Preliminaries

In this chapter we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to Harary [22], Parthasarathy [28], Chartrand and Lesniak [20] and Bondy and Murty [16].

Definition 1.1. A graph $G$ is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of $G$, called edges. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively.

If $e = \{u, v\}$ is an edge, we write $e = uv$; we say that $e$ joins the vertices $u$ and $v$; $u$ and $v$ are adjacent vertices; $u$ and $v$ are incident with $e$.

If two vertices are not joined, then we say that they are non-adjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other.

Definition 1.2. The cardinality of the vertex set of a graph $G$ is called the order of $G$ and is denoted by $p$. The cardinality of its edge set is called the size of $G$ and is denoted by $q$. A graph with $p$ vertices and $q$ edges is called a $(p, q)$-graph.

Definition 1.3. A graph $G_1$ is isomorphic to a graph $G_2$ if there exists a bijection $\phi$ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. If $G_1$ is isomorphic to $G_2$, we write $G_1 \cong G_2$. 
Definition 1.4. A graph $H$ is called a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of $G$ is a subgraph $H$ with $V(H) = V(G)$. For any set $S$ of vertices of $G$, the induced subgraph $(S)$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $(S)$ if and only if they are adjacent in $G$. $(S)$ is also denoted by $G[S]$.

Notation 1.5. Let $v$ be a vertex of a graph $G$. The induced subgraph $(V(G) \setminus \{v\})$ is denoted by $G - v$; it is the subgraph of $G$ obtained by the removal of $v$ and edges incident with $v$. If $e \in E(G)$, the spanning subgraph with edge set $E(G) \setminus \{e\}$ is denoted by $G - e$; it is the subgraph of $G$ obtained by the removal of $e$.

Definition 1.6. The degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and is denoted by $\deg_G v$ or $\deg v$. The minimum and maximum degrees of vertices of $G$ are denoted by $\delta$ and $\Delta$ respectively. A vertex of degree 0 in $G$ is called an isolated vertex; a vertex of degree 1 is called a pendant vertex or an end vertex of $G$. Any vertex which is adjacent to a pendant vertex is called a support.

Definition 1.7. A graph $G$ is regular of degree $r$ if every vertex of $G$ has degree $r$. Such graphs are called $r$-regular graphs. Any 3-regular graph is called a cubic graph.

Definition 1.8. A graph $G$ is complete if every pair of its vertices are adjacent. A complete graph on $p$ vertices is denoted by $K_p$. 
Definition 1.9. The complement $\overline{G}$ of a graph $G$ is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

Definition 1.10. A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and the other end in $V_2$; $(V_1, V_2)$ is called a bipartition of $G$. If further, every vertex of $V_1$ is joined to all the vertices of $V_2$, then $G$ is called a complete bipartite graph. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.

Definition 1.11. Let $u$ and $v$ be (not necessarily distinct) vertices of a graph $G$. A $u$-$v$ walk of $G$ is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, e_n, u_n = v$ of vertices and edges beginning with vertex $u$ and ending with vertex $v$ such that $e_i = u_{i-1}u_i$, $i = 1, 2, \ldots, n$. The number $n$ is called the length of the walk. The walk is said to be open if $u$ and $v$ are distinct vertices; it is closed otherwise. A walk $u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_n$ is determined by the sequence $u_0, u_1, u_2, \ldots, u_n$ of its vertices and hence we specify this walk by $(u_0, u_1, u_2, \ldots, u_n)$. A walk in which all the edges are distinct is called a trail. A walk in which all the vertices are distinct is called a path. A closed walk $(u_0, u_1, u_2, \ldots, u_n)$ in which $u_0, u_1, u_2, \ldots, u_{n-1}$ are distinct is called a cycle. A path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$.

Definition 1.12. A graph $G$ is said to be connected if any two distinct vertices of $G$ are joined by a path. A maximal connected subgraph of $G$
is called a *component* of $G$. Thus a disconnected graph has at least two components.

The *distance* $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the length of a shortest path joining them if any; otherwise $d(u, v) = \infty$. A shortest $u$-$v$ path is often called a *geodesic*. The *diameter* $d(G)$ of a connected graph $G$ is the length of any longest geodesic.

**Definition 1.13.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Then *union* of $G_1$ and $G_2$ is the graph $G = G_1 \cup G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

The *join* of $G_1$ and $G_2$ is the graph $G = G_1 + G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

The graph $K_1 + C_p$ ($p \geq 3$) is called a *wheel* and is denoted by $W_p$.

The *Cartesian product* $G_1 \times G_2$ of two graphs $G_1$ and $G_2$ is defined to be the graph whose vertex set is $V_1 \times V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$ are adjacent in $G_1 \times G_2$ if either $u_1 = v_1$ and $u_2$ is adjacent to $v_2$ or $u_2 = v_2$ and $u_1$ is adjacent to $v_1$.

The *composition* $G = G_1[G_2]$ is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, u_2)$ is adjacent to $v = (v_1, v_2)$ whenever $u_1$ is adjacent to $v_1$ or $u_1 = v_1$ and $u_2$ is adjacent to $v_2$.

The *Kronecker* product of $G_1$ and $G_2$ is the graph $G = G_1 \times G_2$ with vertex set $V_1 \times V_2$ and edge set $E = \{(u_1, u_2), (v_1, v_2) : (u_1, v_1) \in E_1 \text{ and } (u_2, v_2) \in E_2\}$. 
Definition 1.14. A vertex $v$ of a graph $G$ is called a *cut-vertex* of graph $G$ if the removal of $v$ increases the number of components. An edge $e$ of a graph $G$ is called a *cut edge* or *bridge* if the removal of $e$ increases the number of components. A set of edges $S$ is called an *edge cut of $G$* if the number of components of $G - S$ is greater than that of $G$. A *block* of a graph is a maximal connected, non-trivial subgraph without cut-vertices.

Definition 1.15. A graph is *acyclic* or a *forest* if it has no cycles. A *tree* is a connected acyclic graph. A forest in which every component is a path is called a *linear forest*.

Definition 1.16. The graph got from $G$ by removing isolated vertices and pendant vertices is called the *foundation* of $G$ and is denoted by $Z_G$.

If $T$ is not a path but $Z_T$ is a path then $T$ is called a *caterpillar*.

If $T$ is not a caterpillar but $Z_T$ is a caterpillar then $T$ is called a *lobster*.

Definition 1.17. The *connectivity* $\kappa = \kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or $K_1$, the trivial graph. The *line connectivity* or *edge connectivity* $\kappa' = \kappa'(G)$ of a graph $G$ is the minimum number of edges whose removal results in a disconnected graph.

Definition 1.18. In a graph $G$, any closed trail containing all vertices and edges of $G$ is called an *Eulerian trail*. A graph $G$ is said to be *Eulerian* if it has an Eulerian trail.
Definition 1.19. A graph $G$ is called *Hamiltonian* if it has a spanning cycle. Any spanning cycle of $G$ is called a *Hamiltonian cycle*.

Definition 1.20 [2]. A *factor* of a graph $G$ is a spanning subgraph of $G$ which is not totally disconnected. If $G$ is the edge disjoint union of a set of its factors, then such a union is called a *factorization* of $G$. An *$n$-factor* is a regular factor of degree $n$. If $G$ is a union of $n$-factors then $G$ is said to be *$n$-factorable*.

Definition 1.21. The minimum number of edge-disjoint spanning linear forests into which $G$ can be decomposed is called *linear arboricity* of $G$ and is denoted by $r(G)$.

Definition 1.22. A set of vertices in $G$ is said to be *independent* if no two of them are adjacent. The largest number of vertices in any independent set of $G$ is called the *independence number* of $G$ and is denoted by $\beta_0$.

Definition 1.23. A vertex and an edge are said to *cover* each other if they are incident. A set of vertices which cover all the edges of a graph $G$ is called a *vertex cover* of $G$. The smallest number of vertices in any vertex cover is called the *vertex covering number* and is denoted by $\alpha_0$.

Definition 1.24. A *colouring* of a graph $G$ is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. The *chromatic number* $\chi(G)$ is defined to be the minimum $n$ for which $G$ has $n$-colouring.
Definition 1.25. An edge-colouring of a graph $G$ is an assignment of colours to its edges so that no two adjacent edges are assigned the same colour. The edge-chromatic number $\chi'(G)$ is the minimum $n$ for which $G$ has $n$-edge colouring.

Definition 1.26 [17]. A decomposition of a graph $G$ is a collection of subgraphs of $G$ whose edge sets partition the edge set of $G$. The subgraphs of the decomposition are called the parts of the decomposition.

Definition 1.27. A graph $G$ is said to be $F$-decomposable or $F$-packable if $G$ has a decomposition in which all of its parts are isomorphic to the graph $F$.

Given a graph $G$ which is $F$-packable, the task of actually performing a packing of copies of $F$ into $G$ will be easier if $G$ has the property that every collection of edge-disjoint copies of $F$ in $G$ can be extended to a $F$-packing of $G$. This motivated Ruiz [30] to introduce the concept of randomly $F$-packable graph.

Definition 1.28. A graph $G$ is said to be randomly $F$-packable if for every proper $F$-packable subgraph $H$ of $G$, $G - E(H)$ is also $F$-packable.

The problem of characterizing randomly $F$-packable graphs for arbitrary $F$ seems to be a difficult problem. Ruiz [30] obtained a characterization of randomly $F$-packable graphs when $F$ is $P_3$ or $2K_2$.

Theorem 1.29 [30]. A graph is randomly $P_3$-packable if and only if it is one of the following: $C_4$, $K_4$, $2K_3$, $K_3 \cup K_{1,3}$, $2K_{1,n}$ or $2nK_2$ ($n \geq 1$).
Theorem 1.30 [30]. A graph $G$ is randomly $K_{1,2}$-packable if and only if each component of $G$ is isomorphic to $C_4$ or $K_{1,2t}$.

Barrientos et al. [9] obtained a characterization of randomly $K_{1,r}$-packable graphs.

Theorem 1.31 [9]. For $r \geq 2$, a connected graph $G$ is randomly $K_{1,r}$-packable if and only if either it is $K_{r,r}$ or it is bipartite with all degrees in one partite set being multiples of $r$ and all degrees in other set being less than $r$.

Beineke et al. [10] obtained a characterization of randomly $F$-packable graphs when $F$ is either $K_n$, $P_4$, $P_5$ or $P_6$. They also characterized randomly $tK_2$-packable graphs with sufficiently many edges.

Theorem 1.32 [10]. A graph $G$ is randomly $K_n$-packable if and only if every edge lies in precisely one copy of $K_n$ in $G$.

Theorem 1.33 [10]. The only connected randomly $P_4$-packable graphs are $P_4$, $K_4$, $K_{2,3}$, $C_6$ and $C_3 \circ C_3$ where $C_3 \circ C_3$ is the graph obtained by identifying one vertex from each copy of $C_3$.

Theorem 1.34 [10]. The only connected randomly $P_5$-packable graphs are $P_5$, $K_{2,4}$, $C_4 \circ C_4$, $C_8$ and $S_4^{(k)}$ for $k \geq 2$ where $S_4^{(r)}$ denotes the graph obtained from $r$ paths of length $2k$ by identifying their central vertices.

Theorem 1.35 [10]. The only connected randomly $P_6$-packable graphs are $P_6$, $C_{10}$ and the three graphs in Figure 1.1.
Theorem 1.36 [10]. A graph $G$ having at least $2t^3 - t^2$ edges is randomly $tK_2$-packable if and only if it is either $tnK_2$ or $tK_{1,n}$ where $n \geq 1$.

Sumner [32] considered a different form of random packing, in which every partial matching is extendible to a full one.


Definition 1.37. Let $G = (V, E)$ be a graph. A difference labeling of $G$ is an injection $f$ from $V$ to the set of non-negative integers together with the weight function $f^*$ on $E$ given be $f^*(uv) = |f(u) - f(v)|$ for every edge $uv$ in $G$.

Definition 1.38. A decomposition of a labeled graph into parts each part containing the edges having a common weight is called a common-weight decomposition.

Bloom and Ruiz [13] have proved the following theorems.
Theorem 1.39 [13]. Every part in a common-weight decomposition is a linear forest. Further the vertices of minimum and maximum labels are not internal vertices in any path of a part containing it.

Theorem 1.40 [13]. There is a labeling of a cycle $C$ realizing a decomposition of $C$ into parts having $m_1$ and $m_2$ edges respectively if and only if $m_1$ and $m_2$ are relatively prime.

Theorem 1.41 [13]. A labeling exists for every cycle with $2s$ edges ($s \neq 4$) which decomposes it into two perfect matchings.

Theorem 1.42 [13]. A connected graph of maximum degree 3 and diameter 2 cannot have a common-weight decomposition in which all of the component paths in every part have length greater than 2.

Definition 1.43 [13]. A common-weight decomposition of $G$ in which each part contains $m$ edges is called $m$-equitable.

Theorem 1.44 [14]. Let $C$ be a cycle having $(m_1 + m_2 + \cdots + m_k)$ edges with $k > 2$. There is a labeling that will produce a common-weight decomposition of $C$ into paths $P_{m_1+1}, \ldots, P_{m_k+1}$.

Definition 1.45. For any graph $G$, $G^+$ is the graph obtained from $G$ by adjunction of a vertex $v'$ for every vertex $v$ in $G$ and joining $v$ and $v'$.

Akers and Krishnamoorthy [3] introduced the $n$-star graph $S_n$ which has been as an attractive alternative to the $n$-cube, with superior characteristics.
Definition 1.46 [3]. The $n$-star graph $S_n$ is a simple graph whose vertex set is the set of all $n!$ permutations of \{1, 2, \ldots, n\} and two vertices $\alpha$ and $\beta$ are adjacent if and only if $\alpha(1) \neq \beta(1)$ and $\alpha(i) \neq \beta(i)$ for exactly one $i$, $i \neq 1$.

Acharya and Sampathkumar [1] introduced the concept of graphoidal cover.

Definition 1.47. A graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions.

(i) Every path in $\psi$ has at least two vertices.

(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.

(iii) Every edge of $G$ is in exactly one path in $\psi$.

The minimum cardinality of a graphoidal cover of $G$ is called the graphoidal covering number of $G$ is denoted by $\eta(G)$.

Pakkiam and Arumugam [26, 27] determined the graphoidal covering number of several families of graphs.

Arumugam and Suresh Suseela [7] introduced the concept of acyclic graphoidal cover.

Definition 1.48. An acyclic graphoidal cover of $G$ is a graphoidal cover $\psi$ of $G$ such that every element of $\psi$ is a path in $G$. The minimum cardinality of an acyclic graphoidal cover of $G$ is called the acyclic graphoidal covering number of $G$ and is denoted by $\eta_a(G)$ or $\eta_a$. 
Arumugam and Suresh Suseela have proved the following theorems.

Theorem 1.49 [7]. For any tree $T$, $\eta_a(T) = n - 1$ where $n$ is the number of pendant vertices in $T$.

Theorem 1.50 [7]. For any acyclic graphoidal cover $\psi$ of $G$ let $t_\psi$ denote the number of vertices which are not internal to any path in $\psi$. Let $t = \min t_\psi$ where the minimum is taken over all acyclic graphoidal covers $\psi$ of $G$. Then $\eta_a = q - p + t$.

Theorem 1.51 [7]. For any graph $G$ with $\delta \geq 3$, $\eta_a = q - p$.

Theorem 1.52. For any graph $G$, $\eta_a \geq q - p$. Moreover, the following are equivalent.

(i) $\eta_a = q - p$.

(ii) There exists an acyclic graphoidal cover without exterior vertices.

(iii) There exists a set of internally disjoint and edge disjoint paths without exterior vertices. (From such a set of paths required acyclic graphoidal cover can be got by adding the edges which are not covered by paths of this set.)

Theorem 1.53 [7]. Let $G$ be a unicyclic graph with $n$ pendant vertices. Let $C$ be the unique cycle in $G$. Let $m$ be the number of vertices odd degree greater than 2 on $C$. Then

$$\eta_a(G) = \begin{cases} 
2 & \text{if } m = 0, \\
n + 1 & \text{if } m = 1, \\
n & \text{otherwise.}
\end{cases}$$
Harary [23] introduced the concept of path partition of a graph $G$.

**Definition 1.54.** A path partition of a graph $G$ is a collection $\mathcal{P}$ of paths in $G$ such that every edge of $G$ lies in exactly one path in $\mathcal{P}$.

The minimum cardinality of a path partition of $G$ is called path partition number of $G$ and is denoted by $\pi$.

**Theorem 1.55** [7]. $\eta_a = \pi$ if and only if $\Delta \leq 3$.

**Theorem 1.56** [7]. For any 3-regular graph, $\pi = \frac{p}{2}$.

**Theorem 1.57** [7]. Let $\psi$ be any path partition of $G$. Then

$$|\psi| = \frac{k}{2} + \sum_{v \in V(G)} \left\lceil \frac{\deg v}{2} \right\rceil - \sum_{P \in \psi} i(P)$$

where $k$ is the number of odd vertices of $G$ and for any $P \in \psi$, $i(P)$ is the number of internal vertices of $P$.

**Theorem 1.58** [7]. $\pi(K_{2n}) = n$.

**Theorem 1.59** [7]. For any tree $T$, $\pi = \frac{k}{2}$ where $k$ is the number of odd vertices of $T$.

**Theorem 1.60** [7]. Let $G$ be a unicyclic graph with unique cycle $C$. Let $m$ denote the number of vertices of degree greater than 2 on $C$. Let $k$ denote the number of odd vertices of $G$. Then

$$\pi(G) = \begin{cases} 2 & \text{if } m = 0, \\ \frac{k+2}{2} & \text{if } m = 1, \\ \frac{k}{2} & \text{otherwise.} \end{cases}$$
The concept of path double cover was introduced by Bondy [15].

**Definition 1.61.** A *path double cover* of a graph $G$ is a collection $\mathcal{P}$ of paths in $G$ such that each edge of $G$ belongs to exactly two paths in $\mathcal{P}$.

**Definition 1.62.** A one-to-one mapping $f$ from $V(G)$ into $\{0, 1, 2, \ldots, q\}$ is called a *β-valuation* if the induced function $\bar{f}$ on $E(G)$ given by $\bar{f}(uv) = |f(u) - f(v)|$ is one-to-one.

A β-valuation $f$ is called an *α-valuation* if there exists a non-negative integer $\lambda$ such that for every $uv \in E(G)$ with $f(u) < f(v)$, $f(u) \leq \lambda < f(v)$.

**Definition 1.63** [17]. A decomposition $\mathcal{R}$ of a graph $H$ into subgraphs is said to be *cyclic* if there exists an automorphism $f$ of $H$ which induces a cyclic permutation $f_V$ of the set $V = V(H)$ and satisfies the following implication: If $G \in \mathcal{R}$ then $fG \in \mathcal{R}$. (Here $fG$ is the subgraph of $H$ with vertex set $\{f(u) : u \in V(G)\}$ and edge set $\{f(e) : e \in E(G)\}$.)

Rosa [29] established a close connection between α-labeling and cyclic $G$-decomposition.

**Theorem 1.64** [29]. If a graph $G$ with $e$ edges has an α-valuation then for every positive integer $c$ there exists a cyclic decomposition of the complete graph $K_{2ce+1}$ into subgraphs isomorphic to $G$.

**Definition 1.65.** The $n$-cube $Q_n$ is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. $Q_n$ has $2^n$ vertices which may be labeled $(a_1, a_2, \ldots, a_n)$. 

where each $a_i$ is either 0 or 1. Two vertices of $Q_n$ are adjacent if their binary representation differ at exactly one place.

For any positive integer $n$, let $Q_n(G) = G \times K_2 \times \cdots \times K_2$ denote the graph of the $n$-dimensional $G$-cube. $Q_n(G)$ has $p2^{n-1}$ vertices and $(2q + n - 1)2^{n-1}$ edges.

The existence of an $\alpha$-valuation of $G$ need not imply the existence of an $\alpha$-valuation for $Q_n(G)$, $n \geq 2$. For example, the star $K_{1,4m+3}$ admits an $\alpha$-valuation while $Q_2(K_{1,2m+3})$ admits no $\alpha$-valuation.

Kotzig [24] has proved that $Q_n(K_2)$ admits an $\alpha$-valuation.

**Theorem 1.66** [24]. For any positive integer $n$, the $n$-cube $Q_n$ admits an $\alpha$-valuation.

Balakrishnan and Sampathkumar [8] have proved the following theorems.

**Theorem 1.67** [8]. For any positive integer $n$, the graph $Q_n(K_{3,3})$ admits an $\alpha$-valuation.

**Theorem 1.68** [8]. For any positive integer $n$, the graph $Q_n(K_{4,4})$ admits an $\alpha$-valuation.

**Theorem 1.69** [8]. For any positive integer $n$, the graph $Q_n(P_k)$, $k \geq 2$ admits an $\alpha$-valuation.