma_r(S(G)) \leq 2p + q - (\Delta_1 + \Delta_2 + \cdots + \Delta_k)\) where \(\Delta_i\) is the maximum degree
in \(G_i\) \(\forall i\) and the bound is sharp.
\end{corollary}

\begin{proof}
Follows from Corollary 5.2.6. Sharpness follows as in Corollary 5.2.6.
\end{proof}

\begin{theorem}
If \(G\) is any connected graph, \(\frac{p+1}{2} \leq \gamma_r(S(G)) \leq p+q\). Further lower
bound is attained if and only if \(G \cong mK_2 + K_1(m \geq 1)\) and upper bound is attained if
and only if \(G \cong K_2\).
\end{theorem}

137
Proof. Let $D$ be a $\gamma_r$-set of $S(G)$ and let $D_1 = D \cap V(G)$ and $D_2 = D - D_1$. Neither $V(G)$ nor $V(S(G)) \setminus V(G)$ can be restrained dominating sets and so $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. Each vertex in $D_1$ dominates only one vertex of $V(G)$ and each vertex in $D_2$ dominates at most 2 vertices of $V(G)$. Hence $|D_1| + 2|D_2| \geq p$. Thus $2\gamma_r(S(G)) = 2(|D|) = |D_1| + (|D_1| + 2|D_2|) \geq |D_1| + p \geq p + 1$ and so $\gamma_r(S(G)) \geq \frac{p+1}{2}$.

Upper bound is obvious since $|V(S(G))| = p + q$.

Clearly lower bound is attained if $G \cong mK_2 + K_1(m \geq 1)$ Conversely, suppose $\gamma_r(S(G)) = \frac{p+1}{2}$. Then $|D_1| + |D_2| = \frac{p+1}{2}$ and hence $2\gamma_r(S(G)) = |D_1| + (|D_1| + 2|D_2|) = p + 1$. But $D_1 \neq \emptyset$ and $|D_1| + 2|D_2| \geq p$ and so $|D_1| + 2|D_2| = p$ and $|D_1| = 1$. Thus $D$ contains one vertex from $V(G)$ say $u$ and $\frac{p-1}{2}$ vertices from $V(S(G)) \setminus V(G)$. $|(V-D) \cap V(G)| = p - 1$ and these $p - 1$ vertices are dominated by $\frac{p-1}{2}$ vertices from $V(S(G)) - V(G)$. Also these $p - 1$ vertices in $V - D$ are adjacent to the corresponding $p - 1$ vertices in $V(S(G)) - V(G)$ which are adjacent to $u \in D$.

Hence $G \cong mK_2 + K_1(m \geq 1)$. Suppose $\gamma_r(S(G)) = p + q$ and $S(G)$ contains a path of length at least 3 or a cycle, then there exists a restrained dominating set of cardinality $p + q - 2$ which is a contradiction. So diam$(S(G)) \leq 2$ and hence $S(G)$ is a star. Clearly $S(G) \cong K_{1,2}$ since no other star can be a subdivision graph of any other graph. Hence $G \cong K_2$. Converse is obvious.

Theorem 5.2.10. Suppose $G$ is a connected $(p, q)$-graph with $\gamma_r(G)$ = number of pendant vertices. Then $\gamma_r(S(G)) = q$.

Proof. Let $S$ be the set of all pendant vertices of $G$. By theorem 2.8.8, every vertex in $V - S$ is a support. Let $A$ be the set of all $q$ newly added vertices in $S(G)$ and $B \subset A$ be the set all new supports in $S(G)$. Let $D = S \cup (A - B)$. Clearly $D$ is an independent restrained dominating set. For every $v \in A - B$, $D - \{v\}$ does not dominate $v$. Thus $\gamma_r(S(G)) = |D| = q$. 

138
Theorem 5.2.11. Let $G \cong K_{1,p-1}$ be a connected graph. Then $\gamma_r(S(G)) \leq p + q - (e + s)$ where $e$ denotes the number of pendant vertices of $G$ and $s$ the number of supports of $G$. Moreover the bound is sharp.

Proof. If $\delta(G) \geq 2$ then $e + s = 0$. Since $\gamma_r(S(G)) \leq p + q$ always, the result follows.

Let $v_1, v_2, \ldots, v_e$ be the pendant vertices of $G$ and $u_1, u_2, \ldots, u_s$ be the supports. Let $w_1, w_2, \ldots, w_e$ be the vertices subdividing the pendant edges of $G$. If $S \subseteq V(S(G))$ is such that, $V - S = \{u_1, u_2, \ldots, u_s, w_1, w_2, \ldots, w_e\}$ then $S$ is a restrained dominating set of $S(G)$ and so $\gamma_r(S(G)) \leq p + q - (e + s)$. The bound is attained if $G \cong G_1$ where $G_1$ is given in Fig 5.1.

Corollary 5.2.12. Let $G \cong K_2$ be a connected graph in which every support is adjacent to exactly one pendant vertex. Then $\gamma_r(S(G)) \leq p + q - 2e$ where $e$ denotes the number of pendant vertices of $G$.

Remark 5.2.13. (i) In corollary 5.2.12, if $G$ is replaced by a tree $T$ then $\gamma_r(S(G)) \leq 2(p - e) - 1$.

(ii) If $G \cong K_2$ is any connected graph, $\gamma_r(G) = \gamma_r(S(G)) = p$ if and only if $G \cong K_{1,p-1}$.
Theorem 5.2.14. If $G \not\cong K_2$ is any graph with $\delta(G) \geq 1, \gamma_r(S(G)) = 2$.

Proof.

Case (i) $G$ does not contain two non-adjacent edges.
In this case $G \cong K_{1,p-1}$ and $\gamma_r(S(G)) = 2$.

Case (ii) $G$ contains two non-adjacent edges.
Let $uv$ and $xy$ be two non-adjacent edges of $G$ and let $w$ and $z$ be the vertices, subdividing these edges respectively. Let $S = \{u, w\} \cdot w$ is adjacent to all vertices of $V(S(G)) - \{u, v\}$ and $u$ is adjacent to $v$ in $S(G)$ so that $S$ is a dominating set in $S(G)$. Similarly $z$ is adjacent to all vertices of $V(S(G)) - \{x, y\}$ and $x$ is adjacent to $y$. Thus $(V(S(G)) - S)$ has no isolated vertices and hence $S$ is a restrained dominating set of $G$. Since $\delta(G) \geq 1, \gamma_r(S(G)) \geq 2$ and so $S$ is a minimum restrained dominating set of $G$. Hence $\gamma_r(S(G)) = 2$. □

Corollary 5.2.15. $\gamma_r(S(G)) + \gamma_r(S(G)) \leq (p+q+2) - \Delta(G)$ and the bound is sharp.

Proof. Inequality follows from theorems 5.2.4 and 5.2.14. Equality holds if $G \cong K_{1,p-1}(p \geq 3)$ and so the bound is sharp. □

Corollary 5.2.16. If $G \not\cong K_2U K_1$ and $\delta(G) = 0$ then $\gamma_r(S(G)) = 1$.

Remark 5.2.17. If $G \cong K_2U K_1$ then $\gamma_r(S(G)) = 2$ and if $G \cong K_2$ then $\gamma_r(S(G)) = 3$.

Theorem 5.2.18. If $T$ is a tree, $\gamma_r(S(T)) = 2p - 3$ if and only if $T \cong P_3$.

Proof. By theorem 1.74, $\gamma_r(T) = p - 2$ if and only if $T \cong P_4, P_5, P_6$ attached with zero or any number of pendant vertices to the supports of the induced path. Here $S(T)$ is a tree with $2p - 1$ vertices and $2p - 2$ edges. Since $2p - 1$ cannot be 4 or 6, we have $2p - 1 = 5$ and so $p = 3$. Hence $T \cong P_3$. Since supports in $S(P_3)$ are the
subdivision points, pendant edges cannot be attached to them.

Converse is obvious.

Theorem 5.2.19. If $T$ is any tree, $\gamma_r(S(T)) \geq \Delta(T) + 1$ and equality holds if and only if $T$ is a star.

Proof. $S(T)$ is also a tree and so by theorem 1.77, $\gamma_r(S(T)) \geq \Delta(S(T)) = \Delta(T)$.

Clearly there can be no tree $T$ with $\gamma_r(S(T)) = \Delta(T)$ and hence $\gamma_r(S(T)) \geq \Delta(T) + 1$.

Suppose $\gamma_r(S(T)) = \Delta(T) + 1$. Let $S$ be a $\gamma_r$-set of $S(T)$. Let $v \in V(T)$ with $\deg_T v = \Delta(T)$ and let $N(v) \cap V(T) = \{v_1, v_2, \ldots, v_{\Delta(T)}\}$.

Let $u_1, u_2, \ldots, u_{\Delta(T)}$ be the vertices of $S(T)$ subdividing $vv_1, vv_2, \ldots, vv_{\Delta(T)}$. $v_1, v_2, \ldots, v_{\Delta(T)}$ are all pendant vertices and so $v_1, v_2, \ldots, v_{\Delta(T)} \in S$. Since $\gamma_r(S(T)) = \Delta(T) + 1$, $\text{diam}(S(T)) = 4$ and so $S(T)$ is a spider. Thus $T$ is a star. Converse is obvious.

Theorem 5.2.20. If $T \not\cong K_2$ is any tree with $e$ number of pendant vertices, $\gamma_r(T) \leq p + e - 2$ and the bound is sharp.

Proof. We first prove that $\gamma_r(S(P_p)) \leq p$ except when $p = 2$. By theorem 1.70, $\gamma_r(P_p) = p - 2\lfloor \frac{p-1}{3} \rfloor$ and so $\gamma_r(S(P_p)) = 2p - 1 - 2\lfloor \frac{2p-2}{3} \rfloor$. If $p = 3$ or 4, $\gamma_r(S(P_p)) \leq p$.

If $2p - 2 \equiv 0 \pmod{3}$, $\gamma_r(S(P_p)) \leq p \ \forall p \geq 1$.

If $2p - 2 \equiv 1 \pmod{3}$, $\gamma_r(S(P_p)) \leq p \ \forall p \geq 3$.

If $2p - 2 \equiv 2 \pmod{3}$, $\gamma_r(S(P_p)) \leq p \ \forall p \geq 5$.

Hence $\gamma_r(S(P_p)) \leq p$ except when $p = 2$.

Let $T$ be a tree with the diametrical path of length $l$. By what we have proved above, it is possible to choose $l$ number of vertices in the path to form a restrained dominating set of $S(P_{l+1})$ say $A$. Let $v$ be an arbitrary vertex in the diametrical
path and consider a branch of $T$ incident at $v$. It is not necessary that $v$ lies in $A$. Let $u \in N(v) \cap V(T)$ and $w$ be the vertex subdividing the edge $vu$. Considering the branch as a path $P'$ from $u$ of length $l_1$ we can choose a restrained dominating set of $S(P')$ with cardinality $l_1$. Repeating this argument for every path, $A \cup A_1 \cup \{w_1\} \cup A_2 \cup \{w_2\} \cup \cdots \cup A_k \cup \{w_k\}$ is a restrained dominating set of $S(T)$ with cardinality $p + e - 2$ where 2 represents the number of pendant vertices in the diametrical path. Thus $\gamma_r(S(T)) \leq p + e - 2$. Bound is attained by $P_3$. \hfill \Box

Remark 5.2.21. If $T \cong K_2$, $\gamma_r(S(T)) = 3 > p + e - 2$.

For larger trees the bound can be improved.

Theorem 5.2.22. For any nontrivial tree $T$, $\gamma_r(S(T)) \leq 2\gamma_r(T) - 1$ and the bound is sharp.

Proof. Let $T \cong K_{1,p-1}$. If $p = 2$ then $\gamma_r(K_2) = 2$ and $\gamma_r(S(K_2)) = 3$ so $\gamma_r(S(K_2)) = 2\gamma_r(K_2) - 1$.

If $p \geq 3$ then $\gamma_r(S(T)) = p$ and $\gamma_r(T) = p$ and so $\gamma_r(S(T)) < 2\gamma_r(T) - 1$.

Let $T \cong K_{1,p-1}$ and $S$ be a $\gamma_r$-set of $T$. Then $V - S \neq \emptyset$. If $|S| = k$ then $|V - S| = p - k$. Consider $uv \in \langle E(V - S) \rangle$ and $w$ be the subdivision vertex of $uv$. Then construct $D = S \cup S'$ where $S' = \{w/w$ is the subdivision vertex of $uv \in \langle E(V - S) \rangle\}$. Now $\langle V(S(T)) \setminus D \rangle$ has no isolated vertices. Obviously $D$ is a restrained dominating set of $S(T)$. Hence $|V - D| \geq 2(p - k)$. So $|D| \leq p + q - 2(p - k) = 2p - 1 - 2p - 2k = 2k - 1$.

So $\gamma_r(S(T)) \leq 2\gamma_r(T) - 1$. Consider $T \cong P_4$. $\gamma_r(S(T)) = 3 = 2\gamma_r(T) - 1$. So the bound is sharp. \hfill \Box
Theorem 5.2.23. If $G$ is a unicyclic graph with cycle $C_n$ and $e$ is the number of pendant vertices in $G$, $\gamma_r(S(G)) \leq \frac{2(n+2)}{3} + p - n + e$. Moreover the bound is sharp.

Proof. By theorem 1.72, $\gamma_r(S(C_n)) = 2n - 2\left\lfloor \frac{2n}{3} \right\rfloor$. We have $\gamma_r(S(C_n)) = \frac{2n}{3}, \frac{2(n+1)}{3}$ or $\frac{2(n+2)}{3}$ according as $n \equiv 0, 1$ or $2 \pmod{3}$. Thus $\gamma_r(S(C_n)) \leq \frac{2}{3}(n + 2)$. If $G$ is a unicyclic graph which is not a cycle, applying the same argument as in theorem 5.2.20, we have $\gamma_r(S(G)) \leq \frac{2}{3}(n + 2) + (p - n) + e$. The bound is attained by $G \cong C_4$. 

5.3 Restrained domination Subdivision number of a graph

We now consider the following problem. Given a graph $G$, what is the minimum number of edges to be subdivided exactly once so that the restrained domination number of the resulting graph exceeds that of $G$? So we define,

Definition 5.3.1. Restrained domination subdivision number $Sd_{\gamma_r}(G)$ of a graph $G$ is defined to be the minimum number of edges that must be subdivided exactly once so as to increase the restrained domination number of $G$.

Example 5.3.2.

(i) $Sd_{\gamma_r}(C_p) = \begin{cases} 
1 & \text{if } p \equiv 0 \pmod{3} \text{ or } p \equiv 1 \pmod{3} \\
3 & \text{otherwise.}
\end{cases}$

By theorem 1.73, $\gamma_r(C_p) = k + r$ where $p = 3k + r(0 \leq r \leq 2)$ and hence the above result follows.

(ii) $Sd_{\gamma_r}(P_p) = \begin{cases} 
1 & \text{if } p \equiv 1 \pmod{3} \text{ or } p \equiv 2 \pmod{3} \\
3 & \text{otherwise.}
\end{cases}$
By theorem 1.71, \( \gamma_r(P_p) = k + 1 \) if \( p = 3k + 1 \) and \( k + 2 \) if either \( p = 3k \) or \( p = 3k + 2 \).

(iii) \( Sd_r(B(n_1, n_2)) = 1 \). Follows since subdivision of the nonpendant edge increases \( \gamma_r(B(n_1, n_2)) \).

(iv) \( Sd_r(K_p) = 1 \). Follows since subdivision of any edge increases \( \gamma_r(K_p) \) to 2.

Remark 5.3.3. Since \( \gamma_r(K_{1,p-1}) \) is \( p \), \( Sd_r(K_{1,p-1}) \) is not defined.

Example 5.3.4. \( Sd_r(K_{m,n}) = 1 \) if \( 2 \leq m \leq n \).

Let \( X = \{u_1, u_2, \ldots, u_m\} \) and \( Y = \{v_1, v_2, \ldots, v_n\} \) be the bipartition of \( G \cong K_{m,n} \).

Let the edge \( u_1v_1 \) be subdivided by \( w_1 \) and let \( G' \) be the resulting graph. For \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), \( S = \{u_i, v_j\} \) is a \( \gamma_r \)-set of \( G \) but \( S \) will not dominate \( w_1 \) in \( G' \) if \( i \neq 1 \) and \( j \neq 1 \). If \( i = 1 \) and \( j = 1 \) then \( w_1 \) has no neighbor in \( V - S \). If \( i = 1 \) and \( j \neq 1 \) or vice versa, then \( v_1 \) or \( u_1 \) is not dominated accordingly and so \( \gamma_r(G') = 3 \). Thus \( Sd_r(K_{m,n}) = 1 \).

Theorem 5.3.5. If \( G \) is any graph with \( \Delta(G) = p - 1 \) and \( \delta(G) \geq 2 \), then \( Sd_r(G) = 1 \).

Proof. Clearly \( \gamma_r(G) = 1 \). Let \( \deg u = p - 1 \). Let \( uv \in E(G) \) and \( w \) be the vertex that subdivided \( uv \). If \( G' \) is the resulting graph, \( \gamma_r(G') = 2 \) since now \( u \) and \( v \) are non-adjacent and \( w \) is adjacent only to \( u \) and \( v \). Hence \( Sd_r(G) = 1 \).

Theorem 5.3.6. Let \( G \) be a connected graph such that \( \delta(G) = 1, G \not\cong K_{1,p-1} \) and \( \gamma_r(G) = \text{number of pendant vertices} \). Then \( Sd_r(G) = 1 \).

Proof. By theorem 2.8.8, every nonpendant vertex is a support. By hypothesis \( G \not\cong K_{1,p-1} \) and \( G \) has at least 2 supports. Subdividing the edge joining any two supports increases \( \gamma_r(G) \) by 1 and so \( Sd_r(G) = 1 \).
Theorem 5.3.7. Let $G$ be a connected bipartite graph with bipartition $(X, Y)$ where $|X| = m, |Y| = n, 3 \leq m \leq n$. If there exists two vertices $u \in X, v \in Y$ with respective degrees $n$ and $m$, $Sd_{\gamma_r}(G) = 1$ if and only if $G$ is not isomorphic to the graph given in Fig. 5.2.

Proof. Suppose $Sd_{\gamma_r}(G) = 1$ and $G$ is isomorphic to a graph of type given in Fig. 5.2. Let $Y = \{v, v_1, v_2, \ldots, v_n\}$ and $A$ be the set of pendant vertices of $G$. We have $\gamma_r(G) = |A| + 2$. Let $e$ be the edge which is subdivided and $G'$ be the resulting graph. If $e = uv$, $A \cup \{u, v\}$ is a $\gamma_r$-set of $G'$. If $e = uv_1, A \cup \{u_1, v_1\}$ is a $\gamma_r$-set of $G'$. If $e = u_1v_1(i \geq 2), A \cup \{u_1, v_1\}$ is a $\gamma_r$-set of $G'$. If $e = u_1v_1$ or $u_2v_1$ then $A \cup \{u_1, u_2\}$ is a $\gamma_r$-set of $G'$. Thus subdivision of any single edge does not change $\gamma_r(G)$ and so $Sd_{\gamma_r}(G) \neq 1$ which is a contradiction.

Conversely suppose that $G$ is not isomorphic to the graph given in Fig. 5.2.

Case (i) $\delta(G) \geq 2$.

In this case $S = \{u, v\}$ is a $\gamma_r$-set of $G$ and so $\gamma_r(G) = 2$. Let $G'$ be the graph obtained by subdividing the edge $uv$ and let $w$ be the subdividing vertex. Clearly $S$ is not a restrained dominating set of $G'$. Let $S'$ be a restrained dominating set of $G'$.

If $w \in S'$, $|S'| \geq 3$. Since $N(w) = \{u, v\}$, $S'$ can contain exactly one of $u$ or $v$. Without loss of generality let $u \in S'$. Since $|X| \geq 3$, $S'$ should contain at least one
vertex from $Y - \{u\}$ but in this case $v$ will not be dominated. Hence $\gamma_r(G') \geq 3$ and so $\text{Sd}_{\gamma_r}(G) = 1$.

**Case (ii).** $\delta(G) = 1$.

Let $A$ be the set of pendant vertices of $G$ and let $B = V(G) \setminus (A \cup \{u, v\})$. Clearly $|B| \neq 1$.

**Subcase (i).** $|B| = 0$.

Now $G$ is a bistar and the theorem follows by example 5.3.2

**Subcase (ii).** $|B| = 2$.

Let $b_1 \in B \cap X$ and $b_2 \in B \cap Y$. $A \cup \{b_1\}(A \cup \{b_2\})$ is a restrained dominating set of $G$ if either both $u$ and $v$ are supports or $u$ alone is a support ($v$ alone is a support). Hence $\gamma_r(G) = |A| + 1$. Let $G'$ be the graph obtained by subdividing the edge $uv$ by the vertex $w$ and let $S'$ be a $\gamma_r$-set of $G'$. Clearly $A \cup \{w\}$ is not a $\gamma_r$-set of $G'$. Since $\deg_{G'} w = 2$, $S'$ contains exactly one of $u$ or $v$. If $u \in S'$, $A \cup \{u\}$ does not dominate $b_1$ and if $v \in S'$, $A \cup \{v\}$ does not dominate $b_2$. Hence $|S'| > |A| + 1$ and so $\text{Sd}_{\gamma_r}(G) = 1$.

**Subcase (iii).** $|B| \geq 3$.

$A \cup \{u, v\}$ is a restrained dominating set of $G$ and hence $\gamma_r(G) \leq |A| + 2$. If $\gamma_r(G) = |A|$ then by theorem 5.3.6, $\text{Sd}_{\gamma_r}(G) = 1$. If $\gamma_r(G) = |A| + 1$ then either $|B \cap X| = 1$ or $|B \cap Y| = 1$. If $|B \cap X| = 1$ then $|B \cap Y| \geq 2$ and $u$ is a support. $u_1 \in B \cap X$ is a vertex of full degree in $\langle B \rangle$. Subdividing an edge $e$ in $\langle B \rangle$ increases $\gamma_r(G)$ and so $\text{Sd}_{\gamma_r}(G) = 1$.

Suppose $\gamma_r(G) = |A| + 2$. Also let both $u$ and $v$ be supports. In this case if either $|B \cap X| = 1$ or $|B \cap Y| = 1$ then $\langle B \rangle$ has a vertex of full degree and so $\gamma_r(G) = |A| + 1$.
which is a contradiction and so \(|B \cap X| \geq 2, |B \cap Y| \geq 2\). If there exists a \(\gamma_r\)-set of \(G\), containing only vertices of \(B \cap X\) or \(B \cap Y\) (other than pendant vertices) say \(B \cap X\) then \(|B \cap X| = 2\) and \(|B \cap Y| \geq 2\). Subdividing edge \(uv\) increases \(\gamma_r(G)\). If there exists no such \(\gamma_r\)-set let \(e = ua_1\) where \(a_1 \in B\). Let \(G'\) be the graph obtained by subdividing \(e\). Let \(S'\) be a \(\gamma_r\)-set of \(G'\). Without loss of generality let \(u \in S'\). In order to dominate \(a_1\), a vertex of \(B \cap X\) should be included in \(S'\) but now other vertices of \(B \cap X\) are not dominated. Hence \(\gamma_r(G') \geq |A| + 3\). So \(Sd_{\gamma_r}(G) = 1\).

Suppose one of \(u\) or \(v\) say \(u\) is a support. If \(|B \cap X| \geq 2, |B \cap Y| \geq 2\) and there exists a restrained dominating set of \(G\) containing only the vertices of \(B \cap X\) along with pendant vertices then \(|B \cap X| = 2\). In this case there can be no \(\gamma_r\)-set of \(G\) containing only vertices of \(B \cap Y\) along with the pendant vertices, since \(v\) will not be dominated. Now conclusion follows as discussed earlier.

Now, \(|B \cap X| = 1\) and \(|B \cap Y| \geq 2\) is not possible since \(|X| \geq 3\). So \(|B \cap Y| = 1\) and \(|B \cap X| \geq 2\). Also \(|B \cap X| \neq 2\) since such a graph is isomorphic to a graph given in Fig. 5.2. So \(|B \cap X| \geq 3\). Let \(u_1, u_2, u_3, \ldots \in B \cap X\) and \(v_1 \in B \cap Y\). Let \(e = uv_1\) and let \(G'\) be the graph obtained by subdividing \(e\). Let \(S'\) be a \(\gamma_r\)-set of \(G'\). \(v_1\) is of full degree in \(\langle B \rangle\). We observe that \(A \cup C\) is not a restrained dominating set of \(G'\) for any two element subset of \(G'\)

So \(|S'| \geq |A| + 3\). Hence \(Sd_{\gamma_r}(G) = 1\).

**Theorem 5.3.8.** Let \(G\) be a complete \(m\)-partite graph \(K(n_1, n_2, \ldots, n_m)\) where \(n_1 \leq n_2 \leq \cdots \leq n_m\) and \(m \geq 3\). Then

\[
Sd_{\gamma_r}(G') = \begin{cases} 
1 & \text{if } n_1 = 1 \\
2 & \text{otherwise}
\end{cases}
\]

**Proof.** If \(n_1 = 1\), \(\gamma_r(G) = 1\). Hence subdivision of any edge increases \(\gamma_r(G)\) and so \(Sd_{\gamma_r}(G) = 1\). Suppose \(n_1 \geq 2\). Let \(V_1, V_2, \ldots, V_m\) be the \(m\) partite sets of \(G\) and
let \( \{u_k, v_k\} \subseteq V_k \) for all \( k \) such that \( 1 \leq k \leq m \). Let \( e = u_iu_j (1 \leq i, j \leq m) \) be any edge of \( G \) and let \( G' \) be the graph obtained by subdividing \( e \). Now \( \{u_i, u_k\} (k \neq j) \) is a \( \gamma_r \)-set of \( G' \) and so \( Sd_{\gamma_r}(G) \geq 2 \). If in addition \( e' = v_iv_j \) is subdivided, there is no 2-element restrained dominating set of \( G' \) and so \( Sd_{\gamma_r}(G) \leq 2 \).

Thus \( Sd_{\gamma_r}(G) = 2 \).

\[ \square \]

**Theorem 5.3.9.** Let \( T \) be any tree such that \( T \not\cong K_2 \). Then \( Sd_{\gamma_r}(T) = 1 \) if and only if \( T \cong K_{1,p-1}(p-1 \geq 2) \), \( B(1,n)(n \geq 1) \) or \( P_5 \).

**Proof.** Suppose \( T \cong K_{1,p-1}(p-1 \geq 2) \).

When \( p-1 = 2 \), \( T \cong K_{1,2} \) and \( \bar{T} \cong K_1 \cup K_2 \) and so \( \gamma_r(\bar{T}) = 3 \). When the edge in \( \bar{T} \) is subdivided, \( \gamma_r(\text{resulting graph}) = 4 \). So \( Sd_{\gamma_r}(\bar{T}) = 1 \). Now \( T \cong K_{1,p-1}(p-1 \geq 3) \).

Let \( v \) be the central vertex of \( T \) and let \( N(v) = \{v_1, v_2, \ldots v_{p-1}\} \). Then \( \bar{T} \cong K_1 \cup K_{p-1} \) and so \( \gamma_r(\bar{T}) = 2 \). By example 5.3.2 (iv), it follows that \( Sd_{\gamma_r}(\bar{T}) = 1 \). Now suppose \( T \cong B(1,n)(n \geq 1) \). If \( n = 1, T \cong P_4, \bar{T} \cong P_4 \) and by example 5.3.2 (ii) we have \( Sd_{\gamma_r}(\bar{T}) = 1 \). Let \( n > 1 \) and let \( u \) and \( v \) be the supports with \( \deg u = 2 \) and \( \deg v = n + 1(n > 1) \). Let \( u_1 \in N(u) \) and \( N(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_n\} \) and let \( S \) be any \( \gamma_r \)-set of \( \bar{T} \). Clearly \( |S| = 2 \) and \( v \in S \). Let \( T' \) be the graph obtained by subdividing the edge \( v_1v_2 \) by the vertex \( w \) in \( \bar{T} \). \( \{v, w\} \) does not dominate \( u \) in \( T' \) and so any \( \gamma_r \)-set of \( T' \) contains either \( v_1 \) or \( v_2 \). But \( \{v, v_1\} \) and \( \{v, v_2\} \) are not \( \gamma_r \)-sets of \( T' \) since \( v_2 \) and \( v_1 \) are not dominated by the respective sets. Both \( \{v_1, u_1\} \) and \( \{v_2, u_1\} \) are not restrained sets as \( v \) is an isolated vertex in \( \langle V - \{v_1, u_1\} \rangle \) and \( \langle V - \{v_2, u_1\} \rangle \). \( \{v_1, u\} \) and \( v_2, u \) do not dominate \( v \). Hence \( \gamma_r(T') \geq 3 \) and so \( Sd_{\gamma_r}(\bar{T}) = 1 \).

Suppose \( T \cong P_5 \). Let \( u \) and \( v \) be the supports and \( u_1 \) and \( v_1 \) be the respective pendants. Let \( w \) be the subdividing vertex of the edge \( u_1v_1 \). Let \( T' \) be the resulting graph. No two element subset of \( V(T') \) is a \( \gamma_r \)-set of \( T' \) and hence \( Sd_{\gamma_r}(\bar{T}) = 1 \).
Conversely suppose $\text{Sd}_{\gamma_r}(\bar{T}) = 1$ and $T \not\cong K_{1,p-1}(p-1 \geq 2)$, $B(1,n)(n \geq 1)$ and $P_5$. Then $T$ has at least 2 distinct supports.

**Case (i).** $T$ has exactly 2 supports which are adjacent.

Now $T \cong B(m,n)(m,n \geq 2)$. Let $u$ and $v$ be the supports and let $N(u) \setminus \{v\} = \{u_1, u_2, \ldots, u_m\}$ and $N(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_n\}$. Let $K_m$ and $K_n$ be the complete graphs induced by $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ respectively. Then, $\bar{T} \cong (K_m + K_n) \cup \{uv_i(1 \leq i \leq n)\} \cup \{vu_j(1 \leq j \leq m)\} \cup \{u_jv_i(1 \leq i \leq n) \text{ and } 1 \leq j \leq m\}$. If $e \in E(K_m)(E(K_n))$ then $e = u_iv_juv_i$ and let $a_{ij}(b_{ij})$ be the vertex subdividing $e$. Now $\{u_i, v_k\}(\{v_i, u_k\})$ where $1 \leq k \leq n(1 \leq l \leq m)$ is a $\gamma_r$-set of the resulting graph. If $e = u_iv_j$ and if $c_{ij}$ is the vertex subdividing $e$ then $\{u_i, u\}$ is a $\gamma_r$-set of the resulting graph. If $e = uv_jvu_j$ and if $w_j(w_k)$ is the vertex subdividing $e$, then $\{w_j, u_i\}, (\{w_k, v_k\})$ where $1 \leq i \leq m(1 \leq k \leq n)$ is a $\gamma_r$-set of the resulting graph. Thus we have proved that subdivision of a single edge does not increase $\gamma_r(\bar{T})$.

Hence $\text{Sd}_{\gamma_r}(\bar{T}) > 1$ which is a contradiction.

**Case (ii).** $T$ has exactly 2 supports which are non-adjacent.

Let $d(u,v) = 2$. Since $T \not\cong P_5$ either $\deg u \geq 3$ or $\deg v \geq 3$ Let $\deg v \geq 3$. Let $u_1 \in N(u)$ with $\deg u_1 = 1$ and let $v_1, v_2, \ldots, v_n \in N(v)$ with $\deg v_j = 1 \forall j (1 \leq j \leq n)$. Let $x \in N(u) \cap N(v)$. When any edge of $\bar{T}$ is subdivided the resulting graph is $T'$. When the edges $v_ju_1, v_ju, v_jx, v_jv_i$ are subdivided then $\{u_1, v_k(k \neq j)\}$, $\{u, v_k(k \neq j)\}$, $\{u_1, v_j\}$ and $\{v_i, u_1\}$ are the $\gamma_r$-sets of $T'$ respectively. When the edges $u_1x, u_1v$ are subdivided then $\{u_1, v_j\}$, $\{v, v_j\}$ are the respective $\gamma_r$-sets of $T'$.

If $uv$ is subdivided then $\{v, v_j\}$ is a $\gamma_r$-set of $T'$. Now let $d(u,v) \geq 3$. Suppose $d(u,v) = k \geq 3$. Let $u = w_1, w_2, w_3, \ldots, w_{k+1} = v$ be the $d(u,v)$ path in $T$. Let $u_i (i \geq 1)$ and $v_j (j \geq 1)$ be the pendant neighbors of $u$ and $v$ respectively. When the edges $u_iv_j, u_iv, u_iw_k, u_iw_l(2 \leq l \leq k-1)$, are subdivided then $\{u_i, w_3\}, \{v, v_1\}$,
\{u, u_1\}, \{u, v\} are the respective \(\gamma_r\)-sets of \(T'\). When \(uv, uw, s > 2\) are subdivided, then \(\{u, u_1\}\) is a \(\gamma_r\)-set of \(T'\). When the edge \(w_iw_j\), where \(i < j\), \(i\) and \(j\) are non-consecutive, is subdivided, then \(\{w_i, u_l\}\) or \(\{w_i, u_t\}\) where \((1 \leq l \leq m)\) and \((1 \leq t \leq n)\) is a \(\gamma_r\)-set of \(T'\) according as \(i \neq 2\) or \(i \neq k\). By symmetry, we can conclude that subdivision of any single edge doesn’t increase \(\gamma_r(T)\). Hence \(Sd_{\gamma_r}(\bar{T}) > 1\), which is a contradiction.

**Case (iii).** Let \(T\) have at least 3 supports.

Let \(a, b, c\) be any three supports and let \(a_i \in N(a), b_j \in N(b), c_k \in N(c)\) for \(1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq l\) where each \(a_i, b_j, c_k\) are of degree 1.

When any one of the edges in \(\bar{T}\) is subdivided the resulting graph is \(T'\). Let \(x\) and \(y\) be any 2 vertices which are neither supports nor pendant vertices.

If the edges \(a_i a_s, a_i b_j, a_i x, a_i b\) are subdivided then \(\{a_i, b_j\}, \{a_i, a\}, \{a_i, c_k\}, \{b, b_j\}\) are the \(\gamma_r\)-sets of \(T'\) respectively. If \(ab, ax\) are subdivided then \(\{a_i, a\}\) is a \(\gamma_r\)-set of \(T'\). Suppose \(xy\) is subdivided.

Since \(T\) is a tree, there exists at least one support which is non-adjacent to either \(x\) or \(y\). Without loss of generality let us assume that \(a \notin N(x)\). Then \(\{x, a_i\}\) a \(\gamma_r\)-set of \(T'\). By symmetry, we can conclude that subdivision of any single edge doesn’t increase \(\gamma_r(\bar{T})\). Hence \(Sd_{\gamma_r}(\bar{T}) > 1\), which is a contradiction. Hence the result follows.

**Theorem 5.3.10.** Let \(G\) be a graph with a pendant edge \(uv\) where \(\deg u = 2\) and \(\deg v = 2\). If \(u\) is \(\gamma_r\) totally free, \(Sd_{\gamma_r}(G) = 1\).

**Proof.** Let \(S\) be any \(\gamma_r\)-set of \(G\). Since \(u\) is \(\gamma_r\) totally free, \(u \in V - S\). Hence \(N(u) \cap S = \{v\}\) and there exists \(w \in N(u) \cap (V - S)\). When the edge \(uv\) is subdivided by \(x\), by choice of \(u\), there is no set with cardinality \(\gamma_r(G)\) that is a restrained dominating set of the resulting graph. Thus \(Sd_{\gamma_r}(G) = 1\).
Theorem 5.3.11. If the set of all edges incident to a vertex $v$ in a graph $G$ is a $Sd_{\gamma_r}$-set of $G$ and $\delta(G) \geq 2$ then $v \notin V^-$. 

Proof. Let $F$ be the set of all edges incident at $v$ such that $F$ is a $Sd_{\gamma_r}$-set of $G$. Let $G'$ be the graph obtained after subdividing all the edges of $F$. By hypothesis $\gamma_r(G') > \gamma_r(G)$. Suppose $v \in V^-$. Then $\gamma_r(G \setminus v) < \gamma_r(G)$ and let $D$ be a $\gamma_r$-set of $G \setminus v$, so that $S = D \cup \{v\}$ is a $\gamma_r$-set of $G$. By theorem 4.2.2, $v$ is an isolated vertex or $N(v) \cap (V - S) = \emptyset$ or $pn[v; S] = \{v\}$. Since $F$ is nonempty, $v$ is not an isolated vertex. Suppose $pn[v; S] = \{v\}$. Let $v_i (1 \leq i \leq k)$ be the vertices in $N(v) \cap (V - S)$ and let $w_i$ be the vertices subdividing the edges $v_i$. Now $D \cup \{v\}$ is a restrained dominating set of $G'$, since $w_i (1 \leq i \leq k)$ are dominated by $v$ and each $w_i$ has corresponding neighbors in $V - S$. Thus $\gamma_r(G') \leq \gamma_r(G)$ which is a contradiction.

Suppose $N(v) \cap (V - S) = \emptyset$. Then all $v_i (1 \leq i \leq k)$ of $N(v)$ lie in $S$. We claim that $D \cup \{w_j\}$ for any $j \geq 1 \leq j \leq k$ is a restrained dominating set of $G'$. Now $w_j$ dominates $v$ and every $w_i (i \neq j)$ is dominated by the corresponding $v_i$. Every $w_i i \neq j$ has a common neighbor $v$ in $\langle V - (D \cup \{w_j\}) \rangle$.

Since $\delta(G) \geq 2$, there exists at least one $w_i$ in $\langle V - D \cup \{w_j\} \rangle$ which is a neighbor of $v$ in $\langle V - D \cup \{w_j\} \rangle$ in $G'$. So our claim is true. But then $\gamma_r(G') \leq \gamma_r(G)$ and so $v \notin V^-$. \qed