Chapter 3

INDUCED PAIRED DOMINATION NUMBER AND CHROMATIC NUMBER OF A GRAPH

In this chapter, we obtain sharp upper bound for the sum of the induced-paired domination number and chromatic number and characterize the corresponding extremal graphs.

Definition 3.1 A set $S \subseteq V$ is an induced-paired dominating set, if $S$ is a dominating set of $G$ and the induced subgraph $\langle S \rangle$ is a set of independent edges. The induced-paired domination number $\gamma_{ip}(G)$ is the minimum cardinality taken over all induced-paired dominating sets of $G$.

Example 3.2 In figure 3.1, $\{v_1, v_2, v_4, v_5\}$ is an induced-paired dominating set of $G_1$ and there is no such set of cardinality less than 4 and

![Figure 3.1](image-url)
hence $\gamma_{ip}(G_1) = 4$. For the graph $G_2$ there is no induced - paired dominating set and hence $G_2$ has no $\gamma_{ip}$ - set.

Haynes [10 ] has obtained the following result.

**Theorem 3.3 [10]** If $G$ is a connected graph with a $\gamma_{ip}$ - set and of order $n \geq 3$, then $\gamma_{ip}(G) \leq n - 1$ and equality holds if and only if $G = P_3, C_3$, $P_5$ or $G'$ were $G'$ is the graph as in figure 3.2

Now, we proceed to obtain an upper bound for the sum of the induced paired domination number and chromatic number and characterize the corresponding the extremal graphs.
Theorem 3.4 For any connected graph G with a $\gamma_{ip}$ - set and of order $n \geq 3$, $\gamma_{ip} + \chi \leq 2n - 1$ and equality holds if any only if $G = C_3$.

Proof By theorem 3.3, for any connected graph $G$ with a $\gamma_{ip}$ - set and of order $n \geq 3$, $\gamma_{ip} \leq n - 1$. Also for any graph $G$, $\chi \leq \Delta + 1$. Hence $\gamma_{ip} + \chi \leq n - 1 + \Delta + 1 = n + \Delta \leq n + (n - 1) \leq 2n - 1$.

Now assume that $\gamma_{ip} + \chi = 2n - 1$. This is possible only if $\gamma_{ip} = n - 1$ and $\chi = n$. If $\gamma_{ip} = n - 1$, then by theorem 3.3, $G$ is the graph $P_3$, $C_3$, $P_5$ or $G'$. Since $\chi = n$, $G = C_3$. The converse is obvious.

Theorem 3.5 For any connected graph $G$ with a $\gamma_{ip}$ - set and of order $n \geq 3$, $\gamma_{ip} + \chi = 2n - 2$ if and only if $G = P_3$ or $K_4$.

Proof If $G$ is $P_3$ or $K_4$, then clearly $\gamma_{ip} + \chi = 2n - 2$.

Conversely, let $\gamma_{ip} + \chi = 2n - 2$. Then $\gamma_{ip} = n - 1$ and $\chi = n - 1$, (or) $\gamma_{ip} = n - 2$ and $\chi = n$.

Case 1 $\gamma_{ip} = n - 1$ and $\chi = n - 1$.

Since $\gamma_{ip} = n - 1$, by theorem 3.3, $G$ is $P_3$, $C_3$, $P_5$ or $G'$. But $\chi = n - 1$ implies that $G = P_3$.

Case 2 $\gamma_{ip} = n - 2$ and $\chi = n$.

Since $\chi = n$, $G$ is complete. Now $\gamma_{ip} = n - 2$ implies that $2 = n - 2$ so that $n = 4$. Hence $G = K_4$. ■
Theorem 3.6  For any connected graph $G$ with a $\gamma_{ip}$ - set and of order $n \geq 3$, $\gamma_{ip} + \chi = 2n - 3$ if and only if $G = K_5, G_1$ or $G_2$ given in Figure 3.3.

Proof  If $G$ is any one of the graphs in the theorem, then $\gamma_{ip} + \chi = 2n - 3$.

Conversely, let $\gamma_{ip} + \chi = 2n - 3$. Then $\gamma_{ip} = n - 1$ and $\chi = n - 2$, (or) $\gamma_{ip} = n - 2$ and $\chi = n - 1$, (or) $\gamma_{ip} = n - 3$ and $\chi = n$.

Case 1  $\gamma_{ip} = n - 1$ and $\chi = n - 2$.

Since $\gamma_{ip} = n - 1$, by theorem 3.3, $G$ is $P_3, C_3, P_5$ or $G'$. But none of the graphs satisfies $\chi = n - 2$. Hence no graph exists.

Case 2  $\gamma_{ip} = n - 2$ and $\chi = n - 1$.

Since $\chi = n - 1$, $G$ contains a clique $K$ on $n - 1$ vertices. Let $x$ be a vertex other than the vertices of $K_{n-1}$. Since $G$ is connected, $x$ is adjacent to at least one $u_i$ of $K_{n-1}$. Then $\{x, u_i\}$ is an induced-paired dominating set of $G$ for some $u_i$ in $K_{n-1}$. Now $\gamma_{ip} = n - 2$ implies that $n - 2 = 2$, so that $n = 4$. 

Figure 3.3
Hence K = C₃. If u₁, u₂, u₃ are the vertices of C₃, then x may be adjacent to exactly one uᵢ or two vertices of uᵢ and uⱼ, for i ≠ j. Hence G is isomorphic to G₁ or G₂.

**Case 3** \( \gamma_{ip} = n - 3 \) and \( \chi = n \).

Since \( \chi = n \), G is complete. But \( \gamma_{ip} = n - 3 \) implies that \( G = K_5 \).

**Theorem 3.7** For any connected graph G with a \( \gamma_{ip} \) - set and of order \( n \geq 3 \), \( \gamma_{ip} + \chi = 2n - 4 \) if and only if G is \( C_4 \), \( P_4, P_5, K_6 \), \( K_{1,3} \), \( G_1, G_2 \) or \( G_3 \) given in figure 3.3.

![Figure 3.3](image)

**Proof** If G is any one of the graphs in the theorem, then \( \gamma_{ip} + \chi = 2n - 4 \). Conversely, let \( \gamma_{ip} + \chi = 2n - 4 \). Then \( \gamma_{ip} = n - 1 \) and \( \chi = n - 3 \), (or) \( \gamma_{ip} = n - 2 \) and \( \chi = n - 2 \), (or) \( \gamma_{ip} = n - 3 \) and \( \chi = n - 1 \) (or) \( \gamma_{ip} = n - 4 \) and \( \chi = n \).
Case 1  \( \gamma_{ip} = n - 1 \) and \( \chi = n - 3 \).

Since \( \gamma_{ip} = n - 1 \), by theorem 3.3, G is \( P_3 \), \( C_3 \), \( P_5 \), or \( G' \). But \( \chi = n - 3 \) implies that \( G = P_5 \).

Case 2  \( \gamma_{ip} = n - 2 \) and \( \chi = n - 2 \).

Since \( \chi(G) = n - 2 \), G contains a sub graph \( H \) with \( \chi(H) = n - 2 \). Then \( |V(H)| = n - 1 \) or \( n - 2 \). In both cases, \( H \) contains a clique \( K \) on \( n - 2 \) vertices of \( G \). Let \( S = \{ x, y \} = V(G) - V(K) \). Then \( < S > = K_2 \) or \( \overline{K}_2 \).

Subcase 1  \( < S > = K_2 \).

Since \( G \) is connected, \( x \) (or equivalently \( y \)) is adjacent to at least one vertex \( u_i \) of \( K_{n-2} \). Then \( \{ x, u_i \} \) is an induced-paired dominating set of \( G \). Since \( \gamma_{ip} = n - 2 \), \( 2 = n - 2 \) so that \( n = 4 \). Hence \( K = uv \). If \( x \) is adjacent to \( u \), and \( d(x) = 2 \), \( d(y) = 1 \), then \( G \equiv P_4 \). If \( d(x) = d(y) = 2 \), then \( G \equiv C_4 \).

Subcase 2  \( < S > = \overline{K}_2 \).

Since \( G \) is connected, \( x \) and \( y \) are adjacent to a common vertex \( u \) of \( K_{n-2} \) or \( x \) is adjacent to \( u \) and \( y \) is adjacent to \( v \) for some \( v \) in \( K_{n-2} \). In both cases, \( \{ u, v \} \) is an induced-paired dominating set of \( G \). Since \( \gamma_{ip} = n - 2 \), \( 2 = n - 2 \) so that \( n = 4 \). Hence \( K = uv \). Now let \( d(x) = d(y) = 1 \). If both are adjacent to a common vertex say \( u \), then \( G \equiv K_{1,3} \); otherwise \( G \equiv P_4 \).
Case 3 \( \gamma_{ip} = n - 3 \) and \( \chi = n - 1 \).

Since \( \chi = n - 1 \), G contains a clique K on \( n - 1 \) vertices. Let \( x \) be a vertex other than the vertices of \( K_{n - 1} \). Since G is connected, \( x \) is adjacent to at least one \( u_i \) of \( K_{n - 1} \). Then \( \{x, u_i\} \) is an induced-paired dominating set of G for some \( u_i \) in \( K_{n - 1} \). Now \( \gamma_{ip} = n - 3 \) implies that \( 2 = n - 3 \) so that \( n = 5 \). Hence \( K = K_4 \). If \( u_1, u_2, u_3, u_4 \) are the vertices of \( K_4 \), then \( x \) may be adjacent to exactly one \( u_i \) or two vertices \( u_i \) and \( u_j, i \neq j \) or three vertices \( u_i, u_j, u_k, i \neq j \neq k \). Hence G is isomorphic to \( G_1, G_2, \) or \( G_3 \).

Case 4 \( \gamma_{ip} = n - 4 \) and \( \chi = n \).

Since \( \chi = n \), G must be complete. But \( \gamma_{ip} = n - 4 \) implies that \( 2 = n - 4 \) so that \( n = 6 \). Hence \( G \cong K_6 \).  

Theorem 3.8 For any connected graph G with a \( \gamma_{ip} \)-set and of order \( n \geq 3 \), \( \gamma_{ip} + \chi = 2n - 5 \) if and only if G is isomorphic to \( K_7 \) or any one of the graphs given in figure 3.5.
Proof For the graph $G_1$, $\gamma_{ip} = 6$ and $\chi = 3$ so that $\gamma_{ip} + \chi = 9 = 2n - 5$. For the graphs $G_2$ to $G_9$, $\gamma_{ip} = 4$ and $\chi = 3$ so that $\gamma_{ip} + \chi = 7 = 2n - 5$. For the graphs $G_{10}$ to $G_{22}$, $\gamma_{ip} = 2$ and $\chi = 3$ so that $\gamma_{ip} + \chi = 5 = 2n - 5$. For the graphs $G_{23}$ to $G_{26}$, $\gamma_{ip} = 2$ and $\chi = 5$ so that $\gamma_{ip} + \chi = 7 = 2n - 5$. If $G$ is any one of the graphs in the theorem, then $\gamma_{ip} + \chi = 2n - 5$. 
Conversely, let $y + x = 2n - 5$. Then $y = n - 1$ and $x = n - 4$, (or)
$y = n - 2$ and $x = n - 3$, (or) $y = n - 3$ and $x = n - 2$, (or) $y = n - 4$ and
$x = n - 1$ (or) $y = n - 5$ and $x = 5$.

Case 1 $y = n - 1$ and $x = n - 4$.

Since $y = n - 1$, by theorem 3.3, G is $P_3$, $C_3$, $P_5$ or $G'$. But $x = n - 4$
implies that G $\neq P_3$, $C_3$, or $P_5$. Hence G is a sub graph of $G'$. Then $x = 2$ or 3.
If $x = 2$, then $n = 6$ which is a contradiction to the order of $G'$. If $x = 3$, then
$n = 7$ and there is exactly one triangle in $G'$. Hence $G \cong G_1$.

Case 2 $y = n - 2$ and $x = n - 3$.

Since $\chi(G) = n - 3$, G contains a sub graph H with $\chi(H) = n - 3$. Then
$|V(H)| = n, n - 1, n - 2, or n - 3$.

If $|V(H)| = n - 2$ or $n - 3$, then H contains a clique $K$ on $n - 3$
vertices. Let $S = \{x, y, z\} = V(G) - V(K)$. Then $< S > = K_3, \overline{K}_3, P_3 or$
$K_2 \cup K_1$.

Subcase 1 $< S > = K_3$.

Since G is connected, at least one of the vertices of $K_3$ is adjacent to
one of the vertices, say $u_i$ of $K_{n-3}$. Without loss of generality, let x be
adjacent to $u_i$ for some i. Then $\{u_i, x\}$ is an induced-paired dominating set of
G. Since $\gamma_{ip} = n - 2$, $2 = n - 2$ so that $n = 4$. Since $\chi = n - 3 = 1$, $G$ is an empty graph, which is a contradiction. Hence no graph exists in this case.

**Subcase 2**  $\langle S \rangle = \overline{K}_3$

Since $G$ is connected, one of the vertices of $K_{n-3}$, say $u_i$ is adjacent to all the vertices of $S$ or two vertices of $S$ or one vertex of $S$. If $u_i$ for some $i$, is adjacent to all the vertices of $S$, then $\{ u_i, v \}$ for some $v$ in $K_{n-3}$, is an induced-paired dominating set of $G$ and by subcase 1, no graph exists. If $u_i$ is adjacent to $x$ and $y$, $z$ is adjacent to $u_j$ for $i \neq j$ in $K_{n-3}$. Then $\{ u_i, u_j \}$ is an induced-paired dominating set of $G$ and as in subcase 1, no graph exists.

If $u_i$ is adjacent to $x$, then since $G$ is connected, $u_j$ for $i \neq j$ in $K_{n-3}$ is adjacent to $y$ and $u_k$ for $i \neq j \neq k$ in $K_{n-3}$ is adjacent to $z$, then the graph has no $\gamma_{ip}$ - set and hence no graph exists.

**Subcase 3**  $\langle S \rangle = P_3 = x y z$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $x$ (or equivalently $z$) or $y$. If $u_i$ is adjacent to $y$, then $\{ u_i, y \}$ is an induced-paired dominating set of $G$, and hence by subcase 1, no graph exists. If $u_i$ is adjacent to both $x$ and $z$, then $\{ u_i, x \}$ is an induced-paired dominating set of $G$, and hence by subcase 1, no graph exists. If $u_i$ is adjacent to $x$ (or equivalently $z$), then $\{ u_i, v, y, z \}$ for some $v$ in $K_{n-3}$, is an
induced-paired dominating set of $G$. Since $\gamma_{ip} = n - 2$, $4 = n - 2$ so that $n = 6$. Hence $K = K_3$. Let $u_1, u_2, u_3$ be the vertices of $K_3$. Now $x$ is adjacent to one of $u_i$ or two of $u_i, u_j$, $i \neq j$. Without loss of generality, let $x$ be adjacent to $u_1$.

If $d(x) = d(y) = 2$ and $d(z) = 1$, then $G \cong G_2$.

If $d(x) = 3$, $d(y) = 2$ and $d(z) = 1$, then $G \cong G_3$.

Let $d(x) = 3$ and $d(y) = d(z) = 2$. Without loss of generality, let $x$ be adjacent to $u_1$ and $u_2$. Then $z$ is adjacent to $u_3$ only, so that $G \cong G_4$.

If $d(x) = d(z) = 3$ and $d(y) = 2$, let $x$ be adjacent to $u_1$ and $u_2$. Then $z$ is adjacent to $u_3$ and one of $\{u_1, u_2\}$ so that $\{u_1, z\}$ or $\{u_2, z\}$ is an induced-paired dominating set of $G$, which is a contradiction. Hence no graph exists.

If $d(x) = d(y) = 2$ and $d(z) = 3$, let $x$ be adjacent to $u_1$. Then $z$ is adjacent to $u_2$ and $u_3$ only so that $G \cong G_4$.

If $d(x) = d(y) = d(z) = 2$, then $z$ is adjacent to $u_2$ or $u_3$. In both cases, $G \cong G_5$.

**Subcase 4** $\langle S \rangle = K_2 \cup K_1$.

Let $xy$ be the edge in $\langle S \rangle$. Since $G$ is connected, $x$ is adjacent to at least one vertex $u_i$ in $K_{n-3}$. If $z$ is adjacent to the same $u_i$ of $K_{n-3}$, then $\{u_i, x\}$
is an induced-paired dominating set and hence by subcase 1, no graph exists.

If \( z \) is adjacent to \( u_j \) for \( i \neq j \), then \( \{ x, y, z, u_j \} \) is an induced-paired dominating set of \( G \). Since \( \gamma_{ip} = n - 2 \), \( 4 = n - 2 \) so that \( n = 6 \). Hence \( K = K_3 \).

Let \( u_1, u_2, u_3 \) be the vertices of \( K_3 \). Let \( x \) be adjacent to \( u_1 \). Since \( G \) is connected, \( z \) is adjacent to \( u_2 \) (or equivalently \( u_3 \)).

If \( d(x) = 2 \) and \( d(y) = d(z) = 1 \), then \( G \cong G_6 \).

If \( d(x) = d(z) = 2 \) and \( d(y) = 1 \), let \( x \) be adjacent to \( u_1 \). Then \( z \) must be adjacent to \( u_2 \) and \( u_3 \), so that \( G \cong G_3 \).

If \( d(x) = d(y) = 2 \) and \( d(z) = 1 \), let \( x \) be adjacent to \( u_1 \) and \( z \) be adjacent to \( u_3 \). Then \( y \) is adjacent to \( u_2 \) only so that \( G \cong G_7 \).

If \( d(x) = 3 \) and \( d(y) = d(z) = 1 \), let \( x \) be adjacent to \( u_1 \) and \( u_2 \). Since \( G \) is connected, \( z \) must be adjacent to \( u_3 \) so that \( G \cong G_8 \).

Now let \( |V(H)| = n - 1 \). Then \( H \) is neither a clique in \( G \) nor contains \( K_{n-3} \). Let \( x \) be the unique vertex in \( V(G) - V(H) \).

If \( H \) has a full vertex say \( w \), then \( x \) is adjacent to at least one of the neighbour say \( w' \) of \( w \) so that \( \{ w, w' \} \) is a \( \gamma_{ip} \)-set. Since \( \gamma_{ip} = n - 2 \), \( 2 = n - 2 \) so that \( n = 4 \). Now \( H \) contains 3 vertices with \( \chi(H) = 1 \) but not containing \( K_1 \). Hence no graph exists.
If H has no full vertex, then since G is connected, x is adjacent to some vertex \( u \in V(H) \). Then there exists a vertex v which is not adjacent to u.

We claim that \( \gamma_{ip} = 4 \).

Let S be any \( \gamma_{ip} \) - set containing u and w for some w in H. Let \( H' = < V(H) - N(u, w) > \). Then it is enough if we prove that \( \gamma_c(H') = 2 \).

Suppose not, then for any edge e = vz in \( H' \), there exists a vertex a which is neither adjacent to v nor adjacent to z. Assume that u is given colour 1 and w is the given colour 2. Then v can be given colour 1 and z can be given colour 2. Since a is not adjacent to v, it can be given colour 1. Thus five vertices u, w, v, z and a are coloured with 2 colours. Hence \( H \) can be coloured by \( n - 4 \) colours which is a contradiction. Hence \( \gamma_c(H') = 2 \) and \( \gamma_{ip} = 4 \). Since \( \gamma_{ip} = n - 2 \), 4 = \( n - 2 \) so that \( n = 6 \).

Now H contains 5 vertices with \( \chi(H) = 3 \) but not containing \( K_3 \). Hence \( H = C_5 \). If x is adjacent to exactly one vertex of \( H \), then \( G \cong G_9 \). The case that x is adjacent to two adjacent vertices of \( C_5 \) falls under sub case 3.

Now if \( | V(H) | = n \), then H is a spanning sub graph of G with \( \chi(H) = n - 3 \). Let x be any vertex of G and \( H_1 = G - \{ x \} \). Clearly \( \chi(H_1) \leq n - 3 \). If \( \chi(H_1) = n - 3 \), then the graphs fall under the set up
| V(H) | = n - 1. Otherwise $\chi(H_1) \leq n - 4$. Then $x$ is a full vertex in $G$ and hence $\gamma_{ip} = 2$. Thus $n = 4$ and $\chi = 1$ so that $G$ is disconnected which is a contradiction.

**Case 3** $\gamma_{ip} = n - 3$ and $\chi = n - 2$.

Since $\chi(G) = n - 2$, $G$ contains a sub graph $H$ with $\chi(H) = n - 2$. Then $|V(H)| = n - 1$ or $n - 2$. In both cases $H$ contains a clique $K$ on $n - 2$ vertices. Let $S = \{x, y\} = V(G) - V(K)$. Then $<S> = K_2$ or $\overline{K}_2$.

**Subcase 1** $<S> = K_2$.

Since $G$ is connected, $x$ (or equivalently $y$) is adjacent to at least one vertex $u_i$ of $K_{n-2}$. Then $\{x, u_i\}$ is an induced-paired dominating set. Since $\gamma_{ip} = n - 3$, $2 = n - 3$ so that $n = 5$. Hence $K = K_3$. Let $u_1, u_2, u_3$ be the vertices of $K_3$.

If $d(x) = 2$ and $d(y) = 1$, then $G \cong G_{10}$.

Let $d(x) = d(y) = 2$. Without loss of generality, let $x$ be adjacent to $u_1$. Then $y$ is adjacent to $u_1$ or $u_2$ (or equivalently $u_3$). If $y$ is adjacent to $u_1$, then $G \cong G_{11}$; if $y$ is adjacent to $u_2$, then $G \cong G_{12}$.

Let $d(x) = 2$ and $d(y) = 3$. Without loss of generality, let $x$ be adjacent to $u_1$. Then $y$ is adjacent to both $u_2$ and $u_3$, or $u_1$ and $u_2$ (or equivalently $u_3$). If $y$ is adjacent to $u_2$ and $u_3$ then, $G \cong G_{13}$; if $y$ is adjacent to $u_1$ and $u_2$, then $G \cong G_{14}$. 

41
Let \( d(x) = 3 \). Let \( x \) be adjacent to \( u_1 \) and \( u_2 \). If \( d(y) = 1 \), then \( G \cong G_{15} \).

If \( d(y) = 2 \), then \( y \) is adjacent to \( u_1 \) (or equivalently \( u_2 \) ) or \( u_3 \). If \( y \) is adjacent to \( u_1 \), then \( G \cong G_{14} \); if \( y \) is adjacent to \( u_3 \), then \( G \cong G_{13} \). If \( d(y) = 3 \), then \( y \) is adjacent to \( u_1 \) and \( u_3 \) ( or equivalently \( u_2 \) and \( u_3 \)) or \( y \) is adjacent to \( u_1 \) and \( u_2 \). In the former case, \( G \cong G_{16} \) and in the later case \( G \cong G_{17} \).

**Subcase 2** \( \langle S \rangle = \overline{K}_2 \).

Since \( G \) is connected, \( x \) and \( y \) are adjacent to a common vertex say, \( u \) of \( K_{n-2} \) or \( x \) is adjacent to \( u \) and \( y \) is adjacent to \( v \) for some \( v \) in \( K_{n-2} \). In both cases, \( \{ u, v \} \) is an induced-paired dominating set of \( G \). Since \( \gamma_{ip} = n - 3 \), \( 2 = n - 3 \) so that \( n = 5 \). Hence \( K = K_3 \). Let \( u_1, u_2, u_3 \) be the vertices of \( K_3 \).

Let \( d(x) = d(y) = 1 \). If both are adjacent to a common vertex, then \( G \cong G_{18} \); otherwise \( G \cong G_{19} \).

Now let \( d(x) = 1 \) and \( d(y) = 2 \). Without loss of generality, let \( x \) be adjacent to \( u_1 \). If \( y \) is adjacent to \( u_1 \) and \( u_2 \), then \( G \cong G_{20} \). If \( y \) is adjacent to \( u_2 \) and \( u_3 \), then \( G \cong G_{15} \).

Now let \( d(x) = d(y) = 2 \). If \( x \) is adjacent to \( u_1 \) and \( u_2 \), then \( y \) is adjacent to \( u_1 \) and \( u_3 \) ( or equivalently \( u_2 \) and \( u_3 \)) or \( u_1 \) and \( u_2 \). In the former case, \( G \cong G_{21} \) and in the later case, \( G \cong G_{22} \).

**Case 4** \( \gamma_{ip} = n - 4 \) and \( \chi = n - 1 \).
Since \( \chi = n - 1 \), G contains a clique \( K \) on \( n - 1 \) vertices. Let \( x \) be a vertex other than the vertices of \( K_{n-1} \). Since G is connected, \( x \) is adjacent to at least one \( u_i \) of \( K_{n-1} \). Then \( \{x, u_i\} \) is an induced-paired dominating set of G. Since \( \gamma_{ip} = n - 4 \), \( 2 = n - 4 \) so that \( n = 6 \). Hence \( K = K_5 \). Let \( u_1, u_2, u_3, u_4, u_5 \) be the vertices of \( K_5 \). Then \( x \) is adjacent to at most 4 vertices of \( K_5 \).

Thus \( G \cong G_23, G_24, G_25, \) or \( G_26 \).

**Case 5** \( \gamma_{ip} = n - 5 \) and \( \chi = n \).

Since \( \chi = n \), G must be complete. Since \( \gamma_{ip} = n - 5 \), \( 2 = n - 5 \) so that \( n = 7 \). Hence \( G \cong K_7 \).

**CONCLUSION**

In this chapter, we obtained an upper bound for the sum of the induced-paired domination number and chromatic number of a graph and characterized the corresponding extremal graphs. Some directions for further research are given below.

1. Characterize the class of graphs for which \( \gamma_{ip} = \chi \).
2. Nordhaus-Gaddum type of results for \( \gamma_{ip} \).