Chapter 5

COMPLEMENTARY PERFECT DOMINATION NUMBER OF A GRAPH

In this chapter, we introduce the concept of complementary perfect domination number of a graph with application. We determine the exact value of this parameter for some standard graphs. Its relation with other graph theoretic parameters are also investigated.

Definition 5.1 A subset $S$ of $V$ of a non-trivial graph $G$ is said to be complementary perfect dominating set, if $S$ is a dominating set and $V - S$ has a perfect matching. The minimum cardinality taken over all complementary perfect dominating sets is called \textit{complementary perfect domination number} and is denoted by $\gamma_{cp}$. Any complementary perfect dominating set with $\gamma_{cp}$ elements is called a $\gamma_{cp}$-set.

Example 5.2 In figure 5.1, $S_1 = \{ v_1, v_2, v_3, v_4 \}$ forms a complementary perfect dominating set of $G_1$. Also $S_2 = \{ v_1, v_4 \}$ forms a complementary perfect dominating set of $G_1$. But $S_2$ is the smallest complementary perfect dominating set and hence $\gamma_{cp}(G) = 2$. 

57
Application  This concept is likely to have good application in the prevailing situations in developing countries. In India, due to poverty and unemployment problems, people from villages move in large scale towards metropolitan cities, which are already overcrowded. Hence the State Government plans to construct model villages far away from the cities and provide all possible facilities by adopting pairs of villages which are connected to each other by means of a single road. Thus in the corresponding road network, the Government wishes to select a minimum number of cities from which facilities can be extended to all pairs of model cities at a minimum cost. This is nothing but the complementary perfect domination number of the graph associated with this problem.

Observation 5.3

1. \( \gamma_{cp}(G) = 1 \) if and only if \( G \) is isomorphic to \( H \vee K_1 \), where \( H \) has a perfect matching.
2. If $H$ has a perfect matching, then $\gamma_{cp}(H \vee K_2) = 2 = \gamma_{cp}(H \vee \overline{K}_2)$.

**Remark 5.4** If $G$ is the disjoint union of the graphs $G_1, G_2, G_3, \ldots, G_n$, then $\gamma_{cp}(G) = \gamma_{cp}(G_1) + \gamma_{cp}(G_2) + \gamma_{cp}(G_3) + \ldots + \gamma_{cp}(G_n)$. Hence we consider only connected graphs.

**Theorem 5.5** Any complementary perfect dominating set of $G$ must contain all the pendant vertices of $G$.

**Proof** Let $S$ be any complementary perfect dominating set of $G$. Let $u$ be a pendant vertex. Suppose $u$ is not in $S$. Since $S$ is a dominating set, $N(u) = \{v\} \in S$. Then $u$ is an isolate in $<V - S>$. Thus $<V - S>$ has no perfect matching. This is a contradiction. Hence $u$ must be in $S$. $\blacksquare$

**Notation 5.6** $p_v(G)$ denotes the number of pendant vertices of $G$.

$G^*$ is the graph obtained from $G$ by identifying the center of a star of any order at each vertex of $G$.

**Theorem 5.7** For any graph $G$, $\gamma_{cp}(G) = p_v$ if and only if $G$ is isomorphic to $H^*$, where $H$ has a perfect matching.

**Proof** Let $G \cong H^*$, where $H$ has a perfect matching. Let $S$ be set of all pendant vertices of $G$. Then $S$ covers all the vertices of $V - S$ and $<V - S> \cong H$, which has a perfect matching. Therefore $\gamma_{cp}(G) \leq |S| = p_v$. 

59
Moreover any \( \gamma_{cp} \) - set contains all the pendant vertices. Therefore 
\[ \gamma_{cp}(G) \geq p_v \] 
and hence \( \gamma_{cp}(G) = p_v \).

Conversely assume that \( \gamma_{cp}(G) = p_v \). Let \( S \) be the \( \gamma_{cp} \) - set. Since \( S \) contains all the pendant vertices of \( G \), we have \( S \) equals the set of all pendant vertices of \( G \). Also \( V - S \) has a perfect matching. Take \( H = <V - S> \), then \( G \cong H* \).

**Theorem 5.8** For any connected graph \( G \) of order \( n \geq 2 \), \( \gamma_{cp} = n \) if and only if \( G \) is a star.

**Proof** If \( G \) is a star, then clearly, \( \gamma_{cp} = n \). Conversely, assume that \( \gamma_{cp} = n \). We claim that \( G \) is a star. Suppose not, let \( u \) be a vertex of maximum degree \( \Delta \) with \( N(u) = \{ u_1, u_2, u_3, \ldots, u_\Delta \} \). If \( < N(u) > \) has an edge \( e = u_iu_j \), then \( V - \{ u_i, u_j \} \) is a complementary perfect dominating set of cardinality \( n - 2 \), which is a contradiction. If \( < N(u) > \) has no edge, then \( G \) has an edge \( e = xy \) which is not incident with \( u \), such that \( u \) is adjacent to \( x \), then \( V - \{ u, x \} \) is a complementary perfect dominating set of cardinality \( n - 2 \), which is a contradiction. Hence \( G \) is a star.

**Remark 5.9** Since \( \gamma \leq \gamma_{cp} \), we have the following result from theorem 1.28.

If \( G \) is not a star, then \( \lceil p / (\Delta + 1) \rceil \leq \gamma_{cp} \leq p - 2 \) and the bounds are sharp. \( K_{2n+1} \) is the graph for which \( \gamma_{cp} = \lceil p / (\Delta + 1) \rceil \) and bistar is the
the graph for which $\gamma_{cp} = p - 2$.

Now we shall determine the exact value of $\gamma_{cp}$ for some standard graphs.

1. $\gamma_{cp}(W_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$

2. $\gamma_{cp}(K_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$

3. If $G$ is the corona $H \circ K_1$, where $H$ contains a perfect matching, then $\gamma_{cp}(G) = n/2$.

4. $\gamma_{cp}(K_{m,n}) = n - m + 2$, if $n \geq 2$.

5. $\gamma_{cp}(P) = 4$, where $P$ is the Petersen graph.

6. $\gamma_{cp}(H_{n,n}) = \begin{cases} 4, & \text{if } n > 2 \\ 2, & \text{if } n = 2. \end{cases}$

7. $\gamma_{cp}(H_n) = \begin{cases} n + 1, & \text{if } n \text{ is even.} \\ n + 2, & \text{if } n \text{ is odd.} \end{cases}$

**Theorem 5.10** For any $n \geq 0$, (i) $\gamma_{cp}(P_{3n}) = n + 2$

(ii) $\gamma_{cp}(P_{3n+1}) = n + 1$

(iii) $\gamma_{cp}(P_{3n+2}) = n + 2$. 

61
Proof (i) Let \( \{u_1, u_2, u_3, \ldots, u_{3n}\} \) be the vertices of \( P_{3n} \). Then \( S = \{ u_1, u_4, u_7, \ldots, u_{3n-5}, u_{3n-2}, u_{3n-1}, u_{3n}\} \) is a complementary perfect dominating set of \( P_{3n} \). Therefore \( \gamma_{cp}(P_{3n}) \leq |S| = n + 2 \). Also, since each vertex of a dominating set \( S \) except the origin and terminus of the path can dominate at most two vertices in \( V - S \). Moreover, each of origin and terminus dominates only one vertex in \( V - S \) as they belong to \( S \). Therefore we have we have \( \gamma_{cp}(P_{3n}) \geq 2 + \lceil (3n-4)/3 \rceil = n + 1 \). But if \( \gamma_{cp}(P_{3n}) = n + 1 \), then \( |V - S| = 2n - 1 \) and hence \( <V - S> \) has no perfect matching. This is not possible. Hence \( \gamma_{cp}(P_{3n}) \geq n + 2 \) and hence \( \gamma_{cp}(P_{3n}) = n + 2 \).

(ii) Let \( \{u_1, u_2, u_3, \ldots, u_{3n+1}\} \) be the vertices of \( P_{3n+1} \). Then \( S = \{ u_1, u_4, u_7, \ldots, u_{3n-2}, u_{3n+1}\} \) is a complementary perfect dominating set of \( P_{3n+1} \). Therefore \( \gamma_{cp}(P_{3n+1}) \leq |S| = n + 1 \). As in the previous case \( \gamma_{cp}(P_{3n+1}) \geq 2 + \lceil (3n+1-4)/3 \rceil = n + 1 \) and hence \( \gamma_{cp}(P_{3n+2}) = n + 1 \).

(iii) Let \( \{u_1, u_2, u_3, \ldots, u_{3n+2}\} \) be the vertices of \( P_{3n+2} \). Then \( S = \{ u_1, u_4, u_7, \ldots, u_{3n-2}, u_{3n+1}, u_{3n+2}\} \) is a complementary perfect dominating set of \( P_{3n+2} \). Therefore \( \gamma_{cp}(P_{3n+2}) \leq |S| = n + 2 \). As in the previous argument, we have \( \gamma_{cp}(P_{3n+2}) \geq 2 + \lceil (3n+2-4)/3 \rceil = n + 2 \), and hence \( \gamma_{cp}(P_{3n+2}) = n + 2 \) ■

Theorem 5.11 For any \( n \geq 1 \), (i) \( \gamma_{cp}(C_{3n}) = n \)

(ii) \( \gamma_{cp}(C_{3n+1}) = n + 1 \)
(iii) \( \gamma_{cp}(C_{3n+2}) = n + 2. \)

**Proof**  
(i) Let \( \{u_1, u_2, u_3, \ldots, u_{3n}\} \) be the vertices of \( C_{3n} \). Then \( S = \{ u_1, u_4, u_7, \ldots, u_{3n-5}, u_{3n-2} \} \) is a complementary perfect dominating set of \( C_{3n} \). Therefore \( \gamma_{cp}(C_{3n}) \leq |S| = n. \) Also, since each vertex of a dominating set \( S \) can dominate at most two vertices in \( V - S \), we have \( \gamma_{cp}(C_{3n}) \geq \lceil 3n/3 \rceil = n. \) Hence \( \gamma_{cp}(C_{3n}) = n. \)

(ii) Let \( \{u_1, u_2, u_3, \ldots, u_{3n+1}\} \) be the vertices of \( C_{3n+1} \). Then \( S = \{ u_1, u_4, u_7, u_{10}, \ldots, u_{3n-2}, u_{3n+1} \} \) is a complementary perfect dominating set of \( C_{3n+1} \). Therefore \( \gamma_{cp}(C_{3n+1}) \leq |S| = n + 1. \) As in the previous case, we have \( \gamma_{cp}(C_{3n+1}) \geq \lceil (3n + 1)/3 \rceil = n + 1, \) and hence \( \gamma_{cp}(C_{3n+1}) = n + 1. \)

(iii) Let \( \{u_1, u_2, u_3, \ldots, u_{3n+2}\} \) be the vertices of \( C_{3n+2} \). Then \( S = \{ u_1, u_4, u_7, \ldots, u_{3n+1}, u_{3n+2} \} \) is a complementary perfect dominating set of \( C_{3n+2} \). Therefore \( \gamma_{cp}(C_{3n+2}) \leq |S| = n + 2. \) Also, since each vertex of a dominating set \( S \) can dominate at most two vertices in \( V - S \), we have \( \gamma_{cp}(C_{3n+2}) \geq (3n + 2)/3 = n + 1. \) If there is a complementary perfect dominating set \( S \) with \( n + 1 \) elements, then \( V - S \) contains \( 2n + 1 \) vertices which is impossible. Hence \( \gamma_{cp}(C_{3n+2}) \geq n + 2 \) and hence \( \gamma_{cp}(C_{3n+2}) = n + 2. \)

**Theorem 5.12** For any two positive integers \( m \) and \( n \) of same parity such that \( m \geq n \geq 1 \), there is a graph \( G \) of order \( m \) for which \( \gamma_{cp} = n. \)
Proof  If \( m = 1 \), then \( n = 1 \) and \( G = K_1 \) is the required graph.

Hence assume that \( m > 1 \). If \( m = n \), then \( G = K_{1,n-1} \) is the required graph.

If \( m > n \), let \( m - n = 2s \). Consider \( K_{1,n-1} \) with vertices \( \{ u, v_1, v_2, v_3, \ldots, v_{n-1} \} \). Introduce \( 2s \) vertices \( u_1, u_2, u_3, \ldots, u_{2s} \) and the edges \( uu_{2i-1}, v_1u_{2i}, u_{2i-1}u_{2i} \), where \( 1 \leq i \leq s \). Then the resultant graph \( G \) is the required graph. The case when \( m = 10 \) and \( n = 4 \) is illustrated in figure 5.2. □

![Diagram](image)

Here \( S = \{ v_1, v_2, v_3, u \} \) is a \( \gamma_{cp} \) - set.

**Figure 5.2**
Notation 5.13 Let \( v \) be any vertex in a graph \( G \). Then the graph obtained by identifying a pendent vertex of \( P_3 \) with \( v \) is denoted by \( G(vP_3) \).

Theorem 5.14 Let \( G \) be a graph and \( v \) be a non-pendent vertex of \( G \) which is in some \( \gamma_{cp} \) - set \( S \) of \( G \) such that \( S - \{ v \} \) is a dominating set for \( G \), then

\[
\gamma_{cp}(G(vP_3)) = \gamma_{cp}(G)
\]

Proof Let \( S \) be a \( \gamma_{cp} \) - set of \( G \). Let \( v \in S \). Let \( P_3 = v_1 v_2 v_3 \). If \( v_3 \) is identified with \( v \), then take \( S_1 = (S - v) \cup \{ v_1 \} \). Clearly \( S_1 \) is a \( \gamma_{cp} \) - set of \( G(vP_3) \). Hence

\[
\gamma_{cp}(G(vP_3)) \leq |S_1| = |S| = \gamma_{cp}(G).
\]

But \( G \subseteq G(vP_3) \) implies that

\[
\gamma_{cp}(G(vP_3)) \geq \gamma_{cp}(G(vP_3)) = \gamma_{cp}(G)
\]

For example in \( C_{6n+5} \), \( n \geq 0 \), any vertex \( v \) satisfies the hypothesis of the theorem 5.14. Hence

\[
\gamma_{cp}(C_{6n+5} (vP_3)) = \gamma_{cp}(C_{6n+5})
\]

The case \( n = 0 \), is illustrated in the following figure 5.3.

For example in \( C_{6n+5} \), \( n \geq 0 \), any vertex \( v \) satisfies the hypothesis of the theorem 5.14. Hence

\[
\gamma_{cp}(C_{6n+5} (vP_3)) = \gamma_{cp}(C_{6n+5})
\]

The case \( n = 0 \), is illustrated in the following figure 5.3.

\[
\gamma_{cp}(C_5) = \gamma_{cp}(C_5(vP_3)).
\]

Figure 5.3
Figure 5.4
Remark 5.15  \( L_n \) denotes the full \( m \)-array tree on \( n \)th level. Then clearly
\[ L_0 \cong K_1 \text{ and } L_1 \cong K_{1,m}. \]
\( L_2 \) and \( L_3 \) are shown in figure 5.4.

Also \( \gamma_{cp}(L_0) = 1, \gamma_{cp}(L_1) = m + 1. \)
\[ \gamma_{cp}(L_2) = m^2 + m - 1. \]
\[ \gamma_{cp}(L_3) = m^3 + m^2 + m + 1. \] \( \gamma_{cp} \)-sets of \( L_2 \) and \( L_3 \) are indicated by dark vertices in the figure 5.4.

Lemma 5.16  Let \( L_n \) denotes the full \( m \)-array tree on the \( n \)th level, then
\[ \gamma_{cp}(L_{n+3}) = \gamma_{cp}(L_n) + (m-1)m^{n+1} + m^{n+3}. \]

Proof  Let \( S \) be a \( \gamma_{cp} \)-set of \( L_{n+3} \). Since any \( \gamma_{cp} \)-set must contain all the pendant vertices, \( S \) contains \( m^{n+3} \) pendant vertices.

Claim 1  \( S \) contains no vertex in the level \( n+1 \).

If not, let \( u \in S \) be a vertex of level \( n+1 \).

Let \( u_1, u_2, u_3, \ldots, u_m \) be the vertices in level \( n+2 \) which are adjacent to \( u \). Let \( u_{11}, u_{12}, u_{13}, \ldots, u_{1m}; u_{21}, u_{22}, u_{23}, \ldots, u_{2m}; \ldots, u_{m1}, u_{m2}, u_{m3}, \ldots, u_{mm} \) be the vertices in the level \( n+3 \), which are adjacent to \( u_1, u_2, u_3, \ldots, u_m \) respectively. Since \( u \in S \) and all the pendant vertices are in \( S \), the vertices \( u_1, u_2, u_3, \ldots, u_m \) are not in \( V - S \). (If it is in \( V - S \), then \( < V - S > \) has isolates \( u_1, u_2, u_3, \ldots, u_m \), which is a contradiction.). Therefore \( \{ u; u_1, u_2, u_3, \ldots, u_m, u_{11}, u_{12}, u_{13}, \ldots, u_{1m}, u_{21}, u_{22}, u_{23}, \ldots, u_{2m}, \ldots, u_{m1}, u_{m2}, u_{m3}, \ldots, u_{mm} \} \).
\( u_{m_3, \ldots, u_{mm}} \) \( \subseteq S \) and hence \( S - \{ u, u_1 \} \) is a complementary perfect dominating set of \( L_{n+3} \) which is a contradiction. Hence the claim.

**Claim 2** \( S \) contains exactly \( (m - 1) m^{n+1} \) vertices in the level \( n + 2 \).

It is enough to show that \( V - S \) contains \( m^{n+1} \) vertices at level \( n + 2 \).

If it contains more than that, then there are two vertices which the same neighbor at level \( n + 1 \) and hence one of them is an isolate in \( <V - S> \) which is a contradiction. Hence the claim.

Now, let \( S \) be \( \gamma_{cp} \) - set of \( L_n \). Then along with \( S \), the \( (m^{n+3}) \) pendant vertices and \( (m - 1)m^{n+1} \) vertices in the level \( n + 2 \) forms a complementary perfect dominating set of \( L_{n+3} \).

Hence \( \gamma_{cp}(L_{n+3}) \leq \gamma_{cp}(L_n) + (m - 1) m^{n+1} + m^{n+3} \)

We claim that \( \gamma_{cp}(L_{n+3}) \geq \gamma_{cp}(L_n) + (m - 1) m^{n+1} + m^{n+3} \).

If not, let \( S \) be a \( \gamma_{cp} \) -set of \( L_{n+3} \) with \( |S| < \gamma_{cp}(L_n) + (m - 1) m^{n+1} + m^{n+3} \). Then \( S \) contains all the \( m^{n+3} \) vertices say \( V_1 \) in the level \( n + 3 \) and \( (m - 1) m^{n+1} \) vertices say \( V_2 \) in the level \( n + 2 \) (By claim 2).

Thus \( S - \{ V_1 \cup V_2 \} \) is a complementary perfect dominating set of \( L_m \).

Also, \( |S - \{ V_1 \cup V_2 \}| < \gamma_{cp}(L_n) \), which is a contradiction.

Therefore, \( \gamma_{cp}(L_{n+3}) = \gamma_{cp}(L_n) + (m - 1) m^{n+1} + m^{n+3} \).
Theorem 5.17 Let $L_n$ denotes the full $m$-array tree on $n^{th}$ level. Then for any $n \geq 0$,

(i) $\gamma_{cp}(L_{3n}) = \frac{1}{(m^3 - 1)} \left\{ m^{3n+1} (m^2 + m - 1) - (m^2 - m + 1) \right\}$

(ii) $\gamma_{cp}(L_{3n+1}) = \frac{1}{(m^3 - 1)} \left\{ m^{3n+2} (m^2 + m - 1) + (m^2 - m - 1) \right\}$

(iii) $\gamma_{cp}(L_{3n+2}) = \left\{ \frac{(m^{3n+3} - 1)}{(m^3 - 1)} \right\} (m^2 + m - 1)$

Proof We prove the theorem by induction on $n$, the $n^{th}$ level of the full $m$-array tree.

(i) When $n = 0$, $\gamma_{cp}(L_0) = 1 = \frac{(m^3 - 1)}{(m^3 - 1)}$.

Hence the result is true for $n = 0$.

Assume the result is true for $k$.

i.e., $\gamma_{cp}(L_{3k}) = \frac{1}{(m^3 - 1)} \left\{ m^{3k+1} (m^2 + m - 1) - (m^2 - m + 1) \right\}$.

To prove the result for $k + 1$

By lemma 5.16 we have $\gamma_{cp}(L_{3k+3}) = \gamma_{cp}(L_{3k}) + (m-1)m^{k+1} + m^{k+3}$.

Replace $k$ by $3k$. Then $\gamma_{cp}(L_{3k+3}) = \gamma_{cp}(L_{3k}) + (m-1)m^{3k+1} + m^{3k+3}$.

Now, $\gamma_{cp}(L_{3k+3}) = \frac{1}{(m^3-1)} \left\{ m^{3k+1} (m^2 + m - 1) - (m^2 - m + 1) \right\} + (m-1)m^{3k+1} + m^{3k+3}$.

$= \frac{1}{(m^3-1)} \left\{ m^{3k+1} (m^2 + m - 1) - (m^2 - m + 1) \right\} + ((m^3-1)/(m^3-1)) \left\{ (m-1)m^{3k+1} + m^{3k+3} \right\}$

69
\[
\begin{align*}
\frac{1}{m^3-1} & \left\{ m^{3k+1}(m^2+m-1)-(m^2-m+1) + \\
& \left( m^3-1 \right)(m-1)m^{3k+1}+(m^3-1)m^{3k+3} \right\} \\
& = \frac{1}{m^3-1}\left\{ m^{3k+6}+m^{3k+5}-m^{3k+4}-(m^2-m+1) \right\} \\
& = \frac{1}{m^3-1}\left\{ m^{3k+4}(m^2+m-1)-(m^2-m+1) \right\}
\end{align*}
\]

Hence by induction hypothesis, the result is true for all \(n\).

(ii) When \(n=0\), the the full \(m\)-array tree is \(L_1\) which is nothing but a star for which \(\gamma_{cp}(L_1) = m + 1\).

Also \(\gamma_{cp}(L_1) = \frac{1}{m^3-1}\left\{ m^2(m^2+m-1)+(m^2-m-1) \right\} \)

\[
= \left( \frac{1}{m^3-1} \right) \left\{ m^4+m^3-m-1 \right\}
\]

\[
= m + 1.
\]

Hence the result is true for \(n=0\).

Assume the result is true for \(n=k\).

To prove it for \(k+1\).

Now by lemma 5.16 we have

\[
\gamma_{cp}(L_{k+3}) = \gamma_{cp}(L_{nk}) + (m-1)m^{k+1} + m^{k+3}.
\]

Replace \(k\) by \(3k+1\). Then

\[
\gamma_{cp}(L_{3k+4}) = \gamma_{cp}(L_{3k+1}) + (m-1)m^{3k+2} + m^{3k+4}
\]

\[
= \left( \frac{1}{m^3-1} \right) \left\{ m^{3k+2}(m^2+m-1)+(m^2-m-1) \right\} + \\
\left( m-1 \right)m^{3k+2} + m^{3k+4}
\]

\[
= \left( \frac{1}{m^3-1} \right) \left\{ m^{3k+2}(m^2+m-1)+(m^2-m-1) \right\} +
\]

70
\[
\begin{align*}
\left\{ \frac{(m^3 - 1)}{(m^3 - 1)} \right\} \left( \frac{(m - 1) \cdot m^{3k + 2} + m^{3k + 4}}{(m - 1) \cdot m^{3k + 2} + m^{3k + 4}} \right) \\
= \left\{ \frac{1}{(m^3 - 1)} \right\} \left\{ m^{3k + 7} + m^{3k + 6} - m^{3k + 5} + (m^2 - m - 1) \right\} \\
= \left\{ \frac{1}{(m^3 - 1)} \right\} \left\{ m^{3k + 5} \cdot (m^2 + m - 1) + (m^2 - m - 1) \right\}
\end{align*}
\]

Hence by induction hypothesis the result is true for all \( n \).

(iii) When \( n = 0 \), the full \( m \)-array tree is \( L_2 \) and we know that
\[
\gamma_{cp}(L_2) = m^2 + m - 1.
\]
Also \( \gamma_{cp}(L_2) = \left\{ \frac{(m^3 - 1)}{(m^3 - 1)} \right\} (m^2 + m - 1) = (m^2 + m - 1) \)

Hence the result is true for \( n = 0 \).

Assume the result is true for \( n = k \).

To prove it for \( k + 1 \).

By lemma 5.16 \( \gamma_{cp}(L_{k+3}) = \gamma_{cp}(L_k) + (m - 1) \cdot m^{k+1} + m^{k+3} \)

Replace \( k \) by \( 3k + 2 \). Then
\[
\gamma_{cp}(L_{3k+5}) = \gamma_{cp}(L_{3k+2}) + (m - 1) \cdot m^{3k+3} + m^{3k+5}
\]
\[
= \left\{ \frac{(m^{3k+2} - 1)}{(m^3 - 1)} \right\} \left( m^2 + m - 1 \right) +
\]
\[
\left( m - 1 \right) \cdot m^{3k+3} + m^{3k+5}
\]
\[
= \left\{ \frac{(m^{3k+3} - 1)}{(m^3 - 1)} \right\} \left( m^2 + m - 1 \right) +
\]
\[
\left\{ \frac{(m^3 - 1)}{(m^3 - 1)} \right\} \left( m - 1 \right) \cdot m^{3k+3} + m^{3k+5}
\]
\[
= \left\{ \frac{1}{(m^3 - 1)} \right\} \left( m^{3k+8} + m^{3k+7} - m^{3k+6} - m^2 + m + 1 \right)
\]
\[
= \left\{ \frac{(m^{3k+6} - 1)}{(m^3 - 1)} \right\} \left( m^2 + m - 1 \right)
\]

Hence by induction hypothesis, the result is true for all \( n \). ■
Corollary 5.18 For any full binary tree on the $n^{th}$ level

(i) $\gamma_{cp}(L_{3n}) = \left( \frac{1}{7} \right) \left( 5 \times 2^{3n+1} - 3 \right)$

(ii) $\gamma_{cp}(L_{3n+1}) = \left( \frac{1}{7} \right) \left( 5 \times 2^{3n+2} + 1 \right)$

(iii) $\gamma_{cp}(L_{3n+2}) = \left( \frac{5}{7} \right) \left( 2^{3n+3} - 1 \right)$

Proof Follows from the theorem 5.17 by taking $m = 2$. ■

Relationship of $\gamma_{cp}(G)$ with $\chi(G)$

Theorem 5.19 For any connected graph $G$, $\gamma_{cp} + \chi \leq 2n$, and equality holds if and only if $G$ is isomorphic to $K_2$.

Proof Clearly for any graph $G$, $\gamma_{cp} \leq n$. Also for any graph $G$, $\chi \leq \Delta + 1$.

Hence $\gamma_{cp} + \chi \leq n + (\Delta + 1) = n + (n - 1 + 1) = 2n$. Now assume that $\gamma_{cp} + \chi = 2n$. This is possible only if $\gamma_{cp} = n$ and $\chi = n$. Since $\gamma_{cp} = n$, by theorem [5.8] $G$ is a star. Since $\chi = n$, $G$ is $K_2$. Converse is obvious. ■

Theorem 5.20 For any connected graph $G$, $\gamma_{cp} + \chi = 2n - 1$ if and only if $G$ is isomorphic to $P_3$.

Proof Assume that $\gamma_{cp} + \chi = 2n - 1$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 1$ and $\chi = n$.

Case 1 $\gamma_{cp} = n$ and $\chi = n - 1$

Since $\gamma_{cp} = n$, by theorem [5.8], $G$ is a star. Since $\chi = n - 1$, $2 = n - 1$ so that $n = 3$. Hence $G \cong K_{1,2} = P_3$
Case 2 \( \gamma_{cp} = n - 1 \) and \( \chi = n \)

Since \( \gamma_{cp} = n - 1 \), there exists a complementary perfect dominating set \( S \) with \( n - 1 \) elements. Hence \( < V - S > \) has isolates, which is a contradiction. Hence no graph exists. Converse is obvious. \( \blacksquare \)

**Theorem 5.21** For any connected graph \( G \), \( \gamma_{cp} + \chi = 2n - 2 \) if and only if \( G \) is isomorphic to \( K_3, K_4, K_{1,3} \).

**Proof** Assume that \( \gamma_{cp} + \chi = 2n - 2 \). This is possible only if \( \gamma_{cp} = n \) and \( \chi = n - 2 \) (or) \( \gamma_{cp} = n - 1 \) and \( \chi = n - 1 \) (or) \( \gamma_{cp} = n - 2 \) and \( \chi = n \).

The case that \( \gamma_{cp} = n - 1 \) and \( \chi = n - 1 \) is not possible.

**Case 1** \( \gamma_{cp} = n \) and \( \chi = n - 2 \).

Since \( \gamma_{cp} = n \), by theorem [5.8] \( G \) is a star. Since \( \chi = n - 2 \), \( 2 = n - 2 \) so that \( n = 4 \). Hence \( G \cong K_{1,3} \).

**Case 2** \( \gamma_{cp} = n - 2 \) and \( \chi = n \).

Since \( \chi = n \), \( G \cong K_n \). If \( G \) has even number of vertices, then \( \gamma_{cp} = 2 \) so that \( n = 4 \). Hence \( G \cong K_4 \). If \( G \) has odd number of vertices then \( \gamma_{cp} = 1 \) so that \( n = 3 \). Hence \( G \cong K_3 \) \( \blacksquare \)

**Theorem 5.22** For any connected graph \( G \), \( \gamma_{cp} + \chi = 2n - 3 \) if and only if \( G \) is isomorphic to \( K_{1,4}, G_1 \) or \( G_2 \) given in figure 5.5
Proof Assume that $\gamma_{cp} + \chi = 2n - 3$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 3$ (or) $\gamma_{cp} = n - 1$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 2$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 3$ and $\chi = n$.

The cases $\gamma_{cp} = n - 1$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 3$ and $\chi = n$ are not possible.

Case 1 $\gamma_{cp} = n$ and $\chi = n - 3$.

Since $\gamma_{cp} = n$, by theorem [5.8] G is a star. Since $\chi = n - 3, 2 = n - 3$ so that $n = 5$. Hence $G \cong K_{1,4}$

Case 2 $\gamma_{cp} = n - 2$ and $\chi = n - 1$.

Since $\chi = n - 1, G$ contains a clique K on $n - 1$ vertices. Let x be a vertex other than the vertices of $K_{n-1}$. Since G is connected, x is adjacent to at least one vertex say $u_i$ of $K_{n-1}$.

If the clique $K_{n-1}$ has even number of vertices, then $\{x, u_i, u_j\}$ for some $u_j$ in $K_{n-1}$ forms a $\gamma_{cp}$-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 5$. 

Figure 5.5
Hence $K = K_4$. Let $u_1, u_2, u_3, u_4$ be the vertices of $K_4$. Let $x$ be adjacent to $u_1$.

If $d(x) = 1$, then $G \cong G_1$.

If $x$ is adjacent to one more vertex say $u_j$ of $K_{n-1}$, then $\{u_j\}$ is a $\gamma_{cp}$-set, which is a contradiction.

If the clique $K$ has an odd number of vertices, then $\{x, u\}$ is a $\gamma_{cp}$-set of $G$. Since $\gamma_{cp} = n - 2$, we have $n = 4$. Hence $K = K_3$. Let $u_1, u_2, u_3$ be the vertices of $K_3$. Let $x$ be adjacent to $u_1$. If $d(x) = 1$, then $G \cong G_2$. If $x$ is adjacent to one more vertex say $u_j$ in $K_{n-1}$, then $\{u_j\}$ is a $\gamma_{cp}$-set which is a contradiction. ■

**Theorem 5.23** For any connected graph $G$, $\gamma_{cp} + \chi = 2n - 4$ if and only if $G$ is isomorphic to $K_5, K_6, K_{1,5}, P_4, C_4$, or any one of the graphs $G_1$ to $G_{10}$ given in figure 5.6.
**Proof** Assume that $\gamma_{cp} + \chi = 2n - 4$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 4$ (or) $\gamma_{cp} = n - 1$ and $\chi = n - 3$ (or) $\gamma_{cp} = n - 2$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 3$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 4$ and $\chi = n$.

The cases for which $\gamma_{cp} = n - 1$ and $\chi = n - 3$ (or) $\gamma_{cp} = n - 3$ and $\chi = n - 1$ are not possible.
Case 1 \( \gamma_{cp} = n \) and \( \chi = n - 4 \).

Since \( \gamma_{cp} = n \), by theorem [5.8] \( G \) is a star. Since \( \chi = n - 4 \), \( 2 = n - 4 \) so that \( n = 6 \). Hence \( G \equiv K_{1,5} \)

Case 2 \( \gamma_{cp} = n - 2 \) and \( \chi = n - 2 \).

Since \( \chi = n - 2 \), \( G \) contains a clique \( K \) on \( n - 2 \) vertices. Let \( S = \{ x, y \} = V(G) - V(K) \). Then \( \langle S \rangle = K_2 \) or \( \overline{K}_2 \)

Subcase 1 \( \langle S \rangle = K_2 \)

Since \( G \) is connected, there exists a vertex say \( u_i \) in \( K_{n-2} \) which is adjacent to \( x \) (or equivalently \( y \)).

Now, Assume that the clique \( K_{n-2} \) has even number of vertices.

Then \( \{ y, u_i \} \) for \( i \neq j \) in \( K_{n-2} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n - 2 \), we have \( n = 4 \). Hence \( K = K_2 = uv \). Let \( x \) be adjacent to \( u \). If \( d(x) = 2 \) and \( d(y) = 1 \), then \( G \equiv P_4 \). If \( d(x) = 3 \), then \( \chi = 3 \), which is a contradiction.

Now let \( d(x) = d(y) = 2 \).

Without loss of generality let \( x \) be adjacent to \( u \). Then \( y \) is adjacent to \( u \) or \( v \). If \( y \) is adjacent to \( u \), then \( \chi = 3 \), which is a contradiction. If \( y \) is adjacent to \( v \), then \( G \equiv C_4 \)

Now assume that the clique \( K \) has odd number of vertices.
Then \( \{ y, x, u_i \} \) forms a \( \gamma_{cp} \) - set of \( G \). Since \( \gamma_{cp} = n - 2 \), we have \( n = 5 \).

Hence \( K = K_3 \). Let \( u_1, u_2, u_3 \) be the vertices of \( K_3 \). Without loss of generality let \( x \) be adjacent to \( u_1 \). If \( d(x) = 2 \) and \( d(y) = 1 \), then \( G \cong G_1 \). Let \( d(x) = 3 \) and \( d(y) = 1 \). Without loss of generality, let \( x \) be adjacent to both \( u_1 \) and \( u_3 \). Then \( G \cong G_2 \). If \( d(x) = 4 \) and \( d(y) = 1 \), then \( \chi = 4 \), which is a contradiction. Let \( d(x) = d(y) = 2 \). Let \( x \) be adjacent to \( u_1 \). Then \( y \) is adjacent to \( u_1 \) or \( u_3 \) (or equivalently \( u_2 \)). If \( y \) is adjacent to \( u_1 \), then \( \{ u_1 \} \) is a \( \gamma_{cp} \) set which is a contradiction. If \( y \) is adjacent to \( u_3 \), then \( G \cong G_3 \). Let \( d(x) = 2 \) and \( d(y) = 3 \). Let \( x \) be adjacent to \( u_1 \). Then \( y \) is adjacent to \( u_1 \) and one of \( \{ u_2, u_3 \} \) (or) \( y \) is adjacent to both \( u_2 \) and \( u_3 \). If \( y \) is adjacent to \( u_1 \) and \( u_2 \), then \( \{ u_1 \} \) is a \( \gamma_{cp} \) set which is a contradiction. If \( y \) is adjacent to \( u_2 \) and \( u_3 \), then \( G \cong G_4 \). Let \( d(x) = 2 \) and \( d(y) = 4 \), then \( \chi = 4 \), which is a contradiction. Let \( d(x) = d(y) = 3 \). Without loss of generality, let \( x \) be adjacent to \( u_1 \) and \( u_2 \). Then \( y \) is adjacent to \( u_1 \) and \( u_2 \) (or) \( y \) is adjacent to \( u_1 \) and \( u_3 \) (or equivalently \( u_2 \)). If \( y \) is adjacent to \( u_1 \) and \( u_2 \), then \( \chi = 4 \), which is a contradiction. If \( y \) is adjacent to \( u_3 \) and \( u_1 \), then \( \{ u_1 \} \) is a \( \gamma_{cp} \) - set, which is a contradiction.

**Subcase 2** \( < S > = K_2 \).

Since \( G \) is connected, \( x \) and \( y \) are adjacent to a common vertex or distinct vertices of \( K_{n-2} \).

**Subcase 2 (a)** Let \( x \) and \( y \) be adjacent to a common vertex say \( u_i \) of \( K_{n-2} \).
Now, assume that the clique $K_{n-2}$ has even number of vertices.
Then \{ $x, y, u_i, u_j$ \} for $i \neq j$ forms a $\gamma_{cp}$ - set of $G$. Since $\gamma_{cp} = n - 2$, we have $n = 6$. Hence $K = K_4$. Let $u_1, u_2, u_3, u_4$ be the vertices of $K_4$. Let $u_1$ be adjacent to both $x$ and $y$. If $d(x) = d(y) = 1$, then $G \cong G_5$. The case that $d(x) = 2$ and $d(y) = 1$ is not possible.

Now assume that the clique $K$ has odd number of vertices. Then \{ $x, y, u_i$ \} forms a $\gamma_{cp}$ - set of $G$. Since $\gamma_{cp} = n - 2$, we have $n = 5$. Hence $K = K_3$. Let $u_1, u_2, u_3$ be the vertices of $K_3$. Let $u_1$ be adjacent to both $x$ and $y$. If $d(x) = d(y) = 1$, then $G \cong G_6$. Let $d(x) = 2$ and $d(y) = 1$. Without loss of generality, let $x$ be adjacent to $u_1$ and $u_2$, then $G \cong G_7$. If $d(x) = 3$ and $d(y) = 1$, then $\chi = 4$, which is a contradiction. Let $d(x) = d(y) = 2$.Without loss of generality, let $x$ be adjacent to $u_1$ and $u_2$. Then $y$ is adjacent to $u_2$ or $u_3$. If $y$ is adjacent to $u_2$, then $G \cong G_8$. If $y$ is adjacent to $u_3$, then $\{ u_1 \}$ is a $\gamma_{cp}$ - set, which is a contradiction.

Subcase 2(b) Let $x$ and $y$ are adjacent to distinct vertices of $K_{n-2}$

Let $x$ be adjacent to $u_i$ and $y$ is adjacent to $u_j$ for $i \neq j$.

Assume that clique $K_{n-2}$ has even number of vertices.
Then \{ $x, y, u_i, u_j$ \} for $i \neq j$ forms a $\gamma_{cp}$ - set of $G$. Since $\gamma_{cp} = n - 2$, we have $n = 6$. Hence $K = K_4$. Let $u_1, u_2, u_3, u_4$ be the vertices of $K_4$. Without loss
of generality, let $x$ be adjacent to $u_1$ and $y$ be adjacent to $u_2$. If $d(x) = d(y) = 1$, then $G \cong G_9$. Let $d(x) = 2$ and $d(y) = 1$. Then $x$ is adjacent to $u_2$ or $u_3$ (or equivalently to $u_4$). If $x$ is adjacent to $u_2$, then $\{y, u_2\}$ forms a \( \gamma_{cp} \)-set of $G$, which is a contradiction. If $x$ is adjacent to $u_3$ (or equivalently $u_4$), then $\{y, u_4\}$ forms a \( \gamma_{cp} \)-set of $G$. If $d(x) = d(y) = 2$, then no graph exists.

Now assume that the clique $K$ has odd number of vertices. Then $\{x, y, u_1\}$ forms a \( \gamma_{cp} \)-set of $G$. Since $\gamma_{cp} = n - 2$, we have $n = 5$. Hence $K = K_3$. Let $u_1, u_2, u_3$ be the vertices of $K_3$. Without loss of generality, let $x$ be adjacent to $u_1$ and $y$ be adjacent to $u_2$. If $d(x) = d(y) = 1$, then $G \cong G_{10}$. Let $d(x) = 2$ and $d(y) = 1$. Then $x$ is adjacent to one more vertex say $u_2$ or $u_3$. If $x$ is adjacent to $u_3$, then $G \cong G_2$; If $x$ is adjacent to $u_2$, then $G \cong G_7$. If $d(x) = 3$ and $d(y) = 1$, then $\chi = 4$, which is a contradiction. If $d(x) = d(y) = 2$, then no new graph exists.

**Case 3** \( \gamma_{cp} = n - 4 \) and $\chi = n$.

Since $\chi = n$, $G$ is $K_n$. If $K_n$ has even number of vertices, then $\gamma_{cp} = 2$ and hence $n = 6$. Hence $G \cong K_6$. If $K_n$ has odd number of vertices then $\gamma_{cp} = 1$ and hence $n = 5$. Hence $G \cong K_5$. \[\square\]
Theorem 5.24  For any connected graph $G$, $\gamma_{cp} + \chi = 2n - 5$ if and only if $G$ is isomorphic to $K_6$, $K_7$, $K_{1,6}$ or any one of the graphs $G_1$ to $G_{33}$ given in figure 5.7.
$G_{19}$

$G_{20}$

$G_{21}$

$G_{22}$

$G_{23}$

$G_{24}$

$G_{25}$

$G_{26}$

$G_{27}$
Figure 6.5

**Proof** If $G$ is any of the graphs given in figure 6.5, then clearly $\gamma_{cp} + \chi = 2n - 5$. Conversely, assume that $\gamma_{cp} + \chi = 2n - 5$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 5$ (or) $\gamma_{cp} = n - 1$ and $\chi = n - 4$ (or) $\gamma_{cp} = n - 2$ and $\chi = n - 3$ (or) $\gamma_{cp} = n - 3$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 4$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 5$ and $\chi = n$.

The cases for which $\gamma_{cp} = n - 1$ and $\chi = n - 4$ (or) $\gamma_{cp} = n - 3$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 5$ and $\chi = n$ are not possible.
Case 1 \( \gamma_{cp} = n \) and \( \chi = n - 5 \).

Since \( \gamma_{cp} = n \), by theorem [5.8] \( G \) is a star. Since \( \chi = n - 5 \), \( 2 = n - 5 \) so that \( n = 7 \). Hence \( G \cong K_{1,6} \).

Case 2 \( \gamma_{cp} = n - 2 \) and \( \chi = n - 3 \).

Then \( G \) contains a clique \( K_{n-3} \), or \( G \) contains no \( K_{n-3} \).

**Let \( G \) contain a clique \( K_{n-3} \).**

Let \( S = \{ x, y, z \} = V(G) - V(K) \).

Then \( <S> = K_3 \) or \( K_3 \) or \( P_3 \) or \( K_2 \cup K_1 \).

**Subcase 1 \( <S> = K_3 \)**

Since \( G \) is connected, \( x \) is adjacent to some \( u_i \) in \( K_{n-3} \).

If \( K_{n-3} \) has even number of vertices, then \( \{ x, u_i, u_j \} \) for \( i \neq j \) in \( K_{n-3} \) forms a \( \gamma_{cp} \) - set of \( G \). Since \( \gamma_{cp} = n - 2 \), we have \( n = 5 \). But \( \chi = n - 3 = 2 \), which is a contradiction. Hence no graph exists in this case.

If \( K_{n-3} \) has odd number of vertices, then \( \{ x, u_i \} \) is a \( \gamma_{cp} \) - set of \( G \). Since a \( \gamma_{cp} = n - 2 \), we have \( n = 4 \). But \( \chi = n - 3 = 1 \), which is a contradiction. Hence no graph exists in this case.

**Subcase 2 \( <S> = K_3 \)**

Since \( G \) is connected, one of the vertices of \( K_{n-3} \) is adjacent to all the vertices of \( S \) or two vertices of \( S \) or one vertex of \( S \).
Subcase 2 (a) Let $u_i$ for some $i$ in $K_{n,3}$ be adjacent to all the vertices of $S$.

Now assume that the clique $K_{n,3}$ has even number of vertices.

Then $\{x, y, z, u_1, u_2\}$ is a $\gamma_{cp}$ - set of $G$. Since $\gamma_{cp} = n - 2$, we have $n = 7$. Hence $K = K_4$. Let $u_1, u_2, u_3, u_4$ be the vertices of $K_4$. Let $u_1$ be adjacent to all of $S$. If $d(x) = d(y) = d(z) = 1$, then $G$ is $G \cong G_1$.

Let $d(x) = 2, d(y) = d(z) = 1$. Let $x$ be adjacent to $u_2$. Then $\{y, z, u_2\}$ forms a $\gamma_{cp}$ - set of $G$, which is a contradiction.

Now assume that $K_{n,3}$ has odd number of vertices.

Then $\{x, y, z, u_1\}$ forms a $\gamma_{cp}$ - set of $G$. Since $\gamma_{cp} = n - 2$, we have $n = 6$. Hence $K = K_3$. Let $u_1, u_2, u_3$ be the vertices of $K_3$. Let $u_1$ be adjacent to all the vertices of $S$.

If $d(x) = d(y) = d(z) = 1$, then $G \cong G_2$.

If $d(x) = 2, d(y) = d(z) = 1$, then $G \cong G_3$.

If $d(x) = 3, d(y) = d(z) = 1$, then $\chi = 4$, which is a contradiction.

Let $d(x) = d(y) = 2$ and $d(z) = 1$.

Now, let $x$ be adjacent to $u_1$ and $u_2$. Then $y$ is adjacent to $u_2$ or $u_3$. If $y$ is adjacent to $u_2$, then $G \cong G_4$. If $y$ is adjacent to $u_3$, then $\{z, u_1\}$ forms a $\gamma_{cp}$ - set of $G$, which is a contradiction.

Now, let $d(x) = d(y) = d(z) = 2$.
If \( d(x) = d(y) = 2 \) and \( d(z) = 1 \), then the graph is \( G_4 \). Now in \( G_4 \), \( z \) is adjacent to \( u_3 \) or \( u_2 \). If \( z \) is adjacent to \( u_3 \), then \( \{ y, u_1 \} \) forms a \( \gamma_{cp} \) - set, which is a contradiction. If \( z \) is adjacent to \( u_2 \), then \( G \cong G_5 \).

**Subcase 2 (b)** Let \( u_i \) for some \( i \) in \( K_{n-3} \) is adjacent to \( x \) and \( y \), and \( u_j \) for some \( i \neq j \) in \( K_{n-3} \) is adjacent to \( z \).

Now assume that \( K_{n-3} \) has even number of vertices.

Then \( \{ x, y, z, u_i, u_j \} \) forms a \( \gamma_{cp} \) - set of \( G \) so that \( n = 7 \). Hence \( K = K_4 \). Let \( u_1, u_2, u_3, u_4 \) be the vertices of \( K_4 \). Let \( u_1 \) be adjacent to \( x \) and \( y \) and let \( u_2 \) be adjacent to \( z \). If \( d(x) = d(y) = d(z) = 1 \), then \( G \cong G_6 \).

Let \( d(x) = 2, d(y) = d(z) = 1 \).

If \( d(x) = d(y) = d(z) = 1 \), then the graph is \( G_6 \). In \( G_6 \), \( x \) is adjacent to \( u_2 \) or \( u_3 \) ( or equivalently \( u_4 \) ). If \( x \) is adjacent to \( u_2 \) then \( \{ y, z, u_1 \} \) forms a \( \gamma_{cp} \) - set of \( G \), which is a contradiction; if \( x \) is adjacent to \( u_3 \), then \( \{ y, z, u_1 \} \) forms a \( \gamma_{cp} \) - set of \( G \) which is a contradiction.

Let \( d(x) = d(y) = 1 \) and \( d(z) = 2 \).

If \( d(x) = d(y) = d(z) = 1 \), then the graph is \( G_6 \). In \( G_6 \), \( z \) is adjacent to \( u_1 \) or \( u_3 \) ( or equivalently \( u_4 \) ). If \( z \) is adjacent to \( u_1 \) then \( \{ x, y, u_1 \} \) forms a \( \gamma_{cp} \) - set of \( G \), which is a contradiction; if \( z \) is adjacent to \( u_3 \), then \( \{ x, y, u_3 \} \) forms a \( \gamma_{cp} \) - set of \( G \) which is a contradiction.

Now assume that \( K_{n-3} \) has odd number of vertices.
Then \{ x, y, z, u_1 \} forms a \( \gamma_{cp} \) - set of G and hence \( n = 6 \). Hence K = K_3. Let \( u_1, u_2, u_3 \) be the vertices of K_3. Let \( u_1 \) be adjacent to both x and y and let \( u_2 \) be adjacent to z. If \( d(x) = d(y) = d(z) = 1 \), then \( G \cong G_7 \).

Let \( d(x) = 2 \), and \( d(y) = d(z) = 1 \).

Now, if \( d(x) = d(y) = d(z) = 1 \), then the graph is \( G_7 \). In \( G_7 \), x is adjacent to \( u_2 \) or \( u_3 \). If x is adjacent to \( u_2 \), then \( G \cong G_7 \); if x is adjacent to \( u_3 \), then \( G \cong G_9 \). If \( d(x) = 3 \) and \( d(y) = d(z) = 1 \). Then \( \chi = 4 \), which is a contradiction.

Now Let \( d(x) = d(y) = 2 \) and \( d(z) = 1 \).

If \( d(x) = 2 \), and \( d(y) = d(z) = 1 \), then the graphs are \( G_8 \) or \( G_9 \). Now in \( G_8 \), y is adjacent to \( u_2 \) or \( u_3 \). If y is adjacent to \( u_2 \), then \( G \cong G_8 \); If y is adjacent to \( u_3 \), then \{z, u_1\} forms a \( \gamma_{cp} \) - set of G, which is a contradiction. In \( G_9 \), y is adjacent to \( u_2 \) or \( u_3 \). If y is adjacent to \( u_2 \), then \{z, u_1\} forms a \( \gamma_{cp} \) - set of G, which is a contradiction; If y is adjacent to \( u_3 \), then \( G \cong G_{10} \).

Now Let \( d(x) = d(y) = d(z) = 2 \)

If \( d(x) = d(y) = 2 \) and \( d(z) = 1 \), then the graph is \( G_4 \) or \( G_{10} \). Now in \( G_4 \), z is adjacent to \( u_2 \) or \( u_3 \). If z is adjacent to \( u_2 \), then \( G \cong G_4 \); If z is adjacent to \( u_3 \), then \{y, u_1\} forms a \( \gamma_{cp} \) set of G, which is a contradiction. In \( G_{10} \), z is adjacent to \( u_1 \) or \( u_3 \). If z is adjacent to \( u_1 \), then \{u_1, y\} forms a \( \gamma_{cp} \)
set of $G$, which is a contradiction. If $z$ is adjacent to $u_3$, then $\{x, u_3\}$ forms a $\gamma_{cp}$-set of $G$, which is a contradiction.

Now let $d(x) = d(y) = 1$ and $d(z) = 2$.

If $d(x) = d(y) = d(z) = 1$, then the graph is $G_7$. In $G_7$, $z$ is adjacent to $u_1$ or $u_3$. If $z$ is adjacent to $u_1$, then $G \cong G_3$; if $z$ is adjacent to $u_3$, then $G \cong G_{11}$.

Now let $d(x) = 2, d(y) = 1$ and $d(z) = 2$.

Now if $d(x) = 2, d(y) = d(z) = 1$, then the graphs are $G_8$ or $G_9$. In $G_8$, $z$ is adjacent to $u_1$ or $u_3$. If $z$ is adjacent to $u_1$, then $G \cong G_4$; if $z$ is adjacent to $u_3$, then $\{y, u_2\}$ forms a $\gamma_{cp}$ set of $G$, which is a contradiction. In $G_9$, $z$ is adjacent to $u_1$ or $u_3$. If $z$ is adjacent to $u_1$, then $\{y, u_1\}$ forms a $\gamma_{cp}$ set of $G$, which is a contradiction. If $z$ is adjacent to $u_3$, then $\{y, u_3\}$ forms a $\gamma_{cp}$ set of $G$, which is a contradiction.

If let $d(x) = d(y) = 1$ and $d(z) = 3$. Then $\chi = 4$, which is a contradiction.

**Subcase 2 (c)** Let $u_i$ be adjacent to $x$ and $u_j$ for $i \neq j$ be adjacent to $y$ and $u_k$ for $i \neq j \neq k$ be adjacent to $z$.

Assume that the clique $K_{n,3}$ has even number of vertices.

Then $\{x, y, z, u_i, u_j\}$ forms a $\gamma_{cp}$-set of $G$ so that $n = 7$. 

89
Hence $K = K_4$. Let $u_1, u_2, u_3, u_4$ be the vertices of $K_4$. Let $u_1$ be adjacent to $x$ and $u_2$ be adjacent to $y$ and $u_3$ be adjacent to $z$.

If $d(x) = d(y) = d(z) = 1$, then $G \simeq G_{12}$.

Let $d(x) = 2$ and $d(y) = d(z) = 1$. Then clearly no graph exists satisfying the hypothesis.

Assume that the Clique $K_{n-3}$ has odd number of vertices.

Then $\{ x, y, z, u_i \}$ forms a $\gamma_{cp}$-set of $G$ and hence $n = 6$. Hence $K = K_3$. Let $u_1, u_2, u_3$ be the vertices of $K_3$. Let $u_1$ be adjacent to $x$ and $u_2$ be adjacent to $y$ and $u_3$ be adjacent to $z$. If $d(x) = d(y) = d(z) = 1$, then $G \simeq G_{13}$.

If $d(x) = 2$ and $d(y) = d(z) = 1$, then $G \simeq G_{14}$.

If $d(x) = 3$ and $d(y) = d(z) = 1$, then $\chi = 4$, which is a contradiction.

If $d(x) = d(y) = 2$ and $d(z) = 1$, then $G \simeq G_{10}$.

If $d(x) = d(y) = d(z) = 2$, then no graph exists satisfying the hypothesis.

**Subcase 3** $\langle S \rangle = P_3 = x \ y \ z$.

Assume that the clique $K = K_{n-3}$ have even number of vertices.

Since $G$ is connected, at least one of the vertices say $u_i$ of $K_{n-3}$, is adjacent to $x$ (or equivalently $z$) or $y$.

If $u_i$ is adjacent to $x$, then $\{ z, u_i, u_j \}$ for $i \neq j$ forms a $\gamma_{cp}$ set of $G$.

Since $\gamma_{cp} = n - 2$, we have $n = 5$. Hence $K = K_2 = uv$. Let $x$ be adjacent to $u$.
If \( d(x) = d(y) = 2 \) and \( d(z) = 1 \), then \( G \cong P_5 \). If \( d(x) = 3 \), then \( \chi = 3 \), which is a contradiction. If \( d(x) = d(y) = d(z) = 2 \), then \( G \cong G_{15} \).

If \( u_i \) is adjacent to \( y \), then \( \{ x, z, u_j \} \) for \( i \neq j \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n - 2 \), \( n = 5 \). Hence \( K = K_2 = uv \). Let \( u \) be adjacent to \( y \). If \( d(x) = d(z) = 1 \) and \( d(y) = 3 \), then \( G \cong G_{16} \). In all other cases on the degrees of the vertices of \( x, y \) and \( z \), no new graph exists satisfying the hypothesis.

Assume that the clique \( K_n \) have odd number of vertices.

Since \( G \) is connected, at least one of the vertices \( u_i \) of \( K_n \), is adjacent to \( x \) (or equivalently \( z \)) or \( y \).

If \( u_i \) is adjacent to \( x \), then \( \{ z, u_i \} \) forms a \( \gamma_{cp} \) - set of \( G \). Since \( \gamma_{cp} = n - 2 \), we have \( n = 4 \). Hence \( K = K_1 \) which is a contradiction.

If \( u_i \) is adjacent to \( y \), then \( \{ x, z, u_j, u_k \} \) forms a \( \gamma_{cp} \) - set of \( G \) and hence \( n = 6 \). Hence \( K = K_3 \). Let \( u_1, u_2, u_3 \) be the vertices of \( K_3 \). Let \( u_1 \) be adjacent to \( y \).

Let \( d(y) = 3 \).

If \( d(x) = d(z) = 1 \), then \( G \cong G_{17} \). In all other cases on the degrees of \( x \) and \( z \), no new graph exists satisfying the hypothesis.

Let \( d(y) = 4 \).
If \( d(x) = d(z) = 1 \), then \( G \cong G_{11} \). In all other cases on the degrees of \( x \) and \( z \), no new graph exists satisfying the hypothesis.

If \( d(y) = 5 \), then \( \chi = 5 \) which is a contradiction.

**Subcase 4** \( \langle S \rangle = K_2 \cup K_1 \).

Let \( xy \) be the edge in \( \langle S \rangle \).

Assume that the clique \( K_{n-3} \) have even number of vertices.

Since \( G \) is connected \( x \) (or equivalently \( y \)) is adjacent to atleast one of the vertices say \( u_i \) of \( K_{n-3} \). Without loss of generality let \( x \) be adjacent to \( u_i \).

Then \( z \) is adjacent to the same \( u_i \) or \( u_j \) for \( i \neq j \).

If \( z \) is adjacent to \( u_i \), then \( \{ y, z, u_j \} \) for \( i \neq j \) forms a \( \gamma_{cp} \) - set of \( G \) and hence \( n = 5 \). Hence \( K = K_2 = uv \). Let \( u \) be adjacent to both \( x \) and \( z \). If \( d(x) = 2 \) and \( d(y) = d(z) = 1 \), then \( G \cong G_{16} \); If \( d(x) = d(y) = 2 \) and \( d(z) = 1 \), then \( G \cong G_{15} \). Let \( d(x) = d(y) = d(z) = 2 \). Then \( \chi = 3 \), which is a contradiction.

If \( z \) is adjacent to \( u_i \) for \( i \neq j \), then \( \{ y, z, u_j \} \) forms a \( \gamma_{cp} \) - set of \( G \) and hence \( n = 5 \). Hence \( K = K_2 = uv \). Let \( x \) be adjacent to \( u \) and \( z \) be adjacent to \( v \). If \( d(x) = 2 \) and \( d(y) = d(z) = 1 \), then \( G \cong P_5 \). All other cases on the degrees of \( x, y \) and \( z \) no new graph exists satisfying the hypothesis.

Assume that the clique \( K_{n-3} \) has odd number of vertices.
Since $G$ is connected $x$ (or equivalently $y$) is adjacent to at least one of the vertices say $u_i$ of $K_{n,3}$. Without loss of generality let $x$ be adjacent to $u_i$. Then $z$ is adjacent to the same $u_i$ or $u_j$ for $i \neq j$.

If $z$ is adjacent to $u_i$, then $\{y, z, u_j, u_k\}$ for $i \neq j \neq k$ forms a $\gamma_{cp}$-set of $G$ and hence $n = 6$. Hence $K = K_3$. Let $u_1, u_2, u_3$ be the vertices of $K_3$. Let $u_1$ be adjacent to both $x$ and $z$. If $d(x) = 2$ and $d(y) = d(z) = 1$, then $G \cong G_{18}$.

Let $d(x) = d(y) = 2$ and $d(z) = 1$.

If $d(x) = 2$ and $d(y) = d(z) = 1$, then the graph is $G_{18}$. In $G_{18}$, $y$ is adjacent to $u_1$ or $u_2$ (or equivalently $u_3$). If $y$ is adjacent to $u_1$, then $\{z, u_1\}$ forms a $\gamma_{cp}$-set of $G$, which is a contradiction; if $y$ is adjacent to $u_2$, then $G \cong G_{19}$.

Let $d(x) = d(y) = d(z) = 2$.

Now if $d(x) = d(y) = 2$ and $d(z) = 1$, then the graph is $G_{19}$. In $G_{19}$, $z$ is adjacent to $u_2$ or $u_3$. If $z$ is adjacent to $u_2$, then $G \cong G_{20}$; if $z$ is adjacent to $u_3$, then $\{x, u_1\}$ forms a $\gamma_{cp}$-set of $G$, which is a contradiction.

If $d(x) = d(y) = 2$ and $d(z) = 3$, then $\chi = 4$, which is a contradiction.

If $d(x) = 3$ and $d(y) = d(z) = 1$, then $G \cong G_9$.

Now let $d(x) = 3$, $d(y) = 2$ and $d(z) = 1$. 

93
Now if \( d(x) = d(y) = 2, \ d(z) = 1 \), then the graph is \( G_{19} \). In \( G_{19} \), \( x \) is adjacent to \( u_2 \) or \( u_3 \). If \( x \) is adjacent to \( u_2 \), then \( \{ z, u_2 \} \) forms a \( \gamma_{cp} \)-set of \( G \), which is a contradiction; if \( x \) is adjacent to \( u_3 \), then \( G \cong G_{21} \).

If \( d(x) = 3, \ d(y) = 2 \) and \( d(z) = 2 \), then no new graph exists.

If \( d(x) = 3, \ d(y) = 3 \) and \( d(z) = 1 \), then no new graph exits.

If \( d(x) = 2, \ d(y) = 3 \) and \( d(z) = 1 \), then \( G \cong G_{22} \).

If \( d(x) = 2, \ d(y) = 3 \) and \( d(z) = 2 \), then no new graph exits.

If \( z \) is adjacent to \( u_j \), then \( \{ y, z, u_j, u_k \} \) for \( i \neq j \neq k \) forms a \( \gamma_{cp} \)-set of \( G \) and hence \( n = 6 \). Hence \( K = K_3 \). Let \( u_1, u_2, u_3 \) be the vertices of \( K_3 \). Let \( u_1 \) be adjacent to \( x \) and \( u_2 \) be adjacent to \( z \).

If \( d(x) = 2 \) and \( d(y) = d(z) = 1 \), then \( G \cong G_{23} \).

Now let \( d(x) = d(y) = 2 \) and \( d(z) = 1 \).

Now if \( d(x) = 2 \) and \( d(y) = d(z) = 1 \), then the graph is \( G_{23} \). In \( G_{23} \), \( y \) is adjacent to \( u_1 \) or \( u_2 \) or \( u_3 \). If \( y \) is adjacent to \( u_1 \), then \( \{ u_1, z \} \) is a \( \gamma_{cp} \) set of \( G \) which is a contradiction; if \( y \) is adjacent to \( u_2 \), then \( G \cong G_{19} \); If \( y \) is adjacent to \( u_3 \), then \( G \cong G_{24} \).

If \( d(x) = d(y) = d(z) = 2 \), then no new graph exists satisfying the hypothesis.

Let \( d(x) = 3 \) and \( d(y) = d(z) = 1 \).
Now, if \( d(x) = 2 \) and \( d(y) = d(z) = 1 \), the graph is \( G_{23} \). In \( G_{23} \), \( x \) is adjacent to \( u_2 \), or \( u_3 \). If \( x \) is adjacent to \( u_2 \), then \( G \cong G_{25} \); If \( x \) is adjacent to \( u_3 \), then \( G \cong G_{26} \).

Let \( d(x) = 3 \), \( d(y) = 2 \) and \( d(z) = 1 \).

Now, if \( d(x) = 3 \) and \( d(y) = d(z) = 1 \), then the graphs are \( G_{25} \) or \( G_{26} \).

In \( G_{25} \), \( y \) is adjacent to \( u_1 \), or \( u_2 \), or \( u_3 \). If \( y \) is adjacent to \( u_1 \), then \( \{ u_1, z \} \) forms a \( \gamma_{cp} \) set of \( G \), which is a contradiction; If \( y \) is adjacent to \( u_2 \), then \( \{ z, u_2 \} \) forms a \( \gamma_{cp} \)-set of \( G \), which is a contradiction; If \( y \) is adjacent to \( u_3 \), then \( \{ z, u_3 \} \) forms a \( \gamma_{cp} \) set of \( G \), which is a contradiction. In \( G_{26} \), \( y \) is adjacent to \( u_1 \) or \( u_2 \) or \( u_3 \). If \( y \) is adjacent to \( u_1 \), then \( \{ u_1, z \} \) forms a \( \gamma_{cp} \)-set of \( G \), which is a contradiction. If \( y \) is adjacent to \( u_2 \), then \( \{ z, u_3 \} \) forms a \( \gamma_{cp} \)-set of \( G \), which is a contradiction; if \( y \) is adjacent to \( u_3 \), then \( \{ z, u_3 \} \) forms a \( \gamma_{cp} \)-set of \( G \), which is a contradiction.

If \( d(x) = 3 \), \( d(y) = 2 \) and \( d(z) = 2 \), then no new graph exists.

Now let \( G \) contain no \( K_{n-3} \).

Then clearly \( n \geq 6 \).

If \( n = 6 \), then \( \gamma_{cp} = 4 \), and \( \chi = 3 \) and \( G \) contains no \( K_3 \). Therefore \( G \) contains \( C_5 \), since \( \chi = 3 \). Let \( x \) be a vertex of \( G \) which is not in \( C_5 \). Since \( G \) is connected and since \( G \) contains no \( K_3 \), \( x \) cannot be adjacent to two adjacent

95
vertices of $C_5$. i.e., $d(x) = 1$ or $2$ and hence the only possible graphs are isomorphic to $G_{27}$ or $G_{28}$.

If $n \geq 8$, then $\gamma_{cp} = n - 2$ and $\chi \geq 5$ and $G$ contains no $K_5$. In this case, if $S$ is a $\gamma_{cp}$- set of $G$, then $\langle S \rangle$ cannot contain $K_3$ or $P_4$ (otherwise $\gamma_{cp}(G) \leq n - 2$). Therefore $\langle S \rangle$ is acyclic and hence $\chi(\langle S \rangle) = 2$. This implies that $\chi(G) \leq 4$, which is a contradiction.

If $n = 7$, then $\gamma_{cp} = 5$ and $\chi = 4$, $G$ contains no $K_4$.

If $S$ is a $\gamma_{cp}$- set of $G$, then $\langle S \rangle$ is any one of the following graphs given in figure 5.8.

![Graphs H1 to H7](image-url)
If \( < S > \geq H_7 \), then \( \chi(G) \leq 3 \), which is a contradiction.

If \( < S > = H_1 \) to \( H_6 \), then \( \chi( < S > ) = 2 \), and since \( \chi(G) = 4 \), \( G \) contains \( K_4 \), which is a contradiction.

**Case 3** \( \gamma_{cp} = n - 4 \) and \( \chi = n - 1 \).

Since \( \chi = n - 1 \), \( G \) contains a clique \( K \) on \( n - 1 \) vertices. Let \( x \) be the vertex other than the vertices of \( K_{n-1} \). Since, \( G \) is connected, \( x \) is adjacent to at least one of the vertices say \( u_i \) of \( K_{n-1} \).

Now assume that the clique \( K_{n-1} \) has even number of vertices.

Then \( \{x, u_i, u_j\} \) for \( i \neq j \) forms a \( \gamma_{cp} \) - set of \( G \). Since \( \gamma_{cp} = n - 4 \), we have \( n = 7 \). Hence \( K = K_6 \). Let \( u_1, u_2, u_3, u_4, u_5, u_6 \) be the vertices of \( K_6 \). Let \( x \) be adjacent to \( u_1 \). If \( d(x) = 1 \), then \( G \cong G_{29} \). If \( d(x) = 2 \), then \( \{u_1\} \) forms a \( \gamma_{cp} \) - set of \( G \), which is a contradiction.

Now assume that the clique \( K_{n-1} \) has odd number of vertices

Then \( \{x, u_i\} \) forms a \( \gamma_{cp} \) - set of \( G \). Since \( \gamma_{cp} = n - 4 \), we have \( n = 6 \). Hence \( K = K_5 \). Let \( u_1, u_2, u_3, u_4, u_5 \), be the vertices of \( K_5 \). If \( d(x) = 1 \), then \( G \cong G_{30} \). If \( d(x) = 2 \), then \( G \cong G_{31} \). If \( d(x) = 3 \), then \( G \cong G_{32} \). If \( d(x) = 4 \), then \( G \cong G_{33} \).

**CONCLUSION**

In this chapter, we introduced the concept of complementary perfect dominating set which is likely to have a good application. The
corresponding parameter was determined for some standard class of graphs.

We also found an upper bound for the sum of this parameter and chromatic number of a graph and characterized the corresponding extremal graphs. This concept has lot of scope for further research. The characterization of the class of graphs for which \( \gamma_{cp} = 2 \) is an interesting one. The relation of this concept with other graph theoretic parameter will produce several results. Some of the possibilities are given below.

1. Characterize the class of graphs for which \( \gamma_{cp} = \chi \).

2. Relation of \( \gamma_{cp} \) with \( \gamma_{pr} \).

3. Nordhaus – Gaddum type of results for \( \gamma_{cp} \).