CHAPTER 2

Harmonic Oscillator potential and Deformed Magic Numbers

The alpha cluster model is an approximation to the shell model as for as the alpha clustering is concerned. In the alpha cluster model a group of two protons and two neutrons are treated as an entity called alpha cluster. Similarly in the shell model the nucleons are treated individually. It has been demonstrated that the occurrence of clustering phenomena can be related to the appearance of shell structure in highly deformed nuclei. The shell structure in the single particle energy spectrum plays a crucially important role in the stability of the alpha cluster configurations. The magic numbers at certain super-deformations appear to correspond to definite cluster configurations. This can be confirmed by comparing with the Nilsson Strutinsky calculations /1/. Zhang and Rae /2/ generalize another way to explore the connection between cluster model and shell model. They are taking the shell model limit for the cluster model as Brink showed /3/.

2-1. Shell Model

The basic model for the description of nuclear properties is the shell model. The strong nuclear force binds together protons
and neutrons, called by the single name nucleons. Each individual nucleon moves in the average potential generated by all the others. The states of a single nucleon in the average potential cluster together into layers or shells, much like the single particle states in atoms. This type of model of non-interacting particles in a mean potential is often called the independent particle model. The most important piece of experimental information on shell structure is the existence of magic numbers. Nuclei having magic numbers of both neutrons and protons have been found to have spherical equilibrium shapes and special stability. Since the total binding energy of the nuclei having magic numbers are large, a larger energy is required to separate a single nucleon. A few special characters of this type of nuclei are

(i) a higher energy of the lowest excited states, and
(ii) a large number of isotopes or isotones with the same magic number of protons (neutrons).

The lower magic numbers are the same for protons and neutrons, namely 2, 8, 20, 28, 50 and 82, whereas the next number 126 is established only experimentally for neutrons. Theoretically one would expect additional magic number 114 for protons and 184 for neutrons leading to super-heavy nuclei /4-7/ but these have not been confirmed in experiment.
A phenomenological shell model thus is based on the Schrödinger equation for the single-particle levels

\[
\left\{ \frac{\hbar^2}{2m} \nabla^2 + V(r) \right\} \Psi_i(r) = \varepsilon_i \Psi_i(r)
\]

...(2.01)

with a prescribed potential \( V(r) \). Inside heavier nuclei, this potential should be relatively constant. This is to explain the constant density suggested by the fact that the nuclear radius behaves as \( R = r_0 A^{1/3} \) but goes to zero quite rapidly outside the nuclear surface.

Assuming the potential \( V(r) \) is spherically symmetrical there are three types of potentials are followed. They are,

(i) the Woods-Saxon potential

(ii) the harmonic-oscillator potential

(iii) the square-well potential

The Woods-Saxon potential is used usually in the cases where the asymptotic form of the wave functions is important. This potential is expressed as

\[
V(r) = -\frac{V_0}{1 + \exp\left(\frac{r - R}{a}\right)}
\]

...(2.02)

where the parameters, depth \( V_0 \approx 50\text{MeV} \)

radius \( R \approx 1.1 \text{ fm} A^{1/3} \) and

the surface thickness \( a \approx 0.5\text{fm} \)
are the typical values. The potential has the practical disadvantage of not leading to analytic forms for the wave functions. For the Woods-Saxon potential, the radial wave functions can be obtained only numerically. The energies lie between the harmonic oscillator and the square well results and also fail in comparison with experiment.

In the square well potential, the potential $V(r) = -V_0$ for $r \leq R$ and it becomes infinite when $r > R$. For this, the wave function

$$\Psi(r) \propto j \ell \,(kr)Y_{\ell m}(\Omega), \quad r \leq R \quad (2.03)$$

which can be given analytically and which vanish for $r \geq R$. The energies follow from the matching condition $j \ell(kR) = 0$, which can be solved only numerically. The energy eigenvalues are obtained by the boundary condition at $r = R$. By this approach one can get the shell closures at total particle numbers 2, 8, 18, 20, 34, 40, 58,... Since these are not the nuclear magic numbers, this deficiency was approached by considering a particle moving with simple harmonic motion.

Typically the harmonic oscillator potential is

$$V(r) = \frac{1}{2} m\omega^2 r^2$$

with $\hbar\omega \approx 41$(Mev) $\times A^{-1/3}$.
As it goes to infinity instead of zero at large distances, it clearly does not produce the correct large-distance behaviour of the wave functions. The $A^{-1/3}$ dependence of the oscillator constant implies that on the nuclear surface, i.e., for $R = r_0 A^{1/3}$, there is the same potential regardless of the value of $A$, thus simulating the constant depth of the potential well.

Since all these potentials are spherically symmetric, the wave functions contain a factor $Y_{lm}(\Omega)$. For the harmonic oscillator in spherical coordinates the complete wave functions should be familiar from elementary quantum mechanics; they are given by

$$
\Psi_{n\ell m}(r, \Omega) = \frac{2^{n+\ell+2}}{n!(2n+2\ell+1)!} \sqrt{\pi} x_0^3 \frac{r^\ell}{x_0^\ell} \frac{L_n^{\ell+\frac{1}{2}}}{x_0^2} \frac{r}{x_0^2} \frac{e^{-\frac{r^2}{2x_0^2}}}{\sqrt{\pi} x_0^3} Y_{lm}(\Omega) \quad \ldots (2.04)
$$

with $x_0 = \sqrt{\frac{\hbar}{m\omega}}$

The symbol $L_n^{\ell m}(x)$ stands for the generalized Laguerre polynomial. The eigen energies are determined by the principal quantum number $N=2(n-1)+\ell$ as $E_N = \hbar \omega (N+\frac{3}{2})$ and are $\frac{1}{2}(N+1)(N+2)$ - fold degenerate.

A viable tool in Nuclear Physics was the inclusion of a strong spin-orbit force by Mayer and Jensen /8/ in the phenomenological single particle model. It couples the spin and orbital angular momentum of each individual nucleon. The additional term in the
single particle potential is $C \hat{\ell} \cdot \hat{s}$. In the spherical case this term is diagonal and its value can be computed from

$$\hat{\ell} \cdot \hat{s} = \frac{1}{2} \left\{ \left( \hat{\ell} + \hat{s} \right)^2 - \hat{\ell}^2 - \hat{s}^2 \right\}$$

$$= \frac{1}{2} \hbar^2 \left\{ j(j+1) - \ell(\ell+1) - s(s+1) \right\}$$

... (2.05)

The splitting of the two levels with $j = \ell \pm \frac{1}{2}$ is interested one. It is given by,

$$E_{j=\ell+\frac{1}{2}} - E_{j=\ell-\frac{1}{2}} = C \hbar^2 (\ell + \frac{1}{2})$$

... (2.06)

Because of the experimental proof of the state with $j = \ell + \frac{1}{2}$ is lower in energy, the spin-orbit coupling term must have a negative sign. There is still the possibility to have an $r$-dependent coefficient $C$. For the purely phenomenological approach the experimental data do not provide enough information to fix such an $r$ dependence, so that usually a constant is assumed. Typical values for $|C|$ are in the range of 0.3 to 0.6 MeV/\(\hbar^2\). Generally in the spectrum for the harmonic oscillator with spin-orbit coupling, the levels are labelled by the radial quantum number $n$, orbital angular momentum $\ell$, and total angular momentum $j$ as $ntj$. Each of these states is degenerate $2j+1$-fold with projections $\Omega = -j, \ldots, +j$. Clearly the magic numbers are now described correctly. The spherical phenomenological single particle model succeeds in the
explanation of the magic shell closures and of the properties of nuclei nearby. But it fails to explain these for the nuclei between major shell closures. Considering the rotational states, it appears that the model must be modified to account for nuclear deformations.

2-2. Deformation of the nuclear potential

The extension of the shell model to deformed nuclear shapes was first given by S.G.Nilsson/9/ and also in /10,11/, so this version of deformed shell model is often referred as the Nilsson model. The principal idea is to make the oscillator constants different in the different spatial directions;

\[ V(r) = \frac{1}{2} m \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) \quad ... (2-07) \]

For the spherical-oscillator shell model, the nuclear surface has a constant potential value independent of A. In the deformed case the density distribution should follow the potential in the same way, by defining a geometric nuclear surface as consisting of all the points \((x,y,z)\) with

\[ \frac{1}{2} m \omega_o^2 R^2 = \frac{1}{2} m \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) \quad ... (2-08) \]

where \( \hbar \omega_o = 41 \text{MeV} \times A^{-1/3} \) is the oscillator constant for the equivalent spherical nucleus. This describes an ellipsoid with axes \(X, Y\) and \(Z\) given by
The condition of incompressibility of nuclear matter requires that the volume of both ellipsoid and that of the sphere should be same, implying $R^3=XYZ$, and this imposes a condition on the oscillator frequencies:

$$\omega_0^3 = \omega_x \omega_y \omega_z \quad \text{(2.10)}$$

Now assuming axial symmetry around the z-axis, i.e., $\omega_x = \omega_y$ and a small deviation from the spherical shape given by a small parameter $\delta$. Thus one can define,

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(1 + \frac{2}{3} \delta\right) \quad \text{(2.11)}$$

$$\omega_z^2 = \omega_0^2 \left(1 - \frac{4}{3} \delta\right) \quad \text{(2.12)}$$

which clearly fulfills the volume conservation condition of eqn. (2.10) to first order with $\omega_0=\omega_0$. Also for the second order, the volume conservation can be fulfilled using

$$\omega_0^6 = \left(1 - \frac{4}{3} \delta\right) \left(1 + \frac{2}{3} \delta\right) \omega_0^6 \quad \text{(2.13)}$$

leading in second order to

$$\omega_0 \approx \left(1 + \frac{2}{9} \delta^2\right) \omega_0 \quad \text{(2.14)}$$
Utilizing the explicit expression for the spherical harmonic \( Y_{20} \), the potential can be written as

\[
V(r) = \frac{1}{4} m \omega_r^2 r^2 - \beta_0 m \omega_0^2 r^2 Y_{20}(\theta, \phi)
\]  

...(2.15)

where the deformation parameter \( \delta \) of Nilsson is related to the parameter \( \beta_0 \) of Bohr and Mottelson as

\[
\beta_0 = 4 \frac{4\pi}{3} \frac{5}{\delta}
\]  

...(2.16)

The deformed shapes obtained in this way are different from those in the collective model because here the potential was expanded in spherical harmonics. At large deformations, the collective model shapes develop a neck and even separate. At \( \theta = \pi/2 \), the largest negative value obtained

\[
Y_{20}(\pi/2, \phi) = -\frac{1}{2} \frac{5}{\sqrt{4\pi}}
\]  

...(2.17)

\[
R(\pi/2, \phi) = R_0 \left[ 1 + \alpha_{20}(\pi/2, \phi) \right]
\]  

...(2.18)

which becomes zero for \( \alpha_{20} = 2 \frac{4\pi}{5} \) thereby producing a fissioned shape. In contrast, the shapes in the deformed shell model always remain ellipsoidal with arbitrarily long stretching.

Now the Hamiltonian of the deformed shell model can be written as

\[
\hat{H} = -\frac{\hbar}{2m} \nabla^2 + \frac{m \omega_r^2}{2} r^2 - \beta_0 m \omega_0^2 r^2 Y_{20}(\theta, \phi) - \hbar \omega_r \kappa(2 \hat{L} \cdot \hat{S} + \mu \hat{L}^2)
\]  

...(2.19)
To lower the energy of the single-particle states closer to the nuclear surface, ie, in order to correct for the steep rise in the harmonic oscillator potential, the spin-orbit term \((\hat{\ell} \cdot \hat{s})\) parameterized by a constant \(\kappa\), and \(\ell^2\) term parameterized by \(\mu\) are introduced. Both \(\kappa\) and \(\mu\) may be different for protons and neutrons and also depend on the nucleon number.

Depending upon the application, the Hamiltonian may be diagonalized in the basis of the harmonic oscillator using either spherical or cylindrical coordinates. In spherical coordinates the spin-orbit and \(\ell^2\) terms are diagonal, but the \(Y_{20}\) term couples orbital angular momenta differing by \(\pm 2\). But in cylindrical coordinates, the deformed oscillator potential is diagonal and the angular momentum terms must be diagonalized numerically.

Considering the spherical oscillator without spin effects, the energy levels can be written as

\[
\varepsilon = \hbar \omega_o (N + \frac{\ell}{2}) \tag{2.20}
\]

with \(N = 2 (n_r - 1) + \ell\)

where \(N\) is the principal quantum number,
\(n_r\) is the radial quantum number,
\(\ell\) is the angular momentum quantum number and
m is the projection. The energy levels are given by

\[ \varepsilon = \hbar \omega_z (n_z + \frac{1}{2}) + \hbar \omega_p (2n_p + |m| + 1) \] ...(2-21)

In the cylindrical basis, where \( n_z \) is the number of quanta in the z direction, \( n \) is that of radial excitations and \( m \) is the angular momentum projection on z-axis. For the spherical shape the levels will be grouped according to the principal quantum number \( N \), but the behaviour with deformation depends on how much of the excitation is in the z direction. For very large deformations, the influence of the spin-orbit and \( \ell^2 \) terms become less important and one may classify the levels according to the cylindrical quantum number.

One important point to note here is how the deformation parameter \( \beta_0 \) is related to that of the collective model corresponding to \( \alpha_{20} \). For small deformations the two models, collective and single-particle agree sufficiently, even though they are quite different. Instead of identifying the deformation parameters leading to the same quadrupole moment the simpler way is comparing the axis ratios in the two models.

In the collective model, the axis lengths \( R_x = R_y \) and \( R_z \) are given by
\[ R_x = R_y = R_0 \{1 + \alpha_{20} Y_{20} (\theta = \frac{\pi}{2})\} \]
\[ = R_0 \left\{ 1 - \frac{1}{2} \alpha_{20} \sqrt{\frac{5}{4\pi}} \right\} \]
\[ R_z = R_0 \{1 + \alpha_{20} Y_{20} (\theta = 0)\} \]
\[ = R_0 \left\{ 1 + \alpha_{20} \sqrt{\frac{5}{4\pi}} \right\} \]  \hspace{1cm} \text{(2.22)}

To first order, the ratio becomes,
\[ \frac{R_z}{R_x} = \frac{1 + \alpha_{20} \sqrt{\frac{5}{4\pi}}} {1 - \frac{1}{2} \alpha_{20} \sqrt{\frac{5}{4\pi}}} \approx 1 + \frac{3}{2} \alpha_{20} \sqrt{\frac{5}{4\pi}} \]  \hspace{1cm} \text{(2.24)}

but in the Nilsson model, the axis ratio is given by the relation,
\[ \frac{1}{2} \omega_0^2 \left[ 1 - 2 \beta_0 Y_{20} (\theta = 0) \right] R_x^2 = \frac{1}{2} \omega_0^2 \left[ 1 - 2 \beta_0 Y_{20} (\theta = \frac{\pi}{2}) \right] R_x^2 \]  \hspace{1cm} \text{(2.25)}

and the ratio can be written as
\[ \frac{R_z^2}{R_x^2} = \frac{1 + \frac{1}{2} \beta_0 \sqrt{\frac{5}{\pi}}} {1 - \beta_0 \sqrt{\frac{5}{\pi}}} \approx 1 + \frac{3}{2} \alpha_{20} \sqrt{\frac{5}{\pi}} \]  \hspace{1cm} \text{(2.26)}

Taking the square root adds a factor of \(\frac{1}{2}\) to the first order term, so that the two expressions agree to first order. This shows that the notation \(\beta_0\) was appropriate to show the first-order equality to the \(\beta_0\) of the collective model.
If we take off the \( \hat{\ell} \cdot \hat{s} \) term from the Nilsson model mentioned above the harmonic oscillator will be anisotropic. Using this anisotropic harmonic oscillator potential we are generating the energy levels and relate the axes ratios with that of the cluster models in order to study the probability of formation of clusters.

2-3. Link between cluster structure and Super and hyper deformations

One of the successful approximation to the realistic nuclear shell model is the harmonic oscillator model because of its simplicity. In an anisotropic harmonic oscillator model, the single particle energy is given by

\[
E_{sp} = (\frac{v}{2} + n_x) \hbar \omega_x + (\frac{v}{2} + n_y) \hbar \omega_y + (\frac{v}{2} + n_z) \hbar \omega_z \tag{2.27}
\]

Considering \( \omega_x > \omega_y, \omega_y \), and if one choose \( n_z = 0 \) for all the single particle levels, then all the quanta lie in only two directions \( x \) and \( y \). The average oscillator frequency

\[
\bar{\omega} = \frac{1}{2} (\omega_x + \omega_y) \tag{2.28}
\]

and the deformation parameter

\[
\varepsilon = (\omega_y - \omega_x) / \bar{\omega} \tag{2.29}
\]

are the two new parameters defined by Zhang and Rae/2/. Ignoring the zero point energy, which is a constant, the single
particle energy in units of $\hbar \omega$, for a given pair of $(n_x, n_y)$ is expressed as

$$E_{sp} = n_x + n_y - \frac{1}{2} (n_x n_y) \varepsilon$$

...(2.30)

The single particle levels bunch at integer ratios of the oscillator frequencies in two directions and the spectrum is given in fig. (2.01). The magic numbers associated with large degeneracies at major integer ratios of $\omega_y/\omega_x$ are tabulated in (table 2.1.).

The importance of this table here is many of these magic numbers are having close relation with stable alpha cluster configurations.

Theoretical interpretations showed that if the distance between two cluster centers goes to zero, the cluster wave function reduce to harmonic oscillator shell model wave function /3/.

To obtain a shell model limit to the alpha-cluster model, Zhang and Rae /2/ approached systematically to associate an individual alpha-cluster configuration with a particular shell-model state in terms of an overlap probability/9/. For spherical alpha cluster wave functions with a single oscillator frequency $\omega$, overlaps with spherical shell model wave functions only considered; where $\omega_x = \omega_y = \omega_z = \omega$. 

30
The normalized single-particle wave function for one cluster can be obtained by omitting the subscripts which label nucleons and cluster centers and introduce \( R = (X, Y, Z) \) and \( x = (x, y, z) \) so that,

\[
\phi(x) = \left( \frac{\omega_0}{\pi} \right)^{3/4} \exp\left\{ -\frac{1}{2} \omega_0 (x - X)^2 \right\} \exp\left\{ -\frac{1}{2} \omega_0 (y - Y)^2 \right\} \exp\left\{ -\frac{1}{2} \omega_0 (z - Z)^2 \right\}
\]

\( \cdots (2.31) \)

The normalized one-dimensional harmonic oscillator wave function is

\[
\Theta_n(x) = \left( 2^n n! \sqrt{\pi} \right)^{-1/2} \sqrt{\omega} \exp\left\{ -\frac{1}{2} \omega x^2 \right\} H_n(\sqrt{\omega} x)
\]

\( \cdots (2.32) \)

The overlap between shell model single particle wave function \((n_x, n_y, n_z)\) and the cluster model single particle wave function \( \phi(x) \) can be obtained by expanding the exponent terms in eqn.(2-31), and making use of the Hermite Polynomials. Thus the result is

\[
\langle n_x, n_y, n_z | \phi(x) \rangle = B \exp\left\{ -\frac{1}{4} \omega (X^2 + Y^2 + Z^2) \right\}
\]

\[
\times \left( \sqrt{\omega} X \right)^{n_x} \left( \sqrt{\omega} Y \right)^{n_y} \left( \sqrt{\omega} Z \right)^{n_z} \delta_{n_0 n_0} \quad \cdots (2.33)
\]

where

\[
B = \frac{1}{\sqrt{2^{(n_x+n_y+n_z)} n_x! n_y! n_z!}}
\]

Because of the orthogonality, the oscillator frequency \( \omega \) of the shell model is equal to that of the cluster model \( \omega_0 \). So one can
use both of them as \( \omega \). By introducing the new parameters, i.e., the internal angles \( u, v, \) and \( w \) \( /9/ \) by

\[
X_i = \mu_x u_i, \quad Y_i = \mu_y v_i, \quad Z_i = \mu_z w_i
\]

The overlap between the shell model configuration \( |n_x, n_y, n_z, \ldots, n_N \rangle \) and cluster configuration \( |\phi(\beta)\rangle (\beta=R_1, R_2, R_3, \ldots, R_N) \) is finally,

\[
\langle n'_x, n'_y, n'_z, \ldots n'_N | \phi(\beta) \rangle = N \exp \left\{ -\frac{1}{4} \left( \mu_x^2 + \mu_y^2 + \mu_z^2 \right) \right\}
\]

\[
\times \left( \frac{\sqrt{\omega}}{\sqrt{2}} \right)^{N_x} \left( \frac{\sqrt{\omega}}{\sqrt{2}} \right)^{N_y} \left( \frac{\sqrt{\omega}}{\sqrt{2}} \right)^{N_z}
\]

\[
\times \det \left( \begin{array}{ccc}
\frac{n_x^{ij}}{n_y^{ij}} & \frac{n_x^{ij}}{n_z^{ij}} & \frac{n_y^{ij}}{n_z^{ij}} \\
\sqrt{n_x^{ij}} & \sqrt{n_y^{ij}} & \sqrt{n_z^{ij}} \\
\end{array} \right)_{N \times N}
\]

where \( N \) is the normalization constant of the cluster wave function which depends on the cluster’s positions. If \( \mu_x, \mu_y \) and \( \mu_z \) approaches zero the overlap remain finite. The determinant term depends only on the internal angles \( u, v, \) and \( w \) in the shell model limit.

In general the Nilsson-Strutinsky calculation \( /1/ \) and the cranked Hartree-Fock calculation \( /10/ \) are more accurate than the harmonic oscillator since it may be oversimplified by not considering the nucleon-nucleon interaction, for heavier nuclei. There is a clear link between the stable cluster configuration and the minima in the potential energy surface predicted by Nilsson-
Strutinsky calculations. Zhang and Rae relates these minima with the BB alpha cluster model configurations/2/. They have given different axes ratios corresponding to the cluster shapes of the nuclei. For the even-even nuclei below \( A=28 \), the superdeformed configurations predicted by the simple harmonic oscillator model is in good agreement with the BB alpha cluster model. But for \( A>28 \) the two models do not always agree.

The axially symmetric hyperdeformed states with 3:1 major to minor axes ratios predicted by the pure harmonic oscillator model was related with the Cranked cluster model by Rae and Merchant/11/. They have taken \(^{36}\text{Ar}\) and \(^{48}\text{Cr}\) for their study of cluster configurations and hyperdeformed states. By taking sufficiently small incremental increases of the function of \( \omega \), the rotational frequency, in the cranking model, rotational bands of hyperdeformed states have been followed up to very high spins and excitation energies. The results obtained by this way are at odds with conventional liquid drop model expectations, which suggest that these nuclei should fission at much angular momenta and excitation energies. If the distances between the alpha clusters are shrunk to zero it is possible to make a connection between the cranked cluster model wave functions and those obtained from shell model orbitals. The hyperdeformed bands predicted in \(^{36}\text{Ar}\)
and $^{48}\text{Cr}$ with major to minor axes ratios 3:1 are analogous to similar bands in $^{12}\text{C}$ and $^{24}\text{Mg}/3,11/$

The alpha cluster model has had a long history in nuclear physics dating from the observation of alpha decay from the nucleus. The local minima found in the total energy surface were compared with other theoretical calculations/1,12-17/ and with experiment/18-24/. In order to confirm the true local minima in the total energy surface, the local minimum candidates were subjected to an unconstrained variation of all parameters using Brink-Boeker B1 nucleon-nucleon force/11/ and they got six excited configurations in $^{24}\text{Mg}$, several of which are highly deformed. An alpha chain configuration is identified in $^{24}\text{Mg}$ similar to those predicted in $^{12}\text{C}$ /25/ and $^{16}\text{O}$ /26-29/. When these cluster wave functions are compared with Nilsson-Strutinsky calculations of the $^{24}\text{Mg}$ potential energy surface, a close association can be made between the cluster model configurations and local minima in the potential energy surface namely a connection between clustering and deformation. The Nilsson-Strutinsky calculations reveal that these local minima in the potential energy surface are common across the whole of the s-d shell, and relating clustering phenomena with superdeformation.
References:


Fig. 2.01. Single particle level spectrum for deformed harmonic oscillator (taken from ref. /2/).