Path Double Covering Number of a Graph

A path double cover of a graph $G$ is a collection $\mathcal{P}$ of paths in $G$ such that every edge of $G$ belongs to exactly two paths in $\mathcal{P}$. The minimum cardinality of a path double cover is called the path double covering number of $G$ and is denoted by $\eta_{PD}(G)$. In this chapter we determine the exact value of this parameter for several classes of graphs. We also characterize graphs for which $\eta_{PD} = 2\pi$ where $\pi$ is the path partition number of $G$.

Bondy [15] introduced the concept of path double cover of a graph. This was further studied by Hao Li [25].

Definition 5.1. A path double cover (PDC) of a graph $G$ is a collection $\mathcal{P}$ of paths in $G$ such that every edge of $G$ belongs to exactly two paths in $\mathcal{P}$.

The collection $\mathcal{P}$ may not necessarily consist of distinct paths in $G$ and hence it cannot be treated as a set in the standard sense. For any graph $G = (V, E)$ let $\mathcal{P}$ denote the collection of all paths of length 1, each path appearing twice in the collection. Clearly $\mathcal{P}$ is a path double cover of $G$ and hence the set of all path double covers of $G$ is not non-empty.
Definition 5.2. The minimum cardinality of a path double cover of a graph $G$ is called path double covering number of $G$ and is denoted by $\eta_{PD}$.

We observe that for any graph $G$, $\eta_{PD}(G) \leq 2q$ and equality holds if and only if $G$ is isomorphic to $qK_2$.

Theorem 5.3. Let $\mathcal{P}$ be any path double cover of $G$. Then $|\mathcal{P}| = 2q - i_\mathcal{P}$ where $i_\mathcal{P} = \sum_{P \in \mathcal{P}} i(P)$ and $i(P)$ is the number of internal vertices of $P$.

Proof. For any vertex $v$, let $i(v)$ denote the number of paths in $\mathcal{P}$ having $v$ as an internal vertex. Then $v$ is an end vertex of $2\deg v - 2i(v)$ paths in $\mathcal{P}$. Hence

$$2|\mathcal{P}| = \sum_{v \in V} [2\deg v - 2i(v)] = 4q - 2i_\mathcal{P}. $$

Thus $|\mathcal{P}| = 2q - i_\mathcal{P}$. \qed

Corollary 5.4. $\eta_{PD} = 2q - i$ where $i = \max i_\mathcal{P}$, the maximum being taken over all path double covers $\mathcal{P}$ of $G$.

Corollary 5.5. Let $G$ be a graph with $\delta = 1$. If there exists a path double cover $\mathcal{P}$ such that every non-pendant vertex of $G$ is an internal vertex of $d(v)$ paths in $\mathcal{P}$ then $\mathcal{P}$ is a minimum path double cover and $\eta_{PD} = |\mathcal{P}|$.

Theorem 5.6. For any tree $T$, $\eta_{PD}(T) = n$ where $n$ is the number of pendant vertices of $T$. 
Proof. We first prove by induction on \( n \) that there exists a path double cover \( \mathcal{P} \) such that every non-pendant vertex of \( T \) is an internal vertex of \( d(v) \) paths in \( \mathcal{P} \).

When \( n = 2 \) then \( T \) is a path and the result is trivial. Assume that the result is true for all trees with less than \( n \) pendant vertices.

Let \( T \) be any tree with \( n \) pendant vertices, \( n \geq 3 \). Let \( w \) be a pendant vertex of \( T \). Choose a vertex \( v \) such that \( \deg v \geq 3 \) and \( d(w, v) \) is maximum. Let \( Q \) denote the \((w, v)\)-path. Since \( \deg v \geq 3 \), there exist pendant vertices \( w_1, w_2 \) such that the \((w, w_1)\)-path \( Q_1 \) and the \((w, w_2)\)-path \( Q_2 \) both contain \( Q \). Now let \( P_1 \) and \( P_2 \) denote the \((w, v)\)-section of \( Q_1 \) and \((w_2, v)\)-section of \( Q_2 \) respectively. Since \( v \) is a vertex with \( \deg v \geq 3 \) for which \( d(v, w) \) is maximum, every internal vertex of \( P_1 \) and \( P_2 \) has degree 2. Hence \( v \) is the only vertex of degree greater than 2 on \( P = P_1 \circ P_2^{-1} \). Let \( P = (w_1 = u_0, u_1, u_2, \ldots, u_r (= v), u_k = w_2) \). Let \( T_1 = T - \{u_0, u_1, \ldots, u_{r-1}, u_{r+1}, \ldots, u_k\} \). By induction hypothesis, there exists a path double cover \( \mathcal{P}_1 \) of \( T_1 \) such that every non-pendant vertex of \( T_1 \) is an internal vertex of \( d(v) \) paths in \( \mathcal{P}_1 \).

Case (i) \( \deg_T(v) = 3 \).

Then \( \deg_{T_1}(v) = 1 \). Therefore \( v \) is exterior to two paths, say \( Q_1, Q_2 \) in \( \mathcal{P}_1 \). We may assume that \( v \) is the origin of \( Q_1 \) and \( Q_2 \). Now let \( R_1 = P_1 \circ Q_1 \) and \( R_2 = P_2 \circ Q_2 \). Then \( \mathcal{P} = (\mathcal{P}_1 \cup \{P_1', P_2', P\}) \setminus \{R_1, R_2\} \) is a path double cover of \( T \) such that every non-pendant vertex is an internal vertex of \( d(v) \) paths in \( \mathcal{P} \).
Case (ii) \( \deg_T(v) > 3 \).

Then \( \mathcal{P} = \mathcal{P}_1 \cup \{P, P\} \) is a path double cover of \( T \) such that every non-pendant vertex is an internal vertex of \( d(v) \) paths in \( \mathcal{P} \). Hence it follows that \( \mathcal{P} \) is a minimum path double cover of \( T \) and \( \eta_{PD} = |\mathcal{P}| \). Further \( i_\mathcal{P} = 2q - n \) and it follows from Theorem 5.3 that \( \eta_{PD} = 2q - (2q - n) = n \).

\[ \square \]

**Theorem 5.7.** Let \( G \) be a unicyclic graph with \( n \) pendant vertices. Let \( C = (v_1, v_2, \ldots, v_t, v_1) \) be the unique cycle in \( G \). Let \( m \) be the number of vertices of degree greater than 2 on \( C \). Then

\[
\eta_{PD}(G) = \begin{cases} 
3 & \text{if } m = 0 \\
n + 2 & \text{if } m = 1 \\
n + 1 & \text{if } m = 2 \\
n & \text{otherwise.}
\end{cases}
\]

**Proof.** If \( m = 0 \), then \( G = C \) and \( \eta_{PD} = 3 \). Hence we assume that \( m \geq 1 \).

Case (i) \( m = 1 \).

Let \( v_1 \) be the unique vertex of degree greater than 2 on \( C \). We prove the result by induction on \( n \).

When \( n = 1 \), \( G \) is isomorphic to the graph consisting of the cycle \( C = (v_1, v_2, \ldots, v_t, v_1) \) and a path \( P = (w_1, w_2, \ldots, w_r = v_1) \).

Let \( P_1 = (w_1, w_2, \ldots, w_r = v_1, v_2, \ldots, v_t) \),

\[ P_2 = (w_1, w_2, \ldots, w_r = v_1, v_t, v_{t-1}, \ldots, v_2) \]

and \( P_3 = (v_2, w_r = v_1, v_t) \)
Clearly \( \mathcal{P} = \{P_1, P_2, P_3\} \) is a minimum path double cover of \( G \) and hence \( \eta_{PD}(G) = 3 = n + 2 \).

We now assume that the result is true for all unicyclic graphs with \( m = 1 \) and \( n - 1 \) pendant vertices \( (n \geq 2) \).

Let \( G \) be any unicyclic graph with \( m = 1 \) and having \( n \) pendant vertices.

Let \( w \) be a pendant vertex of \( G \). Choose a vertex \( v \) such that \( \deg v \geq 3 \) and \( d(w, v) \) is maximum. Let \( Q \) denote the \((w, v)\)-path. Since \( \deg v \geq 3 \), there exist pendant vertices \( w_1, w_2 \) such that the \((w, w_1)\)-path \( Q_1 \) and the \((w, w_2)\)-path \( Q_2 \) both contain \( Q \). Now let \( P_1 \) and \( P_2 \) denote the \((w_1, v)\)-section of \( Q_1 \) and \((w_2, v)\)-section of \( Q_2 \) respectively. Let \( P = P_1 \circ P_2^{-1} \).

Clearly \( v \) is the only vertex of degree greater than 2 on \( P \).

Let \( P = (u_0, u_1, \ldots, u_r, \ldots, u_k) \). Now consider the graph \( G_1 = G - \{w_1 = w_0, u_1, u_2, \ldots, u_{r-1}, u_{r+1}, \ldots, u_k = w_2\} \). Using induction hypothesis, \( \eta_{PD}(G_1) = (n - 1) + 2 = n + 1 \). Let \( \mathcal{P}_1 \) be a minimum path double cover of \( G_1 \).

Subcase (a) \( \deg_G u_r = 3 \).

Then \( \deg_{G_1} u_r = 1 \). Therefore \( u_r \) is exterior to two paths in \( \mathcal{P}_1 \), say for the paths \( R_1 \) and \( R_2 \) in \( \mathcal{P}_1 \). Let \( P'_1 = R_1 \circ P_1 \) and \( P'_2 = R_2 \circ P_2 \). Let \( \mathcal{P} = \mathcal{P}_1 \cup \{P'_1, P'_2, P\} - \{R_1, R_2\} \). Then \( \mathcal{P} \) is a path double cover of \( G \). Thus \( \eta_{PD}(G) \leq n + 2 \). In any path double cover of \( G \) all the \( n \) pendant vertices and at least two vertices in \( C \) are exterior points. Thus \( \eta_{PD}(G) \geq n + 2 \). Hence \( \eta_{PD}(G) = n + 2 \).
Case (ii) \( m = 2 \).

Proof is similar to Case (i).

Case (iii) \( m > 2 \).

As in Case (ii) we can prove that \( \eta_{PD}(G) \leq n \).

Since all the pendant vertices are exterior to all paths in any path double cover of \( G \), \( \eta_{PD}(G) \geq n \).

Hence \( \eta_{PD}(G) = n \). \qed

Theorem 5.8. \( \eta_{PD}(K_n) = n \).

Proof. Since the total number of edges to be covered is \( 2 \binom{n}{2} = n(n-1) \) and any path covers at most \( n-1 \) edges it follows that \( \eta_{PD}(K_n) \geq n \).

Now let \( V(K_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\} \) and for \( 0 \leq i \leq n-1 \) let

\[
P_i = \begin{cases} v_i v_{i+1} v_i v_{i+2} v_{i-2} \ldots v_{i+k+1} v_{i+k} & \text{if } n = 2k \\ v_i v_{i+1} v_i v_{i+2} v_{i-2} \ldots v_{i+k} v_{i+k+1} & \text{if } n = 2k + 1 \end{cases}
\]

where the suffixes are integers modulo \( n \). Then \( P = \{P_i \mid 0 \leq i \leq n-1\} \) is a path double cover of \( K_n \). Thus \( \eta_{PD} \leq n \).

Hence \( \eta_{PD}(K_n) = n \). \qed

Theorem 5.9. For any graph \( G \), \( \eta_{PD} \geq \Delta \). Further for any tree \( T \), \( \eta_{PD} = \Delta \) if and only if \( T \) is homeomorphic to a star.

Proof. Let \( v \) be a vertex of degree \( \Delta \) in \( G \). Since we need at least \( \Delta \) paths to cover all the edges incident with \( v \) twice, \( \eta_{PD} \geq \Delta \).
Now suppose $T$ is a tree with $\eta_{PD} = \Delta$. Then it follows from Theorem 5.6 that the number of pendant vertices of $T$ is $\Delta$ and hence $T$ is homeomorphic to a star.

The converse is obvious. \qed

**Theorem 5.10.** For any wheel, $\eta_{PD}(W_n) = \Delta = n$.

**Proof.** Let $V(W_n) = \{v_0, v_1, v_2, \ldots, v_n\}$ with $\deg v_0 = n$.

Case (i) $n$ is even.

Let $P_1 = (v_n, v_1, v_0, v_{n/2+1}, v_{n/2})$, 
$P_2 = (v_{n-1}, v_n, v_0, v_{n/2}, v_{n/2-1})$, 
$P_3 = (v_{n-2}, v_{n-1}, v_0, v_{n/2-1}, v_{n/2-2})$, 
$\vdots$ 
$P_i = (v_{n-(i-1)}, v_{n-(i-2)}, v_0, v_{n/2-(i-2)}, v_{n/2-(i-1)})$, 
$\vdots$ 
$P_{n/2} = (v_{n/2+1}, v_{n/2+2}, v_0, v_{n/2-(n/2-2)}, v_{n/2-(n/2-1)})$.

Then $P = \{P_1, P_2, P_2, \ldots P_{n/2}, P_{n/2}\}$ is a path double cover of $W_n$ with $|P| = \Delta$.

Case (ii) $n$ is odd.

Let $P_1 = (v_n, v_1, v_0, v_{n+1/2}, v_{n-1/2})$
$P_2 = (v_1, v_2, v_0, v_{(n+1)/2+1}, v_{(n+1)/2})$
$P_3 = (v_2, v_3, v_0, v_{(n+1)/2+2}, v_{(n+1)/2+1})$
$\vdots$ 
$\vdots$
\[ P_{(n-1)/2} = (v_{(n-3)/2}, v_{(n-1)/2}, v_0, v_{n-1}, v_{n-2}) \]
\[ R = (v_{(n-1)/2}, v_{(n+1)/2}, v_0, v_{n-1}) \]
\[ Q_1 = (v_n, v_1, v_0, v_{(n+3)/2}, v_{(n+1)/2}) \]
\[ Q_2 = (v_1, v_2, v_0, v_{(n+5)/2}, v_{(n+3)/2}) \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ Q_{(n-1)/2} = (v_{(n-3)/2}, v_{(n-1)/2}, v_0, v_n, v_{n-1}) \]

Then \( \mathcal{P} = \{ P_1, P_2, \ldots, P_{(n-1)/2}, R, Q_1, Q_2, \ldots, Q_{(n-1)/2} \} \) is a path double cover of \( G \) with \( |P| = \Delta \).

Hence \( \eta_{PD}(W_n) = \Delta = n \).

\[ \square \]

**Theorem 5.11.** For every rectangular grid \( P_m \times P_n \),

\[ \eta_{PD}(P_m \times P_n) = \Delta = 4 \forall m, n \geq 3. \]

**Proof.** Let \( V(G) = \{ v_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq m \} \) where \( P_m = (v_1, v_2, \ldots, v_m) \), \( P_n = (u_1, u_2, \ldots, u_n) \) and \( v_{ij} = (v_i, w_j) \).

**Case (i) \( m \) and \( n \) are odd.**

Let \( P_1 = (v_{11}, v_{1n}, v_{2n}, \ldots, v_{21}, v_{31}, \ldots, v_{3n}, \ldots, v_{m1}, \ldots, v_{mn}) \)
\[ P_2 = (v_{11}, v_{21}, \ldots, v_{m1}, v_{m2}, \ldots, v_{12}, \ldots, v_{1n}, v_{2n}, \ldots, v_{mn}) \]
\[ P_3 = (v_{11}, v_{12}, v_{22}, v_{32}, \ldots, v_{m2}, v_{m3}, \ldots, v_{13}, \ldots, v_{1(n-1)}, \ldots, v_{m(n-1)}, v_{mn}) \]
\[ P_4 = (v_{11}, v_{21}, v_{22}, \ldots, v_{2n}, v_{3n}, \ldots, v_{41}, \ldots, v_{4n}, \ldots, v_{(m-1)1}, \ldots, v_{(m-1)n}, v_{mn}) \]

**Case (ii) \( m \) and \( n \) are even.**

Let \( P_1 = (v_{11}, \ldots, v_{1n}, v_{2n}, \ldots, v_{21}, \ldots, v_{mn}, \ldots, v_{mn}) \)
\[ P_2 = (v_{11}, v_{21}, \ldots, v_{m1}, v_{m2}, \ldots, v_{12}, \ldots, v_{1n}, v_{(m-1)n}, \ldots, v_{1n}) \]
\[ P_3 = (v_{11}, v_{12}, v_{22}, v_{32}, \ldots, v_{m2}, v_{m3}, \ldots, v_{13}, \ldots, v_{1(n-1)}, \ldots, v_{m(n-1)}, v_{1n}) \]
\[ P_4 = (v_{11}, v_{21}, v_{22}, \ldots, v_{2n}, v_{3n}, \ldots, v_{31}, \ldots, v_{(m-1)n}, \ldots, v_{(m-1)n}, v_{m1}) \]
Case (iii) \( m \) is odd and \( n \) is even.

Let

\[
P_1 = (v_{11}, \ldots, v_{1n}, v_{21}, \ldots, v_{2n}, \ldots, v_{mn})
\]

\[
P_2 = (v_{11}, v_{21}, \ldots, v_{m1}, v_{m2}, \ldots, v_{12}, \ldots, v_{mn}, v_{m(n-1)}, \ldots, v_{1n})
\]

\[
P_3 = (v_{11}, v_{12}, v_{22}, \ldots, v_{m2}, v_{m3}, \ldots, v_{13}, \ldots, v_{m(n-1)}, \ldots, v_{1(n-1)}, v_{1n})
\]

\[
P_4 = (v_{11}, v_{21}, v_{22}, \ldots, v_{(m-1)1}, v_{(m-1)2}, \ldots, v_{(m-1)n}, v_{mn})
\]

Case (iv) \( m \) is even and \( n \) is odd.

Let

\[
P_1 = (v_{11}, \ldots, v_{1n}, v_{21}, \ldots, v_{2n}, \ldots, v_{mn}, \ldots, v_{m1})
\]

\[
P_2 = (v_{11}, v_{21}, \ldots, v_{m1}, v_{m2}, \ldots, v_{12}, \ldots, v_{2n}, v_{m(n-1)}, \ldots, v_{1n})
\]

\[
P_3 = (v_{11}, v_{12}, v_{22}, \ldots, v_{m2}, v_{m3}, \ldots, v_{13}, \ldots, v_{m(n-1)}, \ldots, v_{1(n-1)}, v_{mn})
\]

\[
P_4 = (v_{11}, v_{21}, v_{22}, \ldots, v_{2n}, v_{3n}, \ldots, v_{31}, \ldots, v_{(m-1)n}, \ldots, v_{(m-1)1}, v_{m1})
\]

In all the cases \( \mathcal{P} = \{P_1, P_2, P_3, P_4\} \) is a path double cover of \( P_m \times P_n \).

Thus \( \eta_{PD}(P_m \times P_n) \leq 4 = \Delta \). By Theorem 5.9, \( \eta_{PD}(P_m \times P_n) \geq \Delta = 4 \).

Hence it follows that \( \eta_{PD}(P_m \times P_n) = \Delta = 4 \).

If \( \psi \) is any path partition of \( G \) then the collection \( \mathcal{P} \) consisting of each path in \( \psi \) twice is a path double cover of \( G \) and hence it follows that \( \eta_{PD} \leq 2\pi \). In the following theorems we characterize all trees and unicyclic graphs for which \( \eta_{PD} = 2\pi \).

Theorem 5.12. Let \( T \) be any tree. Then \( \eta_{PD}(T) = 2\pi \) if and only if pendant vertices of \( T \) are the only vertices of odd degree.

Proof. Let \( T \) be any tree with \( n \) pendant vertices, which are the only vertices of odd degree. Then it follows from Theorem 1.59 that \( \pi(T) = \frac{n}{2} \).

Further by Theorem 5.6, \( \eta_{PD}(T) = n \) and hence \( \eta_{PD} = 2\pi \).
Conversely if $\eta_{PD} = 2\pi$ then it follows from Theorem 1.59 and Theorem 5.6 that the number of vertices of odd degree is equal to the number of pendant vertices of $T$.

**Theorem 5.13.** Let $G$ be a unicyclic graph with unique cycle $C$ and $G \neq C$. Let $m$ be the number of vertices of degree greater than 2 on $C$. Then $\eta_{PD} = 2\pi$ if and only if $m \neq 2$ and pendant vertices of $G$ are the only vertices of odd degree or $m = 2$ and there is exactly one non-pendant vertex of odd degree.

**Proof.** Follows from Theorem 5.7 and Theorem 1.60.