Chapter 1

PRELIMINARIES

In this chapter, we collect some basic definitions, and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to Harary [6], and Balakrishnan [2]. For domination theory terminology we refer to Haynes [25].

Definition 1.1 A graph \( G \) consists of a finite nonempty set \( V \) of vertices together with a set \( E \), disjoint from \( V \), whose elements are unordered pairs of (not necessarily distinct) vertices of \( V \). Each element \( e = \{ u, v \} \) of \( E \) is called an edge of \( G \), and \( e \) is said to join \( u \) and \( v \). We write \( e = uv \) and say that \( u \) and \( v \) are the ends of \( e \) and are incident with \( e \). They are also called adjacent vertices; edges which are incident with a common vertex are called adjacent edges. The number of vertices of \( G \) is called the order of \( G \) and is denoted by \( p \) or \( n \). The number of edges of \( G \) is called the size of \( G \) and is denoted by \( q \) or \( m \).

Definition 1.2 An edge whose ends are identical is called a loop and edges having the same end vertices are called multiple edges. A graph which contains neither loops nor multiple edges is called simple graph. Unless otherwise stated, we consider only simple graphs.
Definition 1.3 A graph $H$ is called a **subgraph** of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A **spanning subgraph** of $G$ is a subgraph $H$ with $V(H) = V(G)$. For any set $S$ of vertices of $G$, the **induced subgraph** $<S>$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $<S>$ if and only if they are adjacent in $G$. $<S>$ is also denoted by $G[S]$.

**Definition 1.4** A **walk** of a graph $G$ is an alternating sequence of vertices and edges $W = v_0 e_1 v_1 e_2 \ldots v_{k-1} e_k v_k$ such that $e_i$ is incident with $v_{i-1}$ and $v_i$ for each $i = 1, 2, \ldots, k$. The number of edges in $W$ is called the **length** of the walk. This walk joins $v_0$ and $v_k$, and is called a $v_0 - v_k$ walk. It is also denoted by $v_0 v_1 v_2 \ldots v_k$. If $v_0 = v_k$, $W$ is called a **closed walk**; otherwise it is called **open walk**. If all the edges of $W$ are distinct, it is called a **trail**. Further if all the vertices are distinct it is called a **path**. A closed path is called a **cycle**. A path of length $n$ is denoted by $P_n$ and a cycle of length $n$ is denoted by $C_n$.

**Definition 1.5** A graph $G$ is **connected** if every pair of vertices are joined by a path. A maximal connected subgraph of $G$ is called a **component** of $G$. Thus a disconnected graph has at least two components.

**Definition 1.6** A graph is **acyclic or a forest** if it has no cycles. A **tree** is a connected acyclic graph. A forest in which every component is a path is
called a linear forest.

**Definition 1.7** A rooted tree is called an **m-array** tree if every internal vertex has no more than m children. The tree is called a full m-array tree if every internal vertex has exactly m children. An m-ary tree with m = 2 is called a **binary tree**.

**Definition 1.8** The removal of a vertex v from a graph G gives the subgraph G - \{v\} consisting of all vertices and edges of G except v and edges incident with v. Thus G - \{v\} is the maximal subgraph of G not containing v.

**Definition 1.9** The removal of an edge e from G gives the spanning subgraph G - e containing all edges of G except e. Thus G - e is the maximal subgraph of G not containing e.

**Definition 1.10** An edge e is said to be **subdivided** when it is deleted and replaced by a path of length two connecting its ends. The internal vertex of this path is a new vertex.

**Definition 1.11** The **degree of a vertex** v in a graph G is the number of edges of G incident with v and is denoted by deg_G v or d(v). The minimum and maximum degrees of vertices of G are denoted by \( \delta \) and \( \Delta \) respectively. A vertex of degree 0 in G is called an **isolated vertex**; a vertex of degree 1 is called a **pendant vertex** or an **end vertex** of G.
vertex which is adjacent to a pendant vertex is called a support. A vertex of degree $n - 1$ is called a full vertex.

**Definition 1.12** A graph $G$ is regular of degree $r$ if every vertex of $G$ has degree $r$. Such graphs are called $r$-regular graphs. Any 3-regular graph is called a cubic graph.

**Definition 1.13** A graph $G_1$ is isomorphic to a graph $G_2$ if there exists a bijection $\phi$ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. If $G_1$ is isomorphic to $G_2$, then we write $G_1 \cong G_2$.

**Definition 1.14** The neighborhood of a vertex $u$ in a graph $G$ is the set of all vertices which are adjacent to $u$. It is denoted by $N(u)$. $N[u] = N(u) \cup \{x\}$ is called the closed neighborhood of $u$.

**Definition 1.15** A vertex $v$ of a graph $G$ is called a cut-vertex of a graph $G$ if the removal of $v$ increases the number of components. An edge $e$ of a graph $G$ is called a cut edge or bridge if the removal of $e$ increases the number of components. A block of a graph is a maximal connected, non-trivial subgraph without cut-vertices.

**Definition 1.16** A simple graph in which every vertex is adjacent to all other vertices is called a complete graph. A complete graph on $n$ vertices is denoted by $K_n$. A maximal complete subgraph of $G$ is called a clique of $G$. 


Definition 1.17 A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and the other end in $V_2; (V_1, V_2)$ is called a bipartition of $G$. If further, every vertex of $V_1$ is joined to all the vertices of $V_2$, then $G$ is called a complete bipartite graph. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. $K_{1,n}$ is called a star.

Definition 1.18 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Then union of $G_1$ and $G_2$ is the graph $G = G_1 \cup G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

The join of $G_1$ and $G_2$ is the graph $G = G_1 \vee G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{ uv: u \in V_1, v \in V_2 \}$

The graph $K_1 \vee C_{p-1}$ ($p \geq 2$) is called a wheel and is denoted by $W_p$.

Definiton 1.19 The connectivity $\kappa = \kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph.

Definition 1.20 A set of vertices in $G$ is said to be independent if no two of them are adjacent. The number of vertices in a largest independent set of $G$ is called the independence number of $G$ and is denoted by $\beta_0$.

Definition 1.21 A vertex and an edge are said to cover each other if they are incident. A set of vertices which cover all the edges of a graph $G$ is
called a **vertex cover** of $G$. The number of vertices in a smallest vertex cover is called the **vertex covering number** and is denoted by $\alpha_0$.

**Definition 1.22** A subset $M$ of $E$ is called a **matching** (edge independent set) in $G$, if no two edges of $M$ are adjacent in $G$. A matching that contains all the vertices of $G$ is called a **perfect matching**. The number of edges in a maximum edge independent set is called the **edge independence number** of $G$ and is denoted by $\beta_1$.

**Definition 1.23** The **chromatic number** of a graph is the assignment of colours to its vertices such that no two adjacent vertices receive the same colour and is denoted by $\chi$. If $\chi(G) = k$, then $G$ is $k$-chromatic. If $\chi(G) = k$, but $\chi(H) < k$ for every proper subgraph $H$ of $G$, then $G$ is **$k$-critical**.

**Theorem 1.24** \cite{6} For any graph $G$, $\chi(G) \leq \Delta(G) + 1$.

**Theorem 1.25** \cite{6} If $G$ is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

**Theorem 1.26** If $G$ is $k$-critical graph, then $\delta(G) \geq k - 1$

**Notation** For any real number $x$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$ and $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.

**Definition 1.27** A set $S \subseteq V$ is said to be a **dominating set** in $G$ if every vertex in $V - S$ is adjacent to some vertex in $S$. A dominating set $S$ in $G$ is a
minimal dominating set if no proper subset \( S_1 \subset S \) is a dominating set. The minimum cardinality taken over all minimal dominating sets is called the domination number of \( G \) and is denoted by \( \gamma \).

**Theorem 1.28 [25]** If \( G \) is a graph of order \( p \), with maximum degree \( \Delta \), then \( \gamma \geq \lceil p / (\Delta + 1) \rceil \).

**Theorem 1.29 [16]** For a connected graph \( G \), \( \gamma = \chi = 2 \) if and only if

(i) \( G \) is bipartite with bipartition \((X,Y)\) and

(ii) \( |X| = 2 \), or there exists \( x \) in \( X \) and \( y \) in \( Y \) such that \( N(x) = Y \) and \( N(y) = X \), or there exists \( x \) in \( X \) and \( y \) in \( Y \) such that \( N(x) = Y - \{y\} \) and \( N(y) = X - \{x\} \).

**Definition 1.30** A dominating set \( S \) is called a **total dominating set** if the induced subgraph \(<S>\) has no isolated vertices. The minimum cardinality taken over all total dominating sets in \( G \) is called the total domination number of \( G \) and is denoted by \( \gamma_t \).

**Definition 1.31** A dominating set \( S \) is called a **connected dominating set** if the induced subgraph \(<S>\) is connected. The minimum cardinality taken over all connected dominating sets in \( G \) is called the connected domination number of \( G \) and is denoted by \( \gamma_c \).

**Definition 1.32** A dominating set \( S \) of a graph is called an **independent dominating set** of \( G \) if \( S \) is independent in \( G \). The cardinality of the
smallest independent dominating set of $G$ is called the independent domination number of $G$ and is denoted by $\text{i}(G)$.

**Definition 1.33** A dominating set $S$ is called an **efficient dominating** set if for every vertex $u \in V$, $|N(u) \cap S| = 1$.

**Some special graphs.**

**Definition 1.34** The $n$-bistar $B_{n,n}$ is the graph with $V(B_{n,n}) = \{u, v, u_1, u_2, u_3, \ldots, u_n; v_1, v_2, v_3, \ldots, v_n\}$ and $E(B_{n,n}) = \{uu_i, vv_i, uv: 1 \leq i \leq n\}$. The graph $B_{4,4}$ is shown in the figure 1.1.

![Figure 1.1](image)

**Definition 1.35** $H_{n,n}$ is the graph with vertex set $V(H_{n,n}) = \{v_1, v_2, v_3, \ldots, v_n; u_1, u_2, u_3, \ldots, u_n\}$ and the edge set $E(H_{n,n}) = \{v_iu_j: 1 \leq i \leq n, n - i + 1 \leq j \leq n\}$. The graph $H_{4,4}$ is shown in the figure 1.2.

![Figure 1.2](image)
Definition 1.36 The graph obtained from a wheel $W_p$ by attaching a pendant edge at each vertex of the $(p-1)$-cycle is called a Helm and is denoted by $H_p$. For example the graph $H_5$ is shown in figure 1.3.
Definition 1.37  The corona of two graphs $G_1$ and $G_2$, is the graph $G = G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$. For example if $G_1 = K_3$ and $G_2 = P_3$, then the corona $G_1 \circ G_2$ is shown in figure 1.4

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1.4}
\end{figure}