Chapter 7

Near Equitable Domination in
Graphs

Reference [38, 39, 41, 42] are based on this chapter.
7.1 Introduction

In this chapter, we define and study a new domination parameter called near equitable domination number of graphs.

**Definition 7.1.1.** Let $D$ be a dominating set of a graph $G$. Then $D$ is called a near equitable dominating set of $G$ if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that $u$ is adjacent to $v$ and $|od_D(u) - od_{V-D}(v)| \leq 1$. The minimum cardinality of such a near equitable dominating set is called the near equitable domination number of $G$ and is denoted by $\gamma_{ne}(G)$.

In section 7.2, we introduce some interesting results of a near equitable domination in graphs. Some bounds for a near equitable domination number are found. Near equitable domatic number of a graph $G$ is also studied. In section 7.3, results involving connected near equitable dominating set are found, some bounds for a connected near equitable domination number are obtained. In section 7.4, we initiate the study of a total near equitable domination parameter. In section 7.5, we introduce the concept of strong total near equitable domination in graphs.
7.2 On near equitable domination in graphs

7.2.1 Main Results

**Definition 7.2.1.** Let $D$ be a near equitable dominating set of a graph $G$. The near equitable neighborhood of $u \in D$, denoted by $N_{ne}^D(u)$ is defined as

$$N_{ne}^D(u) = \{v \in V - D : v \in N(u), |od_D(u) - od_{V - D}(v)| \leq 1\}.$$ 

**Definition 7.2.2.** Let $D$ be a near equitable dominating set of a graph $G$. The maximum and minimum near equitable degree of $D$ are denoted by $\Delta_{ne}^D$ and $\delta_{ne}^D$, respectively. That is $\Delta_{ne}^D = \max_{u \in D}|N_{ne}^D(u)|$ and $\delta_{ne}^D = \min_{u \in D}|N_{ne}^D(u)|$.

For example, let $G \cong nK_2$, $n \geq 1$, if $D$ is a near equitable dominating set of $G$, then $\Delta(G) = \Delta_{ne}^D = \delta(G) = \delta_{ne}^D = 1$.

From the definition 7.2.2, we have the following propositions.

**Proposition 7.2.3.** If $D$ is a near equitable dominating set of a graph $G$, then $\Delta_{ne}^D \leq \Delta(G)$.

**Proposition 7.2.4.** Let $G$ be a graph containing isolated vertices. If $D$ is a near equitable dominating set of $G$, then $\delta(G) = \delta_{ne}^D$.

**Proposition 7.2.5.** If $D$ is a near equitable dominating set of a tree $T$, then $\delta(T) \leq \delta_{ne}^D$. 
It is obvious that any near equitable dominating set of a graph $G$ is also a dominating set, and thus we obtain the obvious bound $\gamma(G) \leq \gamma_{ne}(G)$. Furthermore, the difference $\gamma_{ne}(G) - \gamma(G)$ can be arbitrarily large in a graph $G$. It can be easily checked that $\gamma(K_{1,n}) = 1$, while $\gamma_{ne}(K_{1,n}) = n - 1$.

**Observation 7.2.6.** There exist graphs for which the three parameters $\gamma(G)$, $\gamma_e(G)$ and $\gamma_{ne}(G)$ are distinct. For graph $G$ given in Figure 7.2.1, we have $\gamma(G) = 5$, $\gamma_e(G) = 9$ and $\gamma_{ne}(G) = 6$.

![Figure 7.2.1](image)

**Proposition 7.2.7.** For any connected graph $G$ of order $p$, $p \leq 3$,

$$\gamma_{ne}(G) = \gamma_e(G) = \gamma(G) = 1.$$
Proof. Clearly, $\gamma_{ne}(G) = 1$ only for $G \cong K_{1,n}$, $n \leq 2$. Since for $p \leq 3$, we have $\gamma_e(G) = \gamma(G) = 1$, the proof follows.

We now proceed to compute $\gamma_{ne}(G)$ for some standard graphs.

**Observation 7.2.8.** For a path $P_p$, $\gamma_{ne}(P_p) = \gamma_e(P_p) = \gamma(P_p) = \lceil \frac{p}{3} \rceil$.

**Observation 7.2.9.** For a cycle $C_p$, $\gamma_{ne}(C_p) = \gamma_e(C_p) = \gamma(C_p) = \lceil \frac{p}{3} \rceil$.

**Theorem 7.2.10.** For a complete graph $K_p$, $\gamma_{ne}(K_p) = \lfloor \frac{p}{2} \rfloor$.

**Proof.** Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$ and $D \subset V(K_p)$ be a near equitable dominating set. By the definition of near equitable dominating set, for any $v_i \in V - D$, there exists $v_j \in D$ such that $|od_D(v_j) - od_{V-D}(v_i)| \leq 1$. Therefore, $|D| = \lfloor \frac{p}{2} \rfloor$ or $\lceil \frac{p}{2} \rceil$. Thus, $\gamma_{ne}(K_p) = \lfloor \frac{p}{2} \rfloor$.

**Theorem 7.2.11.** For the double star $S_{n,m}$,

$$\gamma_{ne}(S_{n,m}) = \begin{cases} 
2, & \text{if } n, m \leq 2; \\
n + m - 2, & \text{if } n, m \geq 2 \text{ and } n \text{ or } m \geq 3.
\end{cases}$$

**Proof.** Let $V(S_{n,m}) = \{u, v, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$. We consider the following cases.

**Case 1:** $n, m \leq 2$. Let $D = \{u, v\}$ be a minimum dominating set of $S_{n,m}$. Then $|od_D(u) - od_{V-D}(u_i)| \leq 1$, $1 \leq i \leq n$ and $|od_D(v) - od_{V-D}(v_i)| \leq 1$, $1 \leq i \leq m$. Hence, $D$ is a near equitable dominating set of $S_{n,m}$. Thus, $\gamma_{ne}(S_{n,m}) \leq \gamma(S_{n,m})$. But we
have, $\gamma(S_{n,m}) \leq \gamma_{ne}(S_{n,m})$. Therefore, $\gamma(S_{n,m}) = \gamma_{ne}(S_{n,m})$. Thus, $D$ is a minimum near equitable dominating set, and $\gamma_{ne}(S_{n,m}) = 2$.

**Case 2:** $n, m \geq 2$ and $n$ or $m \geq 3$. Without loss of generality, let $m \geq 3$. Consider a dominating set $D = \{u, v, u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_{m-2}\}$. Then, $od_D(u) = 2$, $od_{V-D}(u_i) = 1, i = n-1, n$, $od_D(v) = 2$ and $od_{V-D}(v_i) = 1, i = n-1, n$, it follows that, $|od_D(u) - od_{V-D}(u_i)| \leq 1$ and $|od_D(v) - od_{V-D}(v_i)| \leq 1$. Therefore, by the definition, $D$ is a near equitable dominating set. Now, if $D_1 = \{u, v, u_1, u_2, \ldots, u_{n-3}, v_1, v_2, \ldots, v_{m-2}\}$ is a near equitable dominating set of $S_{n,m}$, then $|od_{D_1}(u) - od_{V-D_1}(u_i)| = 3$, for every $i$, $i = n-2, n-1, n$, a contradiction. Thus, $D$ is a minimum near equitable dominating set, and $\gamma_{ne}(S_{n,m}) = n + m - 2$.

**Theorem 7.2.12.** For the complete bipartite graph $G \cong K_{n,m}$ with $1 < m \leq n$,

$$
\gamma_{ne}(K_{n,m}) = \begin{cases} 
  m - 1, & \text{if } n = m \text{ and } m \geq 3; \\
  m, & \text{if } n - m = 1 \text{ or } n, m \leq 2; \\
  n - 1, & \text{if } n - m \geq 2.
\end{cases}
$$

**Proof.** Let $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ be the bipartition of $K_{n,m}$. We consider the following cases.

**Case 1:** $n = m \geq 3$. We consider the following subcases.

**Subcase 1.1:** $n = m = 3$. Let $D = \{u_1, v_1\}$ be a minimum dominating set of $K_{n,m}$. Then, $|od_D(u_1) - od_{V-D}(v_1)| = 1$, for all $v_i \in V_2 - D$ and $|od_D(v_1) - od_{V-D}(u_i)| = 1$, for all $u_i \in V_1 - D$. Hence, $D$ is a near equitable dominating set of $K_{n,m}$. Therefore,
\( \gamma_{ne}(K_{n,m}) \leq \gamma(K_{n,m}) \). But \( \gamma(K_{n,m}) \leq \gamma_{ne}(K_{n,m}) \). Hence, \( \gamma(K_{n,m}) = \gamma_{ne}(K_{n,m}) \).

Thus, \( D \) is a minimum near equitable dominating set.

**Subcase 1.2** : \( n = m \geq 4 \). We have the following subsubcases.

**Subsubcase 1.2.1** : \( n \) and \( m \) are odd.

Consider a dominating set \( D = \{u_1, u_2, ..., u_{\lfloor \frac{n}{2} \rfloor}, v_1, v_2, ..., v_{\lfloor \frac{m}{2} \rfloor} \} \) such that \( |D| = m - 1 \).

Then \( od_D(u_i) = \lceil \frac{m}{2} \rceil \), \( od_D(v_i) = \lceil \frac{n}{2} \rceil \), \( od_{v_{1-D}}(u_j) = \lfloor \frac{m}{2} \rfloor \) and \( od_{v_{2-D}}(v_j) = \lfloor \frac{n}{2} \rfloor \). Since \( n = m \), \( |od_D(u_i) - od_{v_{2-D}}(v_j)| \leq 1 \), for all \( v_j \in V_2 - D \) and \( |od_D(v_i) - od_{v_{1-D}}(u_j)| \leq 1 \), for all \( u_j \in V_1 - D \). Therefore, \( D \) is a near equitable dominating set. Now, if \( D_1 = \{u_1, u_2, ..., u_s, v_1, v_2, ..., v_{n-s-2} \} \), \( s < \lfloor \frac{n}{2} \rfloor \) is a near equitable dominating set of \( K_{n,m} \), then \( |od_{D_1}(u_i) - od_{v_{2-D_1}}(v_j)| = 2 \) and \( |od_{D_1}(v_i) - od_{v_{1-D_1}}(u_j)| = 2 \), a contradiction. Therefore, \( D \) is a minimum near equitable dominating set.

**Subsubcase 1.2.2** : \( n \) and \( m \) are even.

Consider a dominating set \( D = \{u_1, u_2, ..., u_{\frac{n}{2}}, v_1, v_2, ..., v_{\frac{m}{2}-1} \} \) such that \( |D| = m - 1 \).

Then \( od_D(u_i) = \frac{n}{2} + 1 \), \( od_D(v_i) = \frac{n}{2} \), \( od_{v_{1-D}}(u_j) = \frac{m}{2} - 1 \) and \( od_{v_{2-D}}(v_j) = \frac{n}{2} \). Since \( n = m \), \( |od_D(u_i) - od_{v_{2-D}}(v_j)| \leq 1 \), for all \( v_j \in V_2 - D \) and \( |od_D(v_i) - od_{v_{1-D}}(u_j)| \leq 1 \), for all \( u_j \in V_1 - D \). Therefore, \( D \) is a near equitable dominating set. Now, if \( D_1 = \{u_1, u_2, ..., u_s, v_1, v_2, ..., v_{n-s-2} \} \), \( s < \lfloor \frac{n}{2} \rfloor \) is a near equitable dominating set of \( K_{n,m} \), then \( |od_{D_1}(u_i) - od_{v_{2-D_1}}(v_j)| = 2 \), and \( |od_{D_1}(v_i) - od_{v_{1-D_1}}(u_j)| = 2 \), a contradiction. Therefore, \( D \) is a minimum near equitable dominating set.
Case 2: $n \neq m$. We consider the following subcases.

Subcase 2.1: $n - m = 1$. Consider $D = \{u_1, u_2, \ldots, u_{\lfloor \frac{n}{2} \rfloor}, v_1, v_2, \ldots, v_{\lfloor \frac{m}{2} \rfloor}\}$, a dominating set such that $|D| = m$. Since $n = m + 1$, $|od_D(u_i) - od_{v_2-D}(v_j)| \leq 1$, for all $v_j \in V_2 - D$ and $|od_D(v_i) - od_{v_1-D}(u_j)| \leq 1$, for all $u_j \in V_1 - D$. Therefore, $D$ is a near equitable dominating set. Now, if $D_1 = \{u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{n-s-2}\}$, $s < \lfloor \frac{n}{2} \rfloor$ is a near equitable dominating set and if $n$ is odd, then $|od_{D_1}(v_i) - od_{v_1-D_1}(u_j)| = 2$. Similarly, if $n$ is even, then $|od_{D_1}(v_i) - od_{v_1-D_1}(u_j)| = 2$, a contradiction. Thus, $D$ is a minimum near equitable dominating set.

Subcase 2.2: $n - m \geq 2$.

Consider a dominating set $D = \{u_1, u_2, \ldots, u_{n-m-1}, v_1, v_2, \ldots, v_m\}$, $|D| = n - 1$. Then, $|od_D(u_i) - od_{v_2-D}(v_j)| = 0$, for all $v_j \in V_2 - D$ and $|od_D(v_i) - od_{v_1-D}(u_j)| = 1$, for all $u_j \in V_1 - D$. Therefore, $D$ is a near equitable dominating set. Now, if $D_1 = D - \{u_{n-m-1}\}$ or $D - \{v_m\}$ is a near equitable dominating set, then we have $D_1 = \{u_1, u_2, \ldots, u_{n-m-2}, v_1, v_2, \ldots, v_m\}$ or $D_1 = \{u_1, u_2, \ldots, u_{n-m-1}, v_1, v_2, \ldots, v_{m-1}\}$. But then, $|od_{D_1}(v_i) - od_{v_1-D_1}(u_j)| = 2$, a contradiction. Thus, $D$ is a minimum near equitable dominating set.

Theorem 7.2.13. For the wheel $W_{1,n}$, $n \geq 5$,

$$\gamma_{ne}(W_{1,n}) = \left\lceil \frac{n}{3} \right\rceil + 1$$

Proof. Let $V(W_{1,n}) = \{u, v_1, v_2, \ldots, v_n\}$, where $u$ is the central vertex of $W_{1,n}$ and
$v_i, 1 \leq i \leq n$ is on the cycle. By Theorem 7.2.9, $\gamma_{ne}(C_n) = \lceil \frac{n}{3} \rceil$. Let $D$ be a minimum near equitable dominating set of $C_n$ and $D_1 = D \cup \{u\}$. Then we have $od_{D_1}(v_i) = 2$, $od_{D_1}(u) = n - \lceil \frac{n}{3} \rceil$ and $od_{V-D_1}(v_j) \leq 3$. Therefore, for any $v_j \in V - D_1$, there exists $v_i \in D_1$ such that $v_j$ adjacent to $v_i$ and $|od_{D_1}(v_i) - od_{V-D_1}(v_j)| \leq 1$. Now, if $D_1 = (D - \{v_k\}) \cup \{u\}$ is a near equitable dominating set of $W_{1,n}$, then there exists $v_{k+1} \in V - D_1$ not dominated by any vertex of $D$, but dominated by $u$. But we have $\vert od_{D_1}(u) - od_{V-D_1}(v_{k+1}) \rvert \geq 2$, a contradiction. Also, if $D_1 = D$ is a near equitable dominating set, then for any $v_i \in D_1$, $|od_{D_1}(v_i) - od_{V-D_1}(u)| \geq 2$, a contradiction. So, $u \in D_1$. Thus, $\gamma_{ne}(W_{1,n}) = \lceil \frac{n}{3} \rceil + 1$. \hfill \qed

**Theorem 7.2.14.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $|V_1| = n$ and $|V_2| = m$, $n \leq m$, $m - n \leq 1$. Then $\gamma_{ne}(G_1 + G_2) = n$.

**Proof.** Let $G = G_1 + G_2$. For any $u \in V_1$ and $v \in V_2$, $u$ and $v$ are adjacent. Since $m - n \leq 1$, it follows that $|od_{V_1}(u) - od_{V_2}(v)| \leq 1$ in $G$. Since $n \leq m$, $V_1$ is a minimum near equitable dominating set of $G$. Thus, $\gamma_{ne}(G) = n$. \hfill \qed

**Theorem 7.2.15.** Let $G$ be a graph with any minimum perfect dominating set $D$ having the following property: for every $u \in D$, $od_D(u) \leq 2$. Then $\gamma_{ne}(G) \leq \gamma_p(G)$.

**Proof.** Let $D$ be a minimum perfect dominating set of a graph $G$. Then every $v \in V - D$ is dominated by exactly one vertex of $D$. Since for every $u \in D$, $od_D(u) \leq 2$, it follows that $|od_D(u) - od_{V-D}(v)| \leq 1$. Hence, $D$ is a near equitable dominating set. Thus, $\gamma_{ne}(G) \leq \gamma_p(G)$. \hfill \qed
We define a near equitable pendant vertex of a graph as follows.

**Definition 7.2.16.** Let $D$ be a near equitable dominating set of a graph $G$. Then $u \in D$ is a near equitable pendant vertex if $od_D(u) = 1$. A set $D$ is called a near equitable pendant dominating set if every vertex of $D$ is a near equitable pendant vertex.

**Theorem 7.2.17.** Let $D$ be a near equitable pendant dominating set of a graph $G$. Then $V - D$ is a minimum near equitable dominating set of $G$.

*Proof.* Suppose that $D$ is a near equitable pendant dominating set. Then for every $u \in D$, $od_D(u) = 1$ and $od_{V-D}(v) \leq 2$ for every $v \in V - D$. Therefore, for any $u \in D$, there exists $v \in V - D$ such that $v$ is adjacent to $u$ and $|od_D(u) - od_{V-D}(v)| \leq 1$, so that $V - D$ is a near equitable dominating set. Since for every $u \in D$, $od_D(u) = 1$ and $od_{V-D}(v) \leq 2$, for every $v \in V - D$, it follows that $|V - D| \leq |D|$. So, $V - D$ is a minimum near equitable dominating set of $G$. 

**Theorem 7.2.18.** Let $T$ be a wounded spider obtained from the star $K_{1,p-1}$, $p \geq 5$ by subdividing $m$ edges exactly once. Then

$$\gamma_{ne}(T) = \begin{cases} p, & \text{if } m = p - 1; \\ p - 1, & \text{if } m = p - 2; \\ p - 2, & \text{if } m \leq p - 3. \end{cases}$$

*Proof.* Let $K_{1,p-1}$ be a star with central vertex $u$. Then $V(K_{1,p-1}) = \{u, u_1, u_2, ..., u_{p-1}\}$ with $deg(u) = p - 1$. Let $v_i$ be the vertex subdividing the edge $uu_i$. Then we consider
the following cases.

**Case 1:** \( m = p - 1 \). \( D = \{u, v_1, v_2, ..., v_{m-1}, v_{p-1}\} \) is a near equitable dominating set and hence \( \gamma_{ne}(T) \leq |D| = p \). But \( \gamma(T) = p - 1 \) and \( \gamma(G) \leq \gamma_{ne}(G) \), it follows that \( p - 1 \leq \gamma_{ne}(T) \leq p \). Now, if \( \gamma_{ne}(T) = p - 1 \), then consider a near equitable dominating set, \( D = \{u, v_1, v_2, ..., v_r, u_1, u_2, ..., u_s\} \) such that \( r + s + 1 = p - 1 \). We consider the following subcases.

**Subcase 1.1:** \( r = 0 \) (or \( s = 0 \)). Then \( s = p - 2 \) (or \( r = p - 2 \)), so that there exists a vertex \( u_i \) which is not dominated by any vertex of \( D \), a contradiction.

**Subcase 1.2:** \( u \notin D \). Then \( D = \{v_1, v_2, ..., v_r, u_1, u_2, ..., u_s\} \) and \( r + s = p - 1 \). Since \( p \geq 5 \), \( od_{\nu-D}(u) \geq 4 \) and for any \( v \in D \), \( od_D(v) \leq 2 \), so that \( |od_D(v) - od_{\nu-D}(u)| \geq 2 \), a contradiction.

So, \( |D| = p \). Hence, \( \gamma_{ne}(T) = p \).

**Case 2:** \( m = p - 2 \). Since \( p \geq 5 \), it follows that \( D = \{u, v_1, v_2, ..., v_{m-1}, v_{p-2}\} \) is a near equitable dominating set and hence \( \gamma_{ne}(T) \leq |D| = p - 1 \). Since \( \gamma(T) = p - 1 \), we have \( \gamma_{ne}(T) \geq p - 1 \) and hence \( \gamma_{ne}(T) = p - 1 \).

**Case 3:** \( m \leq p - 3 \). Since \( p \geq 5 \), it follows that \( D = \{u, v_1, v_2, ..., v_{m-1}, v_{p-3}\} \) is a near equitable dominating set and hence \( \gamma_{ne}(T) \leq |D| = p - 2 \). Since \( \gamma(T) = p - 2 \), we have \( \gamma_{ne}(T) \geq p - 2 \) and hence \( \gamma_{ne}(T) = p - 2 \). 

From Theorem 7.2.18, we have the following Corollary.

**Corollary 7.2.19.** Let \( T \) be a wounded spider obtained from the star \( K_{1,p-1} \), \( p \geq 5 \) by
subdividing $m$ edges exactly once. Then $\gamma_{ne}(T) = \gamma_e(T) = p$ if and only if $m = p - 1$.

**Theorem 7.2.20.** Let $G$ be a connected claw free graph and $D$ be a minimum dominating set of $G$. If $od_D(u) \leq 2$, for any $u \in D$, then $\gamma_{ne}(G) = \gamma(G)$.

**Proof.** Let $D$ be a maximal independent set of minimum cardinality. It follows from Theorem 1.5.6 that $\gamma(G) = |D|$. Now, since $od_D(u) \leq 2$, for any $u \in D$ and $G$ is claw free, it follows that every vertex $u \in D$ has at most two neighbors in $V - D$ and every vertex $v \in V - D$ has either one or two neighbors in $D$. Therefore, for every $v \in V - D$, $|od_D(u) - od_{V - D}(v)| \leq 1$. Hence, $D$ is a near equitable dominating set of $G$. Since $\gamma_{ne}(G) \leq |D| = \gamma(G)$ and $\gamma(G) \leq \gamma_{ne}(G)$, it follows that $\gamma_{ne}(G) = \gamma(G)$. \qed

**Remark 7.2.21.** Let $G \cong mK_2$, $m \geq 1$. Then $\gamma_{ne}(G) = \gamma_e(G) = \gamma(G) = m$.

**Theorem 7.2.22.** Let $D$ be a minimum dominating set of a graph $G$. If $D$ is a perfect dominating set such that for any $u \in D$, $od_D(u) \leq 2$, then $\gamma_{ne}(G) = \gamma(G)$.

**Proof.** Suppose $D$ is a perfect dominating set a graph $G$. Then every vertex of $V - D$ has one neighbor in $D$. Since for any $u \in D$, $od_D(u) \leq 2$, it follows that for every $v \in V - D$, $|od_D(u) - od_{V - D}(v)| \leq 1$. Hence, $D$ is a near equitable dominating set of $G$. Since $\gamma_{ne}(G) \leq |D| = \gamma(G)$ and $\gamma(G) \leq \gamma_{ne}(G)$, it follows that $\gamma_{ne}(G) = \gamma(G)$. \qed

**Theorem 7.2.23.** Let $T$ be a tree in which every non-pendant vertex is either a support or adjacent to a support and every non-pendant vertex which is non support
is adjacent to at most three supports and every support is adjacent to at most one non-support and one pendant vertex. Then $\gamma_{ne}(T) = \gamma(T)$.

**Proof.** Let $D$ denote the set of all supports of $T$. Clearly, $D$ is a minimum dominating set. Since by hypothesis, the out degree of any support vertex is at most two and the out degree of any non-support vertex is at least one and at most three, it follows that $D$ is a minimum near equitable dominating set. So $\gamma_{ne}(G) \leq \gamma(G)$. But, $\gamma(G) \leq \gamma_{ne}(G)$. Thus, $\gamma_{ne}(T) = \gamma(T)$. \hfill \qed

**Definition 7.2.24.** A near equitable dominating set $D$ of a graph $G$ is said to be a minimal near equitable dominating set if no proper subset of $D$ is near equitable dominating set.

**Theorem 7.2.25.** Let $D$ be a near equitable dominating set of a graph $G$. Then for any $v \in D$, $D$ is minimal near equitable dominating set of $G$ if and only if one of the following holds.

(i) $D$ is a minimal dominating set.

(ii) There exists $y \in V - D$ such that for each $x \in N(y)$ in $D$, $od_{V-D}(y) \leq od_D(x)$ and for any $z \neq x \in N(y)$ in $D$, $od_D(x) < od_D(z)$, the set $U_v$ is nonempty, where

$$U_v = \{ x \in N(y) : od_D(x) - od_{V-D}(y) = 0 \text{ and } v \in N(x) \cap N(y) \text{ or }$$

$$od_D(x) - od_{V-D}(y) = 1 \text{ and } v \in N(x) \text{ or } v \in N(y) \}.$$
Proof. Suppose that $D$ is a minimal near equitable dominating set of $G$. Then for any $v \in D$, $D - \{v\}$ is not a near equitable dominating set. Since $D$ is a near equitable dominating set, $D$ is a dominating set. If $D$ is a minimal dominating set, then we are done. If not, then for any $v \in D$, let $U_v = \{x \in N(y) : od_D(x) - od_{V-D}(y) = 0 \text{ and } v \in N(x) \cap N(y) \text{ or } od_D(x) - od_{V-D}(y) = 1 \text{ and } v \in N(x) \text{ or } v \in N(y)\}$. Since $D$ is a minimal near equitable dominating set, it follows that there exists $y \in V - (D - \{v\})$ such that for any $x \in D - \{v\}$, $|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| > 1$. If $v$ is not adjacent to both $x$ and $y$, then $|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) - od_{V-D}(y)| \leq 1$, a contradiction. If $v$ is adjacent to $x$, then using triangle inequality, we obtain,

$$1 < |od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) + 1 - od_{V-D}(y)| \leq |od_D(x) - od_{V-D}(y)| + 1.$$ 

Therefore, $|od_D(x) - od_{V-D}(y)| > 0$. Since $|od_D(x) - od_{V-D}(y)| \leq 1$ it follows that $|od_D(x) - od_{V-D}(y)| = 1$. If $od_{V-D}(y) > od_D(x)$, then $|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| \leq 1$, a contradiction. Therefore, $od_{V-D}(y) \leq od_D(x)$ and for any $z \neq x \in N(y)$ in $D$, $od_D(x) < od_D(z)$. Hence, $od_D(x) - od_{V-D}(y) = 1$. Similarly, if $v$ is adjacent to $y$, we have $od_D(x) - od_{V-D}(y) = 1$. Now, if $v$ is adjacent to both $x$ and $y$, then using triangle inequality, we obtain,

$$1 < |od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) + 1 - od_{V-D}(y) - 1| \leq |od_D(x) - od_{V-D}(y)| + 2.$$
But, $| \text{od}_D(x) - \text{od}_{V-D}(y) | \leq 1$. Hence, we have $| \text{od}_D(x) - \text{od}_{V-D}(y) | = 0$ or $1$. Since $\text{od}_{V-D}(y) \leq \text{od}_D(x)$, it follows that, $\text{od}_D(x) - \text{od}_{V-D}(y) = 0$ or $1$. Thus, $U_v$ is nonempty.

Conversely, let $D$ be a near equitable dominating set and suppose that $D$ is a minimal dominating set. Suppose to the contrary, $D$ is not a minimal near equitable dominating set. Then there exists $v \in D$ such that $D - \{v\}$ is a near equitable dominating set. So, $D$ is not minimal dominating set, a contradiction. Next, suppose that $D$ is a near equitable dominating set and $(ii)$ holds. Then for every $v \in D$, $U_v$ is not empty. So, for every $v \in D$, there exist two adjacent vertices $x \in D$ and $y \in V - D$ such that $\text{od}_D(x) - \text{od}_{(V-D)}(y) = 0$ and $v \in N(x) \cap N(y)$ or $\text{od}_D(x) - \text{od}_{(V-D)}(y) = 1$ and $v \in N(x)$ or $v \in N(y)$. Suppose to the contrary, $D$ is not a minimal near equitable dominating set. Then there exists $v \in D$ such that $D - \{v\}$ is a near equitable dominating set. So,

$$| \text{od}_{D-(v)}(x) - \text{od}_{V-(D-(v))}(y) | \leq 1$$

If $v$ is adjacent to $x$, then using triangle inequality, we obtain

$$1 \geq | \text{od}_{D-(v)}(x) - \text{od}_{V-(D-(v))}(y) | = | \text{od}_D(x) + 1 - \text{od}_{V-D}(y) |$$

$$\leq | \text{od}_D(x) - \text{od}_{V-D}(y) | + 1 = 2$$

Similarly, if $v$ is adjacent to $y$, then using triangle inequality, we obtain

$$1 \geq | \text{od}_{D-(v)}(x) - \text{od}_{V-(D-(v))}(y) | = | \text{od}_D(x) - \text{od}_{V-D}(y) - 1 |$$

$$\leq | \text{od}_D(x) - \text{od}_{V-D}(y) | + 1 = 2$$
Now, if $v$ is adjacent to both $x$ and $y$, then using triangle inequality, we obtain
\[
1 \geq |\text{od}_{D - \{v\}}(x) - \text{od}_{V - (D - \{v\})}(y)| = |\text{od}_D(x) + 1 - \text{od}_{V - D}(y) - 1|
\leq |\text{od}_D(x) - \text{od}_{V - D}(y)| + 2 = 3.
\]
Therefore, if $|\text{od}_{D - \{v\}}(x) - \text{od}_{V - (D - \{v\})}(y)| = 2$ or 3, then we have a contradiction to the fact that $D - \{v\}$ is a near equitable dominating set. If $|\text{od}_{D - \{v\}}(x) - \text{od}_{V - (D - \{v\})}(y)| = 1$, then $|\text{od}_{D - \{v\}}(x) - \text{od}_{V - (D - \{v\})}(y)| = |\text{od}_D(x) - \text{od}_{V - D}(y)|$, and $v$ is not adjacent to $x$ or $y$, a contradiction.

\[\square\]

### 7.2.2 Bounds

In this subsection, we present bounds for $\gamma_{ne}(G)$.

**Theorem 7.2.26.** Let $G$ be a connected graph of order $p$, $p \geq 3$. Then $\gamma_{ne}(G) \leq p - 2$.

**Proof.** It is enough to show that for any minimum near equitable dominating set $D$ of $G$, $|V - D| \geq 2$. Since $G$ is a connected graph, it follows that $\delta(G) \geq 1$. Suppose that $v \in V - D$ and is adjacent to $u \in D$. Since $\text{od}_{V - D}(v) \geq 1$, it follows that $\text{od}_D(u) \geq 2$.

The bound is sharp for $K_{1,n}$, $n \geq 2$.

**Theorem 7.2.27.** Let $G$ be a graph of order $p$ and $D$ be a dominating set of $G$. If $V - D$ is near equitable pendant dominating set, then $\gamma_{ne}(G) \leq \frac{p}{2}$.
Proof. Let $D$ be a dominating set of a graph $G$. Suppose that $V - D$ is a near equitable pendant dominating set. Then by Theorem 7.2.17, $D$ is a minimum near equitable dominating set. Therefore,

$$\gamma_{ne}(G) \leq |V - D|$$
$$= p - |D|$$
$$= p - \gamma_{ne}(G)$$
$$= \frac{p}{2}$$

\[\square\]

**Proposition 7.2.28.** If $T$ is a nontrivial tree with $r$ support vertices such that for any support vertex $u$, $\text{deg}(u) \leq 2$, then $\gamma_{ne}(T) \leq p - r$.

Proof. Let $D$ be the set of all support vertices. Then $|D| = r$. Since every vertex of $D$ has at most two neighbors in $V - D$, one of them being a pendant vertex, it follows that $V - D$ is near equitable dominating set of $T$. Hence,

$$\gamma_{ne}(T) \leq |V - D| = p - |D| = p - r.$$

\[\square\]
7.2.3 Near equitable domatic number of graphs.

In this subsection, we present few basic results on the near equitable domatic number of a graph.

Definition 7.2.29. A near equitable domatic partition of $G$ is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each $V_i$, $1 \leq i \leq k$ is a near equitable dominating set of $G$. The maximum order of a near equitable domatic partition of $G$ is called the near equitable domatic number of $G$ and is denoted by $d_{ne}(G)$.

We now proceed to compute $d_{ne}(G)$ for some standard graphs.

Observation 7.2.30. For any complete graph $K_p$, $p \geq 4$, $d_{ne}(K_p) = 2$.

Observation 7.2.31. For any cycle $C_p$, $p \geq 4$, $d_{ne}(C_p) = 2$.

Observation 7.2.32. For any path $P_p$, $d_{ne}(P_p) = 2$.

Observation 7.2.33. For any star $K_{1,n}$, $n \geq 3$, $d_{ne}(K_{1,n}) = 1$.

Proposition 7.2.34. For any wheel $W_{1,n}$, $n \geq 5$, $d_{ne}(W_{1,n}) = 1$

From Theorem 7.2.12, we have the following Proposition.

Proposition 7.2.35. For the complete bipartite graph $G \cong K_{n,m}$ with $m \leq n$,

$$d_{ne}(K_{n,m}) = \begin{cases} 2, & \text{if } n - m \leq 2; \\ 1, & \text{if } n - m \geq 3, n, m \geq 2. \end{cases}$$
It is obvious that any partition of $V$ into near equitable dominating sets is also a partition of $V$ into dominating set, and thus we obtain the obvious bound $d_{ne}(G) \leq d(G)$.

Furthermore, the difference $d(G) - d_{ne}(G)$ can be arbitrarily large in a graph $G$. For example, for the complete graph $K_p$, $p \geq 4$, it can be easily checked that $d_{ne}(K_p) = 2$, while $d(K_p) = p$.

**Theorem 7.2.36.** For any graph $G$, $d_{ne}(G) \leq \delta(G) + 1$.

**Proof.** Let $D$ be any near equitable dominating set. Then for any $v \in V(G)$, $D \cap N[v] \neq \emptyset$. Let $v \in V(G)$ such that $\text{deg}(v) = \delta(G)$ and $N[v] = \{v, u_1, u_2, ..., u_\delta\}$.

If $d_{ne}(G) > \delta(G) + 1$, then there exist at least $(\delta(G) + 2)$ sets in the near equitable domatic partition of $G$, each containing at least one element of $N[v]$. So we have $\text{deg}(v) \geq \delta(G) + 1$, a contradiction. Hence, $d_{ne}(G) \leq \delta(G) + 1$. \hfill \Box

**Theorem 7.2.37.** For any graph $G$ of order $p$, $d_{ne}(G) \leq \frac{p}{\gamma_{ne}(G)}$.

**Proof.** Suppose that $d_{ne}(G) = t$, for some positive integer $t$. Let $F = \{D_1, D_2, ..., D_t\}$ be a near equitable domatic partition of $G$. Obviously, $|V(G)| = \sum_{i=1}^{t} |D_i|$ and from definition of near equitable domination number $\gamma_{ne}(G)$, we have $\gamma_{ne}(G) \leq |D_i|$, $i = 1, 2, ..., t$. Hence, $p = \sum_{i=1}^{t} |D_i| \geq t\gamma_{ne}(G)$. Thus, $d_{ne}(G) \leq \frac{p}{\gamma_{ne}(G)}$. \hfill \Box

**Theorem 7.2.38.** For any connected graph $G$ of order $p$, $p \geq 2$, $d_{ne}(G) \leq \lceil \frac{p}{2} \rceil$.

**Proof.** Let $G$ be a connected graph of order $p$, $p \geq 2$. If $d_{ne}(G) = 1$. Then $d_{ne}(G) \leq \frac{p}{2}$.

If $\gamma_{ne}(G) \geq 2$, then, by Theorem 7.2.37, $d_{ne}(G) \leq \lceil \frac{p}{2} \rceil$. \hfill \Box
7.3 Connected near equitable domination in graphs

We define a connected near equitable dominating set of a graph as follows.

**Definition 7.3.1.** A near equitable dominating set $D$ of a graph $G$ is said to be a connected near equitable dominating set if the induced subgraph $\langle D \rangle$ is connected. The minimum cardinality of such a connected near equitable dominating set is called the connected near equitable domination number of $G$ and is denoted by $\gamma_{cne}(G)$.

### 7.3.1 Main Results

**Observation 7.3.2.** For any connected graph $G$, $\gamma(G) \leq \gamma_{ne}(G) \leq \gamma_{cne}(G)$.

**Observation 7.3.3.** For any connected graph $G$, $\gamma_{c}(G) \leq \gamma_{cne}(G)$.

Now, we proceed to compute $\gamma_{cne}(G)$ for some standard graphs.

**Observation 7.3.4.** For any path $P_{p}$, $p \geq 3$, $\gamma_{cne}(P_{p}) = \gamma_{c}(P_{p}) = p - 2$.

**Observation 7.3.5.** For any cycle $C_{p}$, $\gamma_{cne}(C_{p}) = \gamma_{c}(C_{p}) = p - 2$.

**Observation 7.3.6.** For any complete graph $K_{p}$, $\gamma_{cne}(K_{p}) = \gamma_{ne}(K_{p}) = \left\lfloor \frac{p}{2} \right\rfloor$.

**Observation 7.3.7.** For any star $K_{1,n}$, $\gamma_{cne}(K_{1,n}) = \gamma_{ne}(K_{1,n}) = n - 1$.

**Observation 7.3.8.** For the double star $S_{n,m}$,

$$
\gamma_{cne}(S_{n,m}) = \gamma_{ne}(S_{n,m}) = \begin{cases} 
2, & \text{if } n, m \leq 2; \\
2n + 2m - 2, & \text{if } n, m \geq 2 \text{ and } n \text{ or } m \geq 3.
\end{cases}
$$
Observation 7.3.9. For the complete bipartite graph $K_{n,m}$ with $1 < m \leq n$,

$$
\gamma_{cne}(K_{n,m}) = \gamma_{ne}(K_{n,m}) = \begin{cases} 
  m - 1, & \text{if } n = m \text{ and } m \geq 3; \\
  m, & \text{if } n - m = 1; \\
  n - 1, & \text{if } n - m \geq 2.
\end{cases}
$$

Observation 7.3.10. For the wheel $W_{1,n}$, $n \geq 5$, $\gamma_{cne}(W_{1,n}) = \gamma_{ne}(W_{1,n}) = \lceil \frac{n}{3} \rceil + 1$.

**Theorem 7.3.11.** Let $T$ be a tree in which every non-pendant vertex is either a support or adjacent to a support and every non-pendant vertex which is support is adjacent to two pendant vertices. Then $\gamma_{cne}(T) = \gamma_{ne}(T) = \gamma_{c}(T)$.

**Proof.** Let $D$ denote the set of all non-pendant vertices of $T$. Clearly, $D$ is a minimum connected dominating set. Since the out degree of any vertex of $D$ is at most two, it follows that $D$ is a minimum near equitable dominating set. Since the induced subgraph $\langle D \rangle$ is connected, $D$ is a connected near equitable dominating set. Therefore, $\gamma_{cne}(T) \leq \gamma_{ne}(T)$. But by Observation 7.3.2, $\gamma_{ne}(T) \leq \gamma_{cne}(T)$. Hence, $\gamma_{cne}(T) = \gamma_{ne}(T)$. Since $D$ is a minimum connected dominating set, it follows that $\gamma_{cne}(T) \leq \gamma_{c}(T)$. But by Observation 7.3.3, $\gamma_{c}(T) \leq \gamma_{cne}(T)$. Hence, $\gamma_{cne}(T) = \gamma_{c}(T)$. Thus, $\gamma_{cne}(T) = \gamma_{ne}(T) = \gamma_{c}(T)$. \qed

**Theorem 7.3.12.** For any connected graph $G$ of order $p$, $p \leq 4$, $\gamma_{cne}(G) = \gamma_{ne}(G)$.

**Proof.** Let $G$ be a connected graph of order $p$, $p \leq 4$. Then $G$ is one of the following graphs: $P_2$, $P_3$, $P_4$, $K_3$, $K_{1,3}$, $C_4$, $K_3 \cdot K_2$, $K_4$ or $K_4 - e$, where $e$ is an edge of $K_4$. 
But for each of these graphs, $\gamma_{cne}(G) = \gamma_{ne}(G)$.

Analogous to the definition of complete graph, we define a near equitably complete graph as follows.

**Definition 7.3.13.** A graph $G$ is called a near equitably complete graph if for any near equitable dominating set $D$ of $G$, $|\text{od}_D(u) - \text{od}_{V-D}(v)| \leq 1$, for all $u \in D$, and $v \in V - D$. Furthermore, if the induced subgraph $\langle D \rangle$ is connected, then $G$ is called a connected near equitably complete graph.

**Example 7.3.14.** The standard graphs $P_n$, $C_n$ and $K_n$ are near equitably complete graphs and connected near equitably complete graphs. But in the graph shown in Figure 7.3.1, $D = \{v_1, v_4, v_6\}$ is a near equitable dominating set and $D_1 = \{v_2, v_3, v_6\}$ is a connected near equitable dominating set. With respect to $D$ and $D_1$, this graph is neither near equitably complete nor connected near equitably complete.
Analogous to the definition of an isolated vertex of a graph, we define an isolated near equitable vertex of a near equitable dominating set $D$ as follows.

**Definition 7.3.15.** Let $D$ be a near equitable dominating set of a graph $G$. Then a vertex $u \in D$ is called an isolated near equitable vertex if $|\text{od}_D(u) - \text{od}_{V-D}(v)| \geq 2$, for every vertex $v \in V - D$. $I_{oe}(D)$ be the set of all isolated near equitable vertices of $D$.

**Example 7.3.16.** The center vertex of the wheel $W_{1,n}$, $n \geq 8$ is an isolated near equitable vertex, with respect to any near equitable dominating set of $W_{1,n}$.

**Proposition 7.3.17.** Let $D$ be a near equitable dominating set of a graph $G$. Then $I_s \subseteq I_{oe} \subseteq D$ if and only if every vertex $v \in V - D$, $\text{od}_{V-D}(v) \geq 2$.

*Proof.* Let $D$ be a near equitable dominating set of $G$. Suppose that every vertex $v \in V - D$, $\text{od}_{V-D}(v) \geq 2$. Then any isolated vertex of $G$ is an isolated near equitable vertex. Therefore, $I_s \subseteq I_{oe} \subseteq D$.

Conversely, if $I_s \subseteq I_{oe} \subseteq D$, then any isolated vertex is an isolated near equitable vertex. Therefore, for any $u \in I_s$ and $v \in V - D$, $|\text{od}_{V-D}(v) - \text{od}_{I_s}(u)| \geq 2$. Thus, $\text{od}_{V-D}(v) \geq 2$. \qed

**Proposition 7.3.18.** A near equitably complete graph $G$ contains no isolated near equitable vertex.

**Remark 7.3.19.** The converse of Proposition 7.3.18 is not true. The graph in Figure 7.3.1 is neither near equitably complete nor contains isolated near equitable vertex.
**Proposition 7.3.20.** Let $G$ be a connected graph such that the connected near equitable dominating set is a connected near equitable pendant dominating set. Then $G$ is a connected near equitably complete graph.

**Proof.** Let $D$ be a connected near equitable dominating set of a connected graph $G$. Since a connected near equitable dominating set is connected near equitable pendant dominating set, it follows that for any $u \in D$ and $v \in V - D$, $od_D(u) = 1$ and $od_{V - D}(v) \leq 2$. Hence, $|od_D(u) - od_{V - D}(v)| \leq 1$. Thus, $G$ is a connected near equitably complete graph. $\square$

**Theorem 7.3.21.** A tree $T$ is a connected near equitably complete graph.

**Proof.** Let $D$ be a connected near equitable dominating set of a tree $T$. Then for every vertex $u \in D$ and $v \in V - D$, $od_D(u) \leq 2$ and $od_{V - D}(v) = 1$. Therefore, $|od_D(u) - od_{V - D}(v)| \leq 1$. Thus, $T$ is a near connected equitably complete graph. $\square$

**Remark 7.3.22.** A tree need not be near equitably complete graph, as shown in Figure 7.3.2.

![Figure 7.3.2](image-url)
$D = \{v_2, v_3, v_6, v_8\}$ is a near equitable dominating set and $D_1 = \{v_2, v_3, v_5, v_6, v_8\}$ is a connected near equitable dominating set. The graph shown in Figure 7.3.2 is connected near equitably complete but not near equitably complete.

**Definition 7.3.23.** Let $G$ be a graph. Then the near equitable dominating set $D$ of $G$ is called a 1- near equitable dominating set if for every vertex $v \in V - D$, there exists exactly one vertex $u \in D$ such that $u$ is adjacent to $v$ and $|\text{od}_D(u) - \text{od}_{V-D}(v)| \leq 1$.

**Example 7.3.24.** A near equitable dominating set $D$ of $tK_2$, $t \geq 1$ is a 1- near equitable dominating set.

**Proposition 7.3.25.** A connected near equitable dominating set of a tree is a 1- near equitable dominating set.

### 7.3.2 Bounds

In this subsection, we present sharp bounds for $\gamma_{cne}(G)$.

**Theorem 7.3.26.** Let $G$ be a connected graph of order $p$, $p \geq 3$. Then

$$\gamma_{cne}(G) \leq p - 2.$$ 

From Observation 7.3.3, Theorem 1.5.10 and Theorem 7.3.26, we have the following theorem.
Theorem 7.3.27. For any connected graph $G$ of order $p$, $p \geq 3$ with maximum degree $\Delta$,
\[
\left\lfloor \frac{p}{\Delta+1} \right\rfloor \leq \gamma_{cne}(G) \leq p - 2.
\]
The bound is sharp for $P_3$.

Theorem 7.3.28. For any connected graph $G$ of order $p$, $p \geq 3$, $\gamma_{cne}(G) \leq 2q - p$.

Proof. By Theorem 7.3.26, $\gamma_{cne}(G) \leq p - 2 = 2(p - 1) - p \leq 2q - p$. □

Theorem 7.3.29. For any tree $T$, $\gamma_{cne}(T) \geq p - e$, where $e$ is the number of pendant vertices.

Proof. Let $D$ be a minimum connected near equitable dominating set of $T$. Then $D$ contains all non-pendant vertices of $T$ and all pendant vertices except two for each support vertex. Therefore, $\gamma_{cne}(T) \geq p - e$. □

Corollary 7.3.30. For any tree $T$, $\gamma_{cne}(T) = p - e$ if and only if any support vertex is adjacent to at most two pendant vertices.
7.4 Total near equitable domination in graphs

We define a total near equitable dominating set of a graph as follows.

**Definition 7.4.1.** A near equitable dominating set $D$ of a graph $G$ is said to be a total near equitable dominating set (tned-set) if every vertex $w \in V$ is adjacent to an element of $D$. The minimum cardinality of tned-set of $G$ is called a total near equitable domination number and is denoted by $\gamma_{tne}(G)$.

**Definition 7.4.2.** A subset $D$ of $V(G)$ is a minimal total near equitable dominating set if $D$ is a total near equitable dominating set but no proper subset of $D$ is total near equitable dominating set.

### 7.4.1 Main Results

We note that $\gamma_{tne}(G)$ is defined only for graphs without isolated vertices and, since each total near equitable dominating set is a near equitable dominating set, we have $\gamma_{ne}(G) \leq \gamma_{tne}(G)$. Since each total near equitable dominating set is a total dominating set, we have $\gamma_t(G) \leq \gamma_{tne}(G)$. The bound is sharp for $nK_2$, $n \geq 1$. In fact, $\gamma_{tne}(G) = \gamma_t(G) = |V|$, for $G = nK_2$, it is easy to see however, that $nK_2$, $n \geq 1$ is the only graph with this property. Furthermore, the difference $\gamma_{tne}(G) - \gamma_t(G)$ can be arbitrarily large in a graph $G$. It can be easily checked that $\gamma_t(K_{1,n}) = 2$, while $\gamma_{tne}(K_{1,n}) = p - 2$. 
We now proceed to compute $\gamma_{tne}(G)$ for some standard graphs.

**Observation 7.4.3.** For any path $P_p$, $p \geq 4$,

$$
\gamma_{tne}(P_p) = \gamma_t(P_p) = \begin{cases} 
\frac{p}{2} + 1, & \text{if } p \equiv 2 \pmod{4}; \\
\left\lceil \frac{p}{2} \right\rceil, & \text{otherwise.}
\end{cases}
$$

**Observation 7.4.4.** For any cycle $C_p$, $p \geq 4$,

$$
\gamma_{tne}(C_p) = \gamma_t(C_p) = \begin{cases} 
\frac{p}{2} + 1, & \text{if } p \equiv 2 \pmod{4}; \\
\left\lceil \frac{p}{2} \right\rceil, & \text{otherwise.}
\end{cases}
$$

**Proposition 7.4.5.** For the complete graph $K_p$, $p \geq 4$, $\gamma_{tne}(K_p) = \gamma_{ne}(K_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

**Proposition 7.4.6.** For the double star $S_{n,m}$,

$$
\gamma_{tne}(S_{n,m}) = \gamma_{ne}(S_{n,m}) = \begin{cases} 
2, & \text{if } n, m \leq 2; \\
2n - 4, & \text{if } n, m \geq 2 \text{ and } n \text{ or } m \geq 3.
\end{cases}
$$

**Proposition 7.4.7.** For the complete bipartite graph $K_{n,m}$ with $2 < m \leq n$,

$$
\gamma_{tne}(K_{n,m}) = \gamma_{ne}(K_{n,m}) = \begin{cases} 
m - 1, & \text{if } n = m \text{ and } m \geq 3; \\
m, & \text{if } n - m = 1; \\
n - 1, & \text{if } n - m \geq 2.
\end{cases}
$$

**Proposition 7.4.8.** For the wheel $W_{1,n}$, $n \geq 5$,

$$
\gamma_{tne}(W_{1,n}) = \gamma_{ne}(W_{1,n}) = \left\lceil \frac{n}{3} \right\rceil + 1
$$
Theorem 7.4.9. Let $T$ be a tree of order $p$, $p \geq 4$ in which every non-pendant vertex is either a support or adjacent to a support and every non-pendant vertex which is support is adjacent to at least two pendant vertices. Then $\gamma_{tne}(T) = \gamma_{ne}(T) = \gamma_{cne}(T)$.

Proof. Let $T$ be a tree of order $p$, $p \geq 4$. Suppose that $D$ is a set of all non-pendant vertices and all pendant vertices except two for each support of $T$. Clearly, $D$ is a minimum near equitable dominating set. Since any support vertex is adjacent to at least two pendant vertices, it follows that $\langle D \rangle$ is connected. Therefore, $D$ is minimum connected near equitable dominating set and hence it is a minimum tned-set. So, $\gamma_{cne}(T) \leq \gamma_{ne}(T)$ and $\gamma_{tne}(T) \leq \gamma_{ne}(T)$. Since $\gamma_{ne}(T) \leq \gamma_{tne}(T)$ and by Observation 7.3.2, $\gamma_{ne}(T) \leq \gamma_{cne}(T)$, it follows that $\gamma_{tne}(T) = \gamma_{ne}(T) = \gamma_{cne}(T)$. \qed

We now proceed to obtain a characterization of a minimal total near equitable dominating set.

Theorem 7.4.10. A total near equitable dominating set $D$ of a graph $G$ is a minimal total near equitable dominating set if and only if one of the following holds:

(i) $D$ is a minimal near equitable dominating set.

(ii) There exist $x, y \in D$ such that $N(y) \cap N(D - \{x\}) = \phi$.

Proof. Suppose that $D$ is a minimal tned-set of $G$. Then for any $u \in D$, $D - \{u\}$ is not tned-set. If $D$ is a minimal near equitable dominating set, then we are done. If not, then there exists a vertex $x \in D$ such that $D - \{x\}$ is a near equitable dominating
set, but not a tned-set. Therefore, there exists a vertex \( y \in D - \{x\} \) such that \( y \) is an isolated vertex in \( \langle D - \{x\} \rangle \). Hence, \( N(y) \cap N(D - \{x\}) = \emptyset \).

Conversely, let \( D \) be a tned-set and (i) holds. Suppose \( D \) is not a minimal tned-set. Then there exists \( u \in D \) such that \( D - \{u\} \) is a tned-set. So, \( D \) is not a minimal near equitable dominating set, a contradiction. Next, suppose that \( D \) is a tned-set and (ii) holds. Then there exist \( x, y \in D \) such that \( N(y) \cap N(D - \{x\}) = \emptyset \). Suppose to the contrary, \( D \) is not a minimal tned-set. Then there exists \( u \in D \) such that \( D - \{u\} \) is a tned-set. So, \( \langle D - \{u\} \rangle \) does not contain any isolated vertex. Therefore, for every \( x, y \in D \), \( N(y) \cap N(D - \{x\}) \neq \emptyset \), a contradiction. \( \blacksquare \)

### 7.4.2 Bounds

Analogous to Theorem 7.2.26, we have the following theorem.

**Theorem 7.4.11.** Let \( G \) be a connected graph of order \( p \), \( p \geq 4 \). Then

\[
\gamma_{tne}(G) \leq p - 2.
\]

The star graph \( G \cong K_{1,n} \) is an example of a connected graph for which \( \gamma_{tne}(G) = 2n - (\Delta(G) + 3) \). The following theorem shows that, this is the best possible upper bound for \( \gamma_{tne}(G) \).
Theorem 7.4.12. If $G$ is connected of order $p$, $p \geq 4$, then

$$\gamma_{tne}(G) \leq 2p - (\Delta(G) + 3).$$

Proof. Let $G$ be a connected graph of order $p$, $p \geq 4$. Then by Theorem 7.4.11,

$$\gamma_{tne}(G) \leq p - 2 = 2p - (p - 1 + 3) \leq 2p - (\Delta(G) + 3).$$

\[\square\]

Theorem 7.4.13. If $G$ is a graph of order $p$, $p \geq 4$ and $\Delta(G) \leq p - 2$ such that both $G$ and $\overline{G}$ are connected, then $\gamma_{tne}(G) + \gamma_{tne}(\overline{G}) \leq 3p - 6$.

Proof. Let $G$ be a connected graph. Since $\Delta(G) \leq p - 2$, by Theorem 7.4.11,

$$\gamma_{tne}(G) \leq 2p - (\Delta(G) + 4) \leq 2p - (\delta(G) + 4).$$

Since $\overline{G}$ is connected, by Theorem 7.4.12,

$$\gamma_{tne}(\overline{G}) \leq 2p - (\Delta(\overline{G}) + 3),$$

it follows that

$$\gamma_{tne}(G) + \gamma_{tne}(\overline{G}) \leq 2p - (\delta(G) + 4) + 2p - (\Delta(\overline{G}) + 3)$$

$$= 4p - (\delta(G) + \Delta(\overline{G})) - 7$$

$$= 3p - 6.$$

\[\square\]

The bound is sharp for $C_4$. 
7.5 Strong total near equitable domination in graphs

We define a strong total near equitable dominating set of a graph as follows.

Definition 7.5.1. A near equitable dominating set $D$ of a graph $G$ is said to be a strong total near equitable dominating set (stned-set) if for every vertex $v \in D$ there exists $u \in D$ such that $u$ is adjacent to $v$ and $|\text{od}_D(u) - \text{od}_D(v)| \leq 1$. The minimum cardinality of stned-set of $G$ is called the strong total near equitable domination number of $G$ and is denoted by $\gamma_{stne}(G)$.

Definition 7.5.2. A subset $D$ of $V(G)$ is a minimal strong total near equitable dominating set if $D$ is strong total near equitable dominating set but no proper subset of $D$ is strong total near equitable dominating set.

7.5.1 Main Results

We note that $\gamma_{stne}(G)$ is defined only for graphs without isolated vertices and, since each total near equitable dominating set is a near equitable dominating set and each strong total near equitable dominating set is a total near equitable dominating set, we have $\gamma_{tne}(G) \leq \gamma_{tne}(G) \leq \gamma_{stne}(G)$.

Proposition 7.5.3. Let $D$ be a strong total near equitable dominating set of $G$. Then for every component $C$ of $G$, $D \cap V(C)$ is a strong total near equitable dominating set of $C$. 
Proof. Let $D$ be a strong total near equitable dominating set of $G$ and $C$ be a component of $G$. Then for every vertex $v \in C$, there exists $u \in D$ such that $u$ and $v$ are adjacent and $|od_D(u) - od_C(v)| \leq 1$. Thus, $D \cap V(C)$ is a strong total near equitable dominating set of $C$.

\[\text{Theorem 7.5.4.} \quad \text{Let } T \text{ be a tree of order } p. \text{ Then } \gamma_{stne}(T) = p - 1 \text{ if and only if } T \text{ is a star.} \]

Proof. Let $T$ be a tree of order $p$. Since $T$ is a star, $\gamma_{stne}(T) = p - 1$.

Conversely, let $T$ be a tree such that $\gamma_{stne}(T) = p - 1$. Suppose to the contrary, $T$ is not star. Then $T$ contains more than one support vertex, so that $\gamma_{stne}(T) \leq p - 2$, a contradiction. Thus, $T$ is a star.

\[\text{Theorem 7.5.5.} \quad \text{For any cycle } C_p, \gamma_{stne}(C_p) = p - 2 \text{ if and only if } p = 4, 5, 6. \]

Proof. Clearly, if $G \cong C_p$, $p = 4, 5, 6$, then $\gamma_{stne}(C_p) = p - 2$.

Conversely, suppose that $\gamma_{stne}(C_p) = p - 2$. Since $G \cong C_p$. Assume that $p \neq 4, 5, 6$.

If $p = 3$, then $\gamma_{stne}(C_3) = 2 \neq 1$. If $p = 7$, then $\gamma_{stne}(C_7) = 4 < 5$. Similarly, for $p \geq 8$, $\gamma_{stne}(C_p) < p - 2$.

\[\text{Definition 7.5.6.} \quad \text{A graph } G \text{ is a near equitably balanced graph if for any near equitable dominating set } D \text{ of } G, \od_D(u) = \od_D(v), \text{ for all } u, v \in D. \]
Example 7.5.7. A cycle $C_4$ is a near equitably balanced graph. But a path $P_4$ is not near equitably balanced graph.

Remark 7.5.8. Let $G$ be a graph such that any near equitable dominating set of $G$ is a near equitable pendant dominating set. Then $G$ is a near equitably balanced graph.

Theorem 7.5.9. Let $G$ be a near equitably balanced graph. Then $D$ is a strong total near equitable dominating set of $G$ if and only if $D$ is a total near equitable dominating set.

Proof. Let $G$ be a near equitably balanced graph. Then for any near equitable dominating set $D$ of $G$, $od_D(u) = od_D(v)$, for all $u, v \in D$. Suppose that $D$ is a total near equitable dominating set of $G$. Then for any $u \in D$, there exists $v \in D$ such that $u$ is adjacent to $v$ and $|od_D(u) - od_D(v)| \leq 1$. Therefore, $D$ is a strong near equitable dominating set.

Conversely, If $D$ is a strong near equitable dominating set, then $D$ is a total near equitable dominating set. 

Theorem 7.5.10. Let $G$ be a near equitably balanced graph and $D$ be a near equitable dominating set of $G$. Then for any $w, w' \in V - D$, $|od_{V-D}(w) - od_{V-D}(w')| \leq 2$.

Proof. Let $D$ be a near equitable dominating set of a near equitably balanced graph $G$. Suppose that $w, w'$ are any two vertices of $V - D$ such that $od_{V-D}(w) \leq od_{V-D}(w')$. Since $D$ is a near equitable dominating set of $G$, it follows that for any $u \in D$, 


od_{V-D}(w) \leq od_D(u) \leq od_{V-D}(w') such that |od_D(u) - od_{V-D}(w)| \leq 1 and
|od_D(u) - od_{V-D}(w')| \leq 1. Therefore, |od_{V-D}(w) - od_{V-D}(w')| \leq 2. \qed

Analogous to the definition of regular graph, we define a near equitably regular graph as follows.

**Definition 7.5.11.** Let $G$ be a near equitably balanced graph and $D$ be a near equitable dominating set of $G$. Then $G$ is a near equitably regular graph if for any $u \in D$ and $v \in V - D$, $od_D(u) = od_{V-D}(v)$.

**Example 7.5.12.** A cycle $C_4$ is a near equitably regular graph.

**Theorem 7.5.13.** Let $G$ be a near equitably regular graph and $D$ be a near equitable dominating set of $G$ such that the induced subgraph $\langle V - D \rangle$ is connected. Then $V - D$ is strong total near equitable dominating set.

**Proof.** Let $G$ be a near equitably regular graph. Then for any near equitable dominating set $D$, $od_D(u) = od_D(v) = od_{V-D}(w) = od_{V-D}(w') \geq 1$ for all $u, v \in D$ and for all $w, w' \in V - D$. Therefore, for any $u \in D$, $od_D(u) \geq 1$. Since the induced subgraph $\langle V - D \rangle$ is connected, it follows that $V - D$ is a strong total near equitable dominating set. \qed

**Definition 7.5.14.** A near equitably regular graph with vertices having out degree $k$ is called a $k$-near equitably regular graph or near equitably regular graph of out degree $k$. 
Definition 7.5.15. A $k$-regular bipartite graph is a bipartite graph $G = (V_1, V_2, E)$ in which all vertices have the same degree $k$.

Theorem 7.5.16. A $k$-regular graph is a $k$-near equitably regular graph if and only if it is a $k$-regular bipartite graph or a totally disconnected graph.

Proof. Let $G$ be a $k$-regular graph. Then $\deg_G(u) = k$, for all $u \in V(G)$. Suppose that $G$ is a $k$-near equitably regular graph. Then $\od_D(u) = \od_D(v) = \od_{V-D}(w) = k$, for all $u, v \in D$ and $w \in V - D$. Therefore, both subgraphs $(D)$ and $(V - D)$ induced by $D$ and $V - D$, respectively are totally disconnected. Thus, $G$ is totally disconnected for $k = 0$ and $k$-regular bipartite graph for $k \geq 1$.

Conversely, if $G$ is a $k$-regular bipartite graph or a totally disconnected graph, then $G$ is $k$-near equitably regular graph. \hfill \qed

Theorem 7.5.17. Let $G$ be a near equitably regular graph and $D$ be a total near equitable dominating set of $G$. Then $D$ is a strong total near equitable set.

Proof. Suppose that $D$ is a total near equitable dominating set of a near equitably regular graph $G$. Then for any $v \in V(G)$, there exists $u \in D$ such that $v$ is adjacent to $u$ and $\od_D(u) = \od_D(v)$ or $\od_D(u) = \od_{V-D}(v)$. Therefore, $D$ is a strong total near equitable set. \hfill \qed

Definition 7.5.18. Let $D$ be a near equitable dominating set of a graph $G$. Then $G$ is said to be a near equitably bi-regular graph if for any $u, v \in D$ and $w \in V - D$, $\od_D(u) = \od_D(v) = \od_{V-D}(w) \pm 1$. 

From definitions of a near equitably regular graph and near equitably bi-regular graph, we have the following propositions.

**Proposition 7.5.19.** Any complete graph $K_p$ is a near equitably bi-regular graph.

**Proposition 7.5.20.** Any near equitably bi-regular graph is a near equitably balanced graph.

**Proposition 7.5.21.** Let $D$ be a near equitable pendant dominating set of a graph $G$. Then for any $u \in D$ and $v \in V - D$, $od_D(u) \leq od_{V - D}(v) \leq 2$.

**Theorem 7.5.22.** Let $D$ be a near equitable pendant dominating set of a graph $G$.

Then

(i) $G$ is a near equitably regular graph if and only if $od_{V - D}(v) = 1$.

(ii) $G$ is a near equitably bi-regular graph if and only if $od_{V - D}(v) = 2$.

**Theorem 7.5.23.** Let $G(p, q)$ be a graph and $D$ be a near equitable dominating set of $G$ such that the subgraphs $\langle D \rangle$ and $\langle V - D \rangle$ induced by $D$ and $V - D$, respectively form bipartite graphs. Then for any $u \in D$, $\sum_{u \in D} od_D(u) = q$.

**Proof.** Suppose that $D$ is a near equitable dominating set of $G$ such that the subgraphs $\langle D \rangle$ and $\langle V - D \rangle$ induced by $D$ and $V - D$, respectively form bipartite graphs, then $od_D(u) = deg(u)$ and $od_{V - D}(v) = deg(v)$, for all $u \in D$ and $v \in V - D$. Since $\sum_{w \in V} deg(w) = 2q$, we have $\sum_{u \in D} od_D(u) = \frac{1}{2} \sum_{w \in V} deg(w) = q$. $\square$
We now proceed to obtain a characterization of minimal stned-sets.

**Theorem 7.5.24.** Let \( D \) be a dominating set of a graph \( G \). If \( D \) is a stned- set, then \( D \) is a minimal stned- set if and only if one of the following holds:

1. \( D \) is a minimal near equitable dominating set.

2. For any two adjacent vertices \( x, y \in D \), \( \text{od}_D(x) > \text{od}_D(y) \) and for any vertex \( v \in D \) different from \( x \) and \( y \), the set \( U_v \) is nonempty, where

\[
U_v = \{ x, y \in D : \text{od}_D(x) - \text{od}_D(y) = 1 \text{ and } v \text{ is adjacent to } x \text{ but not adjacent to } y \}.
\]

**Proof.** Suppose that \( D \) is a minimal strong total near equitable dominating set of \( G \). Then for any \( v \in D \), \( D - \{v\} \) is not strong total near equitable dominating set. If \( D \) is a minimal near equitable dominating set, then we are done. If not, then for any \( v \in D \), let \( U_v = \{ x, y \in D, \text{od}_D(x) - \text{od}_D(y) = 1 \text{ and } v \text{ is adjacent to } x \text{ but not adjacent to } y \} \).

There exist \( x, y \in D - \{v\} \) such that \( |\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| > 1 \). If both \( x, y \) are adjacent to \( v \), then \( |\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = |\text{od}_D(x) - \text{od}_D(y)| \leq 1 \), a contradiction. If both \( x, y \) are not adjacent to \( v \), then \( |\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = |\text{od}_D(x) - \text{od}_D(y)| \leq 1 \), a contradiction. So, \( v \) is adjacent to precisely one vertex of \( \{x, y\} \). Without loss of generality, assume that \( v \) is adjacent to \( x \) but not adjacent to \( y \).

Then,

\[
1 < |\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = |\text{od}_D(x) + 1 - \text{od}_D(y)| \leq |\text{od}_D(x) - \text{od}_D(y)| + 1
\]
So, $|od_D(x) - od_D(y)| > 0$. But $|od_D(x) - od_D(y)| \leq 1$. Hence, $|od_D(x) - od_D(y)| = 1$. Therefore, $od_D(x) - od_D(y) = 1$. Thus, $U_v$ is nonempty.

Conversely, let $D$ be a strong total near equitable dominating set and suppose that $D$ is a minimal near equitable dominating set. Suppose to the contrary, $D$ is not a minimal strong total near equitable dominating set. Then there exists $v \in D$ such that $D - \{v\}$ is a strong total near equitable dominating set. So, $D$ is not a minimal near dominating set, a contradiction. Next, suppose that $D$ is a strong total near equitable dominating set and (2) holds. Then for every $v \in D$, $U_v$ is nonempty. So, for every $v \in D$, there exist $x, y \in D$ such that $v$ is adjacent to precisely one vertex of $\{x, y\}$, and $od_D(x) - od_D(y) = 1$. Suppose to the contrary, $D$ is not a minimal strong total near equitable dominating set. Then there exists $v \in D$ such that $D - \{v\}$ is a strong total near equitable dominating set. So, $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| \leq 1$. Now, if $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = 1$, then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)|$, either $\{x, y\} \subseteq N(v)$, or $\{x, y\} \cap N(v) = \phi$, a contradiction.

\section*{7.5.2 Bounds}

\textbf{Theorem 7.5.25.} For a connected graph $G$ of order $p$, $p \geq 4$, $\gamma_{stne}(G) \leq p - 1$. Furthermore, equality holds for a star graph.

\textit{Proof.} It is enough to show that for any minimum strong total near equitable dominating set $D$ of $G$, $|V - D| \geq 1$. Since $G$ is a connected graph of order $p$,
$p \geq 4$, it follows that $\delta(G) \geq 1$. Suppose $|V - D| = 0$, it follows that $|D| = p$. Therefore, $G$ is totally disconnected, a contradiction. \hfill \Box \\

**Theorem 7.5.26.** Let $G$ be a near equitably regular graph and $D$ be a near equitable dominating set of $G$ such that the induced subgraph $\langle V - D \rangle$ is connected. Then $\gamma_{stne}(G) \leq p - \gamma(G)$. Furthermore, equality holds for $C_4$.

**Proof.** Let $G$ be a near equitably regular graph. By Theorem 7.5.13, $V - D$ is a strong total near equitable dominating set. Therefore, $\gamma_{stne}(G) \leq |V - D| \leq p - \gamma(G)$. \hfill \Box