4. Discontinuous Logistic Map

One dimensional iterative maps on an interval of the real axis have been used in modelling a wide variety of nonlinear systems. The most prominent route to chaos in these maps is through the Feigenbaum period doubling bifurcations. The transition from periodic to chaotic state through an infinite sequence of pitch fork bifurcations has been found to be a common feature of all unimodal functions having negative Schwarzian derivative [1]. One of the most extensively analyzed maps in this context is the logistic map [1,10,15,48]. The evolution of the system from regular to chaotic state has been investigated not only with the control parameter in its pure form, but also in some or other modified forms leading to a class of modulated logistic maps [58-64,75]. A remarkable property of the Feigenbaum scenario of bifurcations is the geometric convergence of the parameter values for successive bifurcations. This is found to be a universal property of almost all one dimensional continuous maps with a single hump. The presence of a discontinuity or asymmetry in the map, however, produces a considerable change in the bifurcation structure of the system. A 'new road to chaos' has been identified by de Souza Vieira et al., based on the numerical investigations on a discontinuous logistic map [65,66]. The existence of inverse cascades of bifurcations in which the periods change arithmetically is a novel aspect in these systems. Certain empirical rules for
the existence of inverse and direct cascades have also been reported [67]. The number of inverse cascades and the type of route to chaos depend on the location of the discontinuity as well [68]. The periods of the cycles depend highly on the precision of the computations [69]. The phenomenon of border-collision bifurcations and the formation of inverse and direct cascades in one-dimensional piecewise smooth maps have also been investigated [70-72]. A possibility for having a discontinuous bifurcation from any selected orbit of the period-doubling cascade to an orbit of the inverse or direct cascade has also been reported [72]. A bifurcation phenomenon different from the standard period-doubling one has been observed for a piecewise cubic map [74] and also in an experimental situation in a He-Ne laser system [73]. Similar bifurcations have been observed for other piecewise continuous quadratic maps like the circle map [76] and the logistic-like sawtooth map [77]. Inverse cascades of bifurcations together with direct cascades had been reported for certain Hamiltonian systems [78]. Such Hamiltonian systems exhibit only a few continuous bifurcations in which the periods change geometrically. But, the discontinuous logistic map shows a number of discontinuous bifurcations in which the periods decrease in an arithmetic progression. The whole set of bifurcations can be viewed as an alternate route to chaos, with scaling laws different from that of Feigenbaum's [66]. Most of the studies in these systems are numerical. We provide an analytical explanation for the bifurcation phenomenon of a discontinuous logistic map. Numerical findings in support of the results are also included.
4.1 Analysis of the Discontinuous Logistic Map; Coexistence of Multiple Attractors and their Basins of Attraction

We consider the logistic map with a discontinuity at the centre. The map is defined as,

\[ x_{n+1} = 4\mu x_n (1 - x_n); \text{for } 0 < x_n \leq 1/2. \]
\[ x_{n+1} = 4\mu x_n (1 - x_n) + C; \text{for } 1/2 < x_n < 1. \] (4.1)

where \( \mu \) is the control parameter and \( C \) is a constant representing the strength of the discontinuity. This map corresponds to a special case of the more general asymmetric map that has been numerically investigated by De Sousa Vieira et al., [65] given by

\[ x_{t+1} = 1 - \epsilon_1 - a_1 |x_t|^{\rho_1}; \quad x_t > 0. \]
\[ x_{t+1} = 1 - \frac{1}{2} (\epsilon_1 + \epsilon_2); \quad x_t = 0 \] (4.2)
\[ x_{t+1} = 1 - \epsilon_2 - a_2 |x_t|^{\rho_2}; \quad x_t < 0. \]

The map given by equation 4.1 differs slightly from the one referred to above in that the value of the function at the point of discontinuity is also included in the left half of the interval of the mapping. Also, the interval of the mapping is (0,1) instead of (-1,1) in equation 4.2. Again it differs from the discontinuous map analyzed by Chia & Tan [67] in that the discontinuity parameter \( C \) is introduced in the right half of the interval in our case instead of in the left.
part as in the case of Chia & Tan. De souza Vieira et al., numerically investigated the effect of three type of asymmetries:

a) $\epsilon_1 = \epsilon_2 = 0$, $z_1 = z_2 \equiv z$, $a_1 \neq a_2$; b) $\epsilon_1 = \epsilon_2 = 0$, $a_1 = a_2 \equiv a$, $z_1 \neq z_2$ and c) $a_1 = a_2 \equiv a$, $z_1 = z_2 \equiv z$, $\epsilon_1 \neq \epsilon_2$. They have verified that the Feigenbaum scenario for one-dimensional one-extremum maps gets strongly modified if asymmetry is introduced in the extremum. Amplitude asymmetry ($a_1 \neq a_2$) and exponent asymmetry ($z_1 \neq z_2$) have relatively minor influence on the bifurcation pattern whereas the discontinuity in the extremum drastically alters the bifurcation phenomenon. With typical choices of $(\epsilon_1, \epsilon_2) = (0, 0.1)$ and $(0.1, 0)$, they have observed various inverse cascades of bifurcations. The periods of the cycles were found to decrease in arithmetic progressions, as $a$ is increased. In the case of $(\epsilon_1, \epsilon_2) = (0, 0.1)$ for example, the first cascade of bifurcations occur immediately above $a = 1$, with periods like $16 \rightarrow 14 \rightarrow 12 \rightarrow 10 \rightarrow 8 \rightarrow 6 \rightarrow 4$. Just above this cascade, some standard pitch-fork bifurcations are observed. Again inverse cascades with periods...

...76 $\rightarrow$ 58 $\rightarrow$ 40 $\rightarrow$ 22; ...92 $\rightarrow$ 70 $\rightarrow$ 48 $\rightarrow$ 26; ...134 $\rightarrow$ 108 $\rightarrow$ 82 $\rightarrow$ 56 ..., etc. are observed. Tan & Chia [67] have given certain empirical rules for the existence of inverse and direct cascades. Note that in all the cases considered for the discontinuous logistic map, the common difference of the progressions are even numbers. This observation is of much significance in our analytical investigations.
Fig 4.1 The discontinuous map defined in eqn 4.1. The discontinuity parameter $C = 0.2$ and the control parameter $\mu = 0.4$. Note that the two fixed points $x_i^*$ and $x_r^*$ coexist in this case.
For convenience, we write equation 4.1 in the form, \( x_{n+1} = T(x_n) \),
where the mapping \( T \) is such that \( T(x) = f(x) = 4\mu x(1 - x) \) for \( 0 < x \leq 1/2 \) and \( T(x) = \phi(x) = 4\mu x(1 - x) + C \) for \( 1/2 < x < 1 \). The qualitative shape of the map function is shown in figure 4.1. In the left half of the interval \((0,1)\), the curve is logistic in nature and the function increases from 0 to \( \mu \) as \( x \) increases from 0 to 1/2. As \( x \) increases from 1/2 to 1, the map function \( \phi(x) \) decreases monotonically from \( (\mu + C) \) to \( C \). The presence of such discontinuities (i.e., different \( x \rightarrow (1/2)_{+} \) and \( x \rightarrow (1/2)_{-} \) behaviour) in physical systems \([10,79]\) perturbs the dynamics of them considerably. For every value of \( C \), the control parameter \( \mu \) is varied from 0 to \( (1 - C) \) so as to confine the iterates within the unit interval. The fixed points of the system are determined by \( T(x^*) = x^* \). Depending on the relative values of \( \mu \) and \( C \), there will be one or more fixed points. The fixed point of the left part is \( x^* = 0 \) for values of \( \mu \) ranging from 0 to 1/4. When \( \mu > 1/4 \), the fixed point from the left part is
\[
x^*_1 = 1 - \frac{1}{4\mu}
\]  
(4.3)

The fixed point arising from the right part is given by \( \phi(x^*) = x^* \). i.e., \( 4\mu x^*(1 - x^*) + C = x^* \). This quadratic equation gives the solution,
\[
x^* = \frac{((4\mu - 1) \pm [(4\mu - 1)^2 + 16\mu C]^{1/2})}{8\mu}
\]  
(4.4)

Since the negative root for \( x^* \) is inadmissible, we have the fixed point to the right of \( x = 1/2 \) as
\[ x^*_i = \frac{1}{2} + \frac{\sqrt{(4\mu - 1)^2 + 16\mu C - 1}}{8\mu} \]  

Thus we have simultaneous occurrence of attractors \( x_i^* \) and \( x_i^* \). In the limit of \( C \to 0 \), we get \( x_i^* \to x_i^* = 1 - (1/4\mu) \). This is understandable since the pure logistic map has only one attractor. The fixed point \( (x_i^*) \) and its stability properties depend on the two parameters \( \mu \) and \( C \) so that one can have a desired dynamics for the system by proper choice of \( \mu \) and \( C \) as in the case of combination maps [80,81]. The asymptotic state of the system is determined by either \( x_i^* \) or \( x_i^* \) or both, depending on the values of \( \mu \) and \( C \). Keeping \( C \) fixed, let \( \mu \) be varied from 0 to \( (1 - C) \). The fixed point of the system is \( x^* = 0 \) for values of \( \mu \) ranging from 0 to 1/4. When \( \mu \) increases beyond 1/4, the zero fixed point loses its stability and becomes a repellent while the only attracting fixed point on the left part is \( x_i^* \). As \( \mu \) increases from 1/4 to 1/2, \( x_i^* \) moves from 0 to 1/2. When the control parameter exceeds the value \( \mu = 1/2 \), the left half of the map lies above the bisector line. Thus the left attractor \( (x_i^*) \) ceases to exist beyond \( \mu = 1/2 \). To the right of \( x = 1/2 \), the map is of the nature of a combination map, obtained by combining the logistic map with an additive constant. The fixed point \( (x_i^*) \) manifests itself whenever the parameter \( \mu \) is so tuned as to make \( x_i^* > (1/2) \). For this we have from eqn.4.5, \[ \{(4\mu - 1)^2 + 16\mu C\}^{1/2} - 1 > 0 \] This necessitates \( \mu + C > 1/2 \). i.e., \( \mu > (1/2 - C) \). This criterion for the appearance of \( x_i^* \) can also be inferred from figure 4.1, as the condition for the height of the extremum when approached
from the right of \( x = 1/2 \) to be greater than 1/2. If \( \mu + C < 1/2 \), then the right branch \( \phi(x) \) of the curve lies wholly below the \( y = x \) line. Thus we see that the \( x^*_1 \) exists for all \( \mu > (1/2 - C) \) and \( x^*_1 \) appears for \( 1/4 < \mu \leq 1/2 \).

Hence the two attractors co-exist in the parameter range \( (1/2 - C) < \mu \leq 1/2 \).

For values of \( \mu < (1/2 - C) \), only \( x^*_1 \) exists and for \( \mu > 1/2 \) only \( x^*_2 \) exists.

When the two attractors co-exist, there are two different basins of attraction.

The bifurcation structure will therefore depend on the initial value used for iteration. Multiple basins of attraction for one dimensional maps of a single hump are usually uncommon. That it is seen in our case is a consequence of the fact that the map is represented by two different functions (\( f \) and \( \phi \)) on either side of \( x = 1/2 \).

The basin of attraction for a fixed point is the set of initial points \( (x_0) \) that converge asymptotically to that fixed point. The basins of attraction of the attractors of the map can be obtained as follows. The mapping \( T \) is such that the left half \( f(x) \) increases monotonically with \( x \), while the right part \( \phi(x) \) decreases monotonically with \( x \). Let \( x_r \) be the pre-image of 1/2 on the right, i.e., \( x_r = \phi^{-1}(1/2) \). Or, \( 4\mu x_r(1 - x_r) + C = 1/2 \). This gives,

\[
x_r = 1/2 + \frac{\sqrt{16\mu(\mu + C - 1/2)}}{8\mu}
\]

(4.6)

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The pre-image of $1/2$ on the left is

$$x_i = 1/2 - \frac{1}{\sqrt{\frac{1}{4} - \frac{1}{8\mu}}} \quad (4.7)$$

For the logistic map $x_{n+1} = 4\mu x_n(1 - x_n)$, the two pre-images of $1/2$ are $1/2 \pm \sqrt{\frac{1}{4} - \frac{1}{8\mu}}$ which are symmetric w.r.t $x = 1/2$. But, for the discontinuous map, the pre-image of $1/2$ on the right of $x = 1/2$ is shifted further to the right (for positive $C$). This asymmetry in the two pre-images play a vital role in the nature of bifurcations of the map, as the iterates approach the point of discontinuity. For values of $x$ in $(1/2, x_r)$, the function $\phi(x)$ decreases from $(\mu + C)$ to $1/2$. For all initial values $x_0$ greater than $x_r$, the first iterate $T(x_0) = \phi(x_0)$ is less than $1/2$ so that the second iterate, $T^2(x_0) = T[\phi(x_0)] = f[\phi(x_0)]$; third iterate, $T^3(x_0) = f^2[\phi(x_0)]$ and so on. Thus the second and higher iterates fall on the left branch and the asymptotic state is $x^*_i$. If we start from an initial value $x_0 \in (0, 1/2)$, the iterates are $f(x_0), f^2(x_0), f^3(x_0), f^4(x_0) \ldots$ which converge to $x^*_i$. For $x_0$ lying in the interval $(1/2, x_r)$, the successive iterates form the sequence $\{\phi(x_0), \phi^2(x_0), \phi^3(x_0), \ldots, \phi^n(x_0), \ldots\}$ which converges to $x^*_r$. Hence, the basin of attraction $(R)$ for the attractor $x^*_r$ is the set of points $R = \{x_0 \mid x_0 \in (1/2, x_r)\}$ where $x_r$ is given by eqn. 4.6. Obviously, the basin of attraction $(L)$ for the attractor $x^*_r$ is the set of points on the unit interval, complimentary to the set $R$. i.e., $L = (0, 1/2) \cup (x_r, 1)$. The 'width' of the basin of attraction $(R)$ is $W = x_r - 1/2 = 1/8\mu\{[16\mu(\mu + C - 1/2)]^{1/2}\}$. From the expression for $x_r$, it is clear that the basin $R$ will be real only if $(\mu + C) > 1/2$. This should be so since the condition $(\mu + C) > 1/2$ is an essential requirement.
for the existence of the attractor $x^*_L$ itself. Keeping $C$ fixed, as $\mu$ increases from $(1/2-C)$ onwards, the width of the basin $R$ increases and that of $L$ decreases. At $\mu = 1/2$, the value of $W$ becomes a maximum $= \sqrt{(C/2)}$. When $\mu$ increases beyond 1/2, there exists only one attractor ($x^*_L$) for the system and all points in (0,1) constitute its basin of attraction. Similarly for values of $\mu < (1/2-C)$, there will be $x^*_L$ only and the entire (0,1) interval forms its basin of attraction.

It is also possible for the right attractor $x^*_R$ to co-exist with the zero fixed point, if $\mu < 1/4$ and $\mu + C > 1/2$. For this, $C$ must be $> 1/4$. Since the stability conditions for the zero fixed point and $x^*_R$ are mutually exclusive ($\mu < 1/4$ and $\mu > 1/4$ respectively), the possibility of coexistence of all the three attractors is ruled out. It is obvious that when $x^*_R$ coexists with the zero fixed point, the basin of attraction of $x^*_R$ is $R = (1/2, x_r)$ and that of the zero fixed point is $(0,1/2) \cup (x_r, 1)$ where $x_r$ is the same as the one given by equation 4.6. When only one attractor exists, the basin of attraction is (0,1).

### 4.2 Bifurcation Scenario for the Discontinuous Map

We now consider the stability of the fixed points, as the system parameters are varied. A fixed point $x^*$ will be stable, if the slope of the tangent to $T(x)$ at $x = x^*$ is less than unity, in magnitude. Keeping $C$ fixed, let the control parameter $\mu$ be varied from 0 to $(1-C)$. For $0 < \mu < 1/4$, the fixed point $x^* = 0$ is stable. For $1/4 < \mu < 1/2$, we have $x^*_L$ as the fixed point from the left part. This fixed point remains stable up to $\mu = 1/2$ and after that it
vanishes. For $\mu > 1/2$, the asymptotic state of the system is solely determined by $x^*_1$. This attractor is not stable throughout the range of variation of $\mu$. The slope of the map function $T'(x) = f'(x) = \phi'(x) = 4\mu(1-2x)$. Therefore, $T(x^*_1) = 1 - \sqrt{(4\mu-1)^2 + 16\mu C}$, where equation 4.5 has been used. At $\mu = (1/2 - C)$, this slope = 0. As $\mu$ increases, the slope becomes negative and its magnitude increases. When the slope becomes = -1, the first period doubling occurs and a 2-cycle is formed. The corresponding value of $\mu$ is given by,

$$\mu_1 = \frac{(1 - 2C) + \sqrt{(2C - 1)^2 + 3}}{4} \quad (4.8)$$

Upto this value of $\mu$, the system exhibits one cycle behaviour. Thus there will be two attractors ($x^*_1$ and $x^*_2$) for $\mu$ in the region $(1/2 - C) < \mu < 1/2$. Beyond $\mu = 1/2$, only $x^*_1$ exists and it remains stable upto the parameter value $\mu = \mu_1$. The attractor $x^*_1$ exhibits period doubling at $\mu = \mu_1$ and then the system shows a two cycle behaviour. In the limit, $C \rightarrow 0, \mu_1 \rightarrow 0.75$, the value for the logistic map.

In the usual Feigenbaum route to chaos, the 2-cycle bifurcates to a 4-cycle at a parameter value $\mu = \mu_2$ and it remains stable for a range of $\mu$ and then the 4-cycle bifurcates to an 8-cycle at $\mu = \mu_3$ and so on and these period doublings take place ad infinitum. The system then enters the chaotic region at a parameter value $\mu = \mu_\infty$, the period doubling accumulation point. But in the case of discontinuous maps, another type of bifurcation (a ‘discontinuous bifurcation’) takes place, when $\mu$ is increased. A different route to chaos is
thus possible. In the 2-cycle region of the map, the slope of $\phi^2(x)$ at the cycle elements $= \phi'(x_1^*)\phi'(x_2^*) = 64\mu^2(x_1^* - 1/2)(x_2^* - 1/2)$, where $x_1^*$ and $x_2^*$ are the elements of the two cycle. It remains positive as long as both the cycle elements fall to the right of $x = 1/2$. For $\mu = \mu_1$, the slope of $\phi(x)$ at $x = x_1^*$ becomes equal to -1, so that the slope of $\phi^2(x) = +1$ at this point. With increase of $\mu$ above $\mu_1$, we have a two cycle and the slope of $\phi^2(x)$ at the cycle elements decreases from 1. Both the cycle elements $x_1^*$ and $x_2^*$ lie within the interval $R = (1/2, x_r)$. As $\mu$ increases, the cycle elements move out. The lower element $x_1^*$ moves towards 1/2 and the upper element $x_2^*$ approaches $x_r$. Let $x_1^* = (1/2 + \epsilon)$. Then $x_2^* = \phi(x_1^*) = (\mu + C - 4\mu \epsilon^2)$. In the limit $\epsilon \to 0$, $x_1^* \to (1/2)$, and $x_2^* \to (\mu + C)_-$. In this limiting case, the slope of $\phi^2(x) \to 0$. Thus the limiting 2-cycle $\{(1/2), (\mu + C)_-\}$ is a stable one. This 2-cycle behavior continues up to $\mu = \mu_r$, at which the right element $x_r$, the pre-image of 1/2 on the right part. Using equation 4.6,

$$\mu_r = \frac{(\frac{1}{2} - C) + \sqrt{1 + (\frac{1}{2} - C)^2}}{2} \quad (4.9)$$

For $\mu_1 < \mu < \mu_r$, a two cycle $(x_1^*, x_2^*)$ is obtained. $\phi(x_1^*) = x_2^*$ and $\phi(x_2^*) = x_1^*$. Thus, $x_1^*$ and $x_2^*$ are solutions of $\phi^2(x) = x$. Now, for $\mu = \mu_r$, the right element $x_2^* = x_r$. The next iterate of $x_r$, say, $x_1 = T(x_r) = \phi(x_r) = 1/2$, falls on the left branch. Consequently, the second iterate of $x_r$ is decided not by the function $\phi(x)$, but by $f(x)$. i.e., $x_2 = T^2(x_r) = T(1/2) = f(1/2) = \mu_r$. Now, since $\mu_r$ is greater than 1/2, (for $C < 1/2$, as is usually the case), the second iterate of $x_r$ comes back to the interval $(1/2, x_r)$. The third and
higher iterates are decided by the function $\phi$. Thus, we get the sequence of iterates, \{ $x_2, \phi (x_2), \phi^2 (x_2), \ldots, \phi^r (x_2)\ldots$ \} and the two cycle behaviour is lost. Clearly, all these iterates lie in the interval $(1/2, x_r)$. These iterates are attracted towards the ‘virtual’ 2-cycle $(1/2, x_r)$ [i.e., the two cycle that the system would have, if the mapping were $\phi(x)$ on both sides of $x = 1/2$]. Once $x_r$ is attained, the sequence is repeated. In this sequence of iterates, the odd iterates $\phi (x_2), \phi^3 (x_2), \phi^5 (x_2), \ldots$ approach one end of the interval $(1/2, x_r)$ and the even iterates $\phi^2 (x_2), \phi^4 (x_2), \phi^6 (x_2), \ldots$ approach the other boundary. The system thus exhibits a large periodicity $n$, which is highly dependent on the precision of the computer. The period $n$ will be even or odd depending on whether $x_2 > x_r^*$ or $x_2 < x_r^*$, where $x_r^*$ is the fixed point (unstable) of $\phi(x)$ and $x_2$ is the second iterate of $x_r$. We consider the two cases separately.

*Case (1).* Second iterate of $x_r$ is greater than the unstable fixed point $x_r^*$. i.e., $x_2 > x_r^*$.

The function $\phi(x)$ decreases monotonically for all $x > 1/2$. Therefore, since $x_2$ lies to the right of $x_r^*$, its iterate, $\phi (x_2) < \phi (x_r^*)$. But, $\phi (x_r^*) = x_r^*$, as it is a fixed point (though unstable). Hence, $\phi (x_2) < x_r^*$. Now, since $\phi (x_2) < x_r^*$, under the next iteration, $\phi^2 (x_2) > \phi(x_r^*)$. i.e., $\phi^2 (x_2) > x_r^*$. Continuing like this, after each stage of iteration, the inequality gets reversed. Thus, the odd iterates $\phi (x_2), \phi^3 (x_2), \phi^5 (x_2), \ldots$ lie to the left of $x_r^*$ and the even iterates $\phi^2 (x_2), \phi^4 (x_2), \phi^6 (x_2), \ldots$ lie to the right of $x_r^*$. 88
Fig 4.2 A plot of $\phi^2(x)$ vs $x$. The value of $C$ is taken as 0.1 and that of $\mu \approx \mu_r$. 
Figure 4.3. Schematic representation of the iterates for $\mu = \mu_r$

Case (a). $x_2 > x^*_r$

Case (b). $x_2 < x^*_r$

Fig 4.3 Schematic representation of the iterates of the discontinuous logistic map.
Since $\phi(x)$ is a decreasing function of $x$ for any $x \in (1/2, 1)$, it is clear that $\phi^2(x)$ is an increasing function of $x$ for all $x$ for which $\phi(x)$ is greater than 1/2. Thus $\phi^2(x)$ is an increasing function of $x$ for all $x$ in $(1/2, x_r)$. The behavior of $\phi^2(x)$ on either side of $x_r^*$ is shown in figure 4.2. It is obvious that, $\phi^2(x) > x$ for $x > x_r^*$; $\phi^2(x) < x$ for $x < x_r^*$ and $\phi^2(x) = x$ for $x = x_r^*$. Thus since $x_2 > x_r^*$, we have $\phi^2(x_2) > x_2$. i.e., $\phi^2(x_2) > x_2 > x_r^*$. Again, since $\phi^2(x_2) > x_r^*$, $\phi^4(x_2) > \phi^2(x_2)$. Similarly, $\phi^6(x_2) > \phi^4(x_2)$ and so on. Likewise, since $\phi(x_2) < x_r^*$, $\phi^3(x_2) < \phi(x_2)$; $\phi^5(x_2) < \phi^3(x_2)$ and so on. A schematic representation of the iterates on either side of $x_r^*$ is presented in figure 4.3. We thus, have an ordering for the iterates as,

$$x_r^* < x_2 < \phi^2(x_2) < \phi^4(x_2) < \phi^6(x_2) < \ldots$$

and

$$x_r^* > \phi(x_2) > \phi^3(x_2) > \phi^5(x_2) > \phi^7(x_2) > \ldots$$

Thus it is clear that the even iterates of $x_2$ tend to $x_r$ and the odd iterates of $x_2$ tend to $(1/2)_+$. Thus at some stage of iteration, $\phi^{2r}(x_2)$ becomes infinitesimally close to $x_r$. This iterate will be considered as $x_r$ itself by the computer and the sequence of iterates will be repeated. The value of $r$ depends on the precision used for the computation. Thus we have a cycle of periodicity $n = 2r + 2$.

Case 2. Second iterate of $x_2$ is less than the unstable fixed point $x_r^*$.

Based upon similar reasons as for case (1), we find that the successive iterates of $x_r$ satisfy the inequalities:

$$x_r^* > x_2 > \phi^2(x_2) > \phi^4(x_2) > \phi^6(x_2) > \ldots$$
Thus the odd iterates of $x_2$ move towards $x_r$ and even iterates approach $(1/2)_+$. Hence, after some stage of iteration, we have $\phi^{2r+1}(x_2)$ almost $= x_r$, leading to a periodicity $n = 2r + 3$.

The outward spiralling of the iterates to an $n$-periodic attractor at the parameter value $\mu = \mu_r$ is shown in figures 4.4 and 4.5 for two typical cases of $C = 0.1$ and $C = 0.2$ corresponding to even and odd periods respectively. It is to be emphasized that the true period at $\mu = \mu_r$ is infinity. But in numerical computations, a finite period is obtained; The period in this case is, in fact, equal to the number of iterations required by $x_r$ to come back to a value 'almost' equal to itself; the degree of closeness being pre fixed by the precision of the computer. The map thus belongs to the class of maps with precision dependent periods [69]: if the map function were $\phi(x)$ throughout the interval $(0,1)$, the role of $C$ would be that of an additive constant applied to the logistic map [82] and the system would still have a stable 2-periodic behavior.

The bifurcation of a 2-cycle to an $n$-cycle, in the case of the discontinuous logistic map, as the cycle elements touch the basin boundary is a discontinuous bifurcation. It is because of the discontinuity in the second iterate $T^2(x)$ near $x = 1/2$ and $x = x_r$. Note that $T^2(x) = \phi^2(x)$ for $x = (x_r - \epsilon)$ and $T^2(x) = f(\phi(x))$, for $x = (x_r + \epsilon)$.

When $\mu$ is slightly greater than $\mu_r$, the 'virtual' 2-cycle $(x_1^*, x_2^*)$ falls outside the interval $[1/2, x_r]$. In this case, the successive iterates of any initial
Fig 4.4 Graphical representation of the outward spiralling of the it-
erates to an attractor of large periodicity at the parameter value \( \mu = \mu_r \), for

\( C = 0.1 \).
Fig 4.5 Graphical representation of the outward spiralling of the iterates to an attractor of large periodicity at the parameter value $\mu = \mu_r$, for $C = 0.2$
value \( x_0 \in (1/2, x_r) \) form a sequence which shows a tendency to settle down at
the 2-cycle. (Also, any initial value outside \((1/2, x_r)\) will come to this interval,
after a few iterations, since the system has no other stable periodic solutions).
In this process, an iterate slightly greater than \( x_r \) is obtained. This point is
mapped by \( \phi(x) \) into the domain of \( f(x) \). The next iteration by the mapping
\( f(x) \) takes it to the interval \([1/2, x_r]\). This point, under repeated iterations
by \( \phi(x) \) comes outside this range and the whole process is continued again
and again. The periodicity \((n)\) will be even or odd depending on whether
\( x_2 = f(\phi(x_r)) \) is \( > x^*_r \) or \( < x^*_r \), where \( x^*_r \) is the fixed point (unstable) of \( \phi(x) \).
The behavior of the iterates of the map for values of \( \mu \) immediately below and
above \( \mu_r \) can be understood from figures 4.6 and 4.7, in which the value of the
iterates are plotted against the iteration number.

The mechanism of the bifurcation in which a 2-cycle directly gives
birth to an \( n \)-cycle is quite different from the usual period doubling bifurca-
tions and it is a characteristic feature of discontinuous maps. In the period
doubling process, the slope (that determines the stability of the cycles) at the
bifurcation point becomes equal to -1. But in the case of bifurcation of the
2-cycle to an \( n \)-cycle, the slope of \( \phi^2(x) = 0 \) at the bifurcation point. The ele-
ments of the \( n \)-cycle are \( \{x_0 = x_r, x_1 = 1/2, x_2 = \mu_r, x_3, x_4, x_5, x_6, ... x_{n-1} = x_r \} \).
These elements are equilibrium points of \( T^n(x) \), for \( \mu = \mu_r \). With increase of
\( \mu \), the cycle elements move out until at some value of \( \mu \), the interval boundary
\( x_r \) is reached after \((n - 2)\) iterations. This should be so, since only alternate
Fig 4.6 The time plot of the system in eqn 4.1. The value of $C = 0.1$ and $\mu$ is slightly less than $\mu_r$. 
Fig 4.7 The time plot of the system in eqn 4.1. The value of $C = 0.1$ and $\mu$ is slightly greater than $\mu_r$. 
iterates move towards $x_r$. The periodicity of the system is thus lowered by 2.

Note that $x_r$ also increases (very slowly) with $\mu$ and that one set of alternate iterates are repelled by the unstable fixed point $x^*_r$ towards one side and the other set of alternate iterates to the other side. With further increase of $\mu$, the outermost element ($x_0$) increases beyond the corresponding value of $x_r$ and the cycle element nearest to $x_r$ within $(1/2, x_r)$ moves towards $x_r$ until at some stage, the basin boundary $x_r$ is attained after $(n - 4)$ iterations. Hence the period is again decreased by 2. Proceeding like this, we infer that as $\mu$ increases beyond $\mu_r$, there exist different ranges of the parameter $\mu$, for which cycles of periods decreasing by 2 exist. In a numerical computation, the periods decrease in an arithmetic progression. The common difference of the progression will be even numbers, since only alternate iterates move towards one boundary of $[1/2, x_r]$. The common difference also depends on the step size with which $\mu$ is increased as well as on the precision used for the computation. Again, the bifurcations within an inverse cascade occur whenever one of the cycle elements approaches the discontinuity of $T(x)$ at $x = 1/2$ and another element approaches $x_r$. [In this case, all the cycle elements approach the discontinuities of the $n^{th}$ iterate of the map where $n$ is the period of the cycle]. A given cycle of period $n$ can exhibit the usual period-doubling bifurcation, if the slope of the $n^{th}$ iterate becomes equal to -1 and the cycle loses stability before any of its elements gets a chance to collide with the discontinuity. The bifurcation process continues until the iterates become aperiodic at a parameter value ($\mu_\infty$) and the system enters the chaotic region.
A parameter space plot \((C, \mu)\) for the system is presented in figure 4.8. The fixed point \(x^*_i\) arising from the right part of the map exists only for points above the straight line, KLM, that represents \(\mu + C = 1/2\). The straight line PQRS, given by the equation \(\mu + C = 1\), determines the limiting values of the parameters upto which the iterates are confined in the unit interval. The straight line NLRN' represents \(\mu = 1/4\). The left part of the map function has only zero as the stable attractor for values of \(\mu < 1/4\). i.e., for all parameter points inside the trapezium, ONRS. The line KQK', represents \(\mu = 1/2\). The fixed point of the logistic part \((x^*_i)\) exists for the region of parameter space formed by the trapezium KQRN. The trapezium PKMS represents the region in which \(x^*_i\) exists. Thus the attractors \(x^*_i\) and \(x^*_r\) co-exist within the parallelogram KLRQ. Similarly, the attractor \(x^*_i\) and the zero fixed point co-exist for the parameter region represented by the parallelogram LMSR. The triangular region PKQ represents the set of \((C, \mu)\) points for which only \(x^*_r\) is present and the triangle KNL denotes the space in which the only attractor for the system is \(x^*_i\). Within the parameter region bounded by the trapezium ONLM, the zero attractor alone is present. Obviously, for the co-existence of \(x^*_r\) and the zero attractor, the value of \(C\) should be > 1/4. The curves EFN' and GQG' represent \(\mu_1\) and \(\mu_r\) respectively, as given by eqns.4.8 and 4.9. All the discontinuous cascades of bifurcations occur within the region PGQ. Note that both \(\mu_1\) and \(\mu_r\) are decreasing functions of \(C\). Hence, by taking negative values for \(C\), one can have a 1-cycle behavior for the map for values of \(\mu\) corresponding to the 2-cycle region of the logistic map.
Fig 4.8 A parameter space plot \((C, \mu)\) for the discontinuous logistic map. The lines \(GQG'\) and \(EFN'\) represent \(\mu_r\) and \(\mu_1\) respectively.
From the parameter space plot, one can select suitable \((C, \mu)\) points to have a desired dynamics for the system. It is possible to have a control over the dynamics of the system by proper choice of \(C\) for each \(\mu\), as in the case of the combination maps [80-82].

### 4.3 Numerical Results

The analytical studies presented above have been well substantiated by numerical computations. This section deals with the numerical investigations carried out to check the validity of the results obtained in the previous section. The discontinuity parameter \(C\) is kept at various fixed values. For each value of \(C\), the bifurcation structure of the map is determined, by varying the parameter \(\mu\) in small steps. A series of bifurcation diagrams are drawn, for different initial values \((x_0)\). Figure 4.9 shows a bifurcation diagram for the system. It is obvious that the map has got multiple attractors. Initial value dependent behavior is seen in the bifurcation diagrams. In the case of \(C = 0.2\), for example, we observe the following facts. For \(\mu\) varying from 0 to 0.25, the only attracting fixed point is zero and all initial points \(x_0 \in (0,1)\) converge asymptotically to this attractor. When \(\mu\) increases beyond 0.25, the system stabilizes to a non zero fixed point \((x_1^*)\) which remains stable upto \(\mu = 0.5\) and after that it vanishes. For values of \(\mu > 0.3\), another fixed point \((x_2^*)\) is observed. The two fixed points \(x_1^*\) and \(x_2^*\) co-exist for values of \(\mu\) varying from 0.3 to 0.5. Initial values like 0.1, 0.2, 0.3, etc. converge to the fixed point \(x_1^*\). But, seed values like 0.6 converge asymptotically to the other fixed point.
Fig 4.9 A Bifurcation diagram of the discontinuous logistic map for various initial points in (0,1). The value of \( C = 0.1 \) and \( \mu \) is varied from 0 to 0.8. Note that the 2-cycle suddenly bifurcates to an n-cycle when \( \mu = \mu_r \).
The fixed point $x_i^*$ undergoes the standard pitchfork bifurcation to a 2-cycle at a value $\mu = 0.608222$. The two cycle behavior continues up to the parameter value $\mu = 0.672055$. These values of $\mu_1$ and $\mu_r$ are obtained accurately, by taking further and further blow ups of the bifurcation diagrams. When $\mu$ increases beyond 0.672055, a sudden bifurcation takes place and the iterates form a cycle of large periodicity. The period depends on the precision of the computer. The zero fixed point does not co-exist with $x_i^*$ for values of $C < 0.25$. The parameter values $\mu_1$ and $\mu_r$ obtained numerically and analytically for various values of $C$ are presented in table 4.1. The close agreement between the numerical and analytical values give a strong support to the theory developed in the previous section. A check for any evidence of chaos was performed, based on the criterion of the positivity of the L.C.E ($\lambda$) given by [14]:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{dT(x)}{dx} \right|_{x=x_k}$$  \hspace{1cm} (4.10)

Accordingly, the values of $\lambda$ are computed for various values of $\mu$ in the neighborhood of $\mu_r$. No trace of chaos is seen at these points; We have observed cycles of periodicities decreasing by 2 for $\mu$ values greater than $\mu_r$. Two typical cases of $C = 0.1$ and $C = 0.2$ are analyzed in detail. Figs. 4.10 and 4.11 shows the bifurcation diagrams for these cases. In the case of $C = 0.1$, cycles of even periods (14, 12, 10, 8, 6, ...) are observed when $\mu$ is increased beyond $\mu_r$; For $C = 0.2$, we observe cycles of odd periods for values of $\mu$ greater than $\mu_r$. The usual period-doubling process is also observed. For certain values of the parameters, different initial points converge to cycles of different periods.
Table 4.1

Parameter values for period doubling and period n-tupling of the discontinuous logistic map for different values of $C$.

<table>
<thead>
<tr>
<th>$C$</th>
<th>Discontinuity parameter</th>
<th>Control parameter ($\mu$) for</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>period doubling ($\mu_1$)</td>
<td>period n-tupling ($\mu_n$)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Analytical</td>
<td>Numerical</td>
<td>Analytical</td>
<td>Numerical</td>
</tr>
<tr>
<td>0.05</td>
<td>0.71298053240</td>
<td>0.712945</td>
<td>0.77329280499</td>
<td>0.7732925</td>
</tr>
<tr>
<td>0.10</td>
<td>0.67696960071</td>
<td>0.676985</td>
<td>0.73851648071</td>
<td>0.7385165</td>
</tr>
<tr>
<td>0.15</td>
<td>0.64203854231</td>
<td>0.642015</td>
<td>0.70474050251</td>
<td>0.7047450</td>
</tr>
<tr>
<td>0.20</td>
<td>0.60825756950</td>
<td>0.608222</td>
<td>0.67201532545</td>
<td>0.6720155</td>
</tr>
<tr>
<td>0.25</td>
<td>0.57569390943</td>
<td>0.575664</td>
<td>0.64038820320</td>
<td>0.6403885</td>
</tr>
<tr>
<td>0.30</td>
<td>0.54440972087</td>
<td>0.544410</td>
<td>0.60990195136</td>
<td>0.6099020</td>
</tr>
<tr>
<td>0.35</td>
<td>0.51445989578</td>
<td>0.514447</td>
<td>0.58059371040</td>
<td>0.5805935</td>
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<tr>
<td>0.40</td>
<td>0.48588989435</td>
<td>0.485876</td>
<td>0.55249378106</td>
<td>0.5524935</td>
</tr>
<tr>
<td>0.45</td>
<td>0.45873376932</td>
<td>0.458677</td>
<td>0.52562460986</td>
<td>0.5256245</td>
</tr>
<tr>
<td>0.50</td>
<td>0.43301270189</td>
<td>0.432997</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>
Fig 4.10 A bifurcation diagram of the discontinuous logistic map for

\[ C = 0.1 \]
Fig 4.11 A bifurcation diagram of the discontinuous logistic map for 

\( C = 0.2 \)
Keeping the value of \( C = 0.1 \) and \( \mu \) at various fixed values in the interval \((1/2 - C)\), the asymptotic value of the iterate \( \{x_n\} \) generated from a seed value \( x_0 \) is determined. This is repeated for various values of \( x_0 \). In this way, the basin of attraction is determined. This procedure is then repeated for \( C = 0.2 \) as well. The basins of attractions for the two fixed points \( x_i^* \) and \( x_r^* \) are presented in figure 4.12 and figure 4.13 corresponding to \( C = 0.1 \) and \( C = 0.2 \) respectively. Here, we plot the values of the fixed points versus the initial values \( x_0 \) for all \( x_0 \) in \((0,1)\). The basin of attraction for \( x_r^* \) is the portion of \( x \)-axis from 0.5 to \( x_r \) and that for \( x_i^* \) is the set of points on the \( x \)-axis in the interval 0 to 0.5 and from \( x_r \) to 1, where \( x_r \) is the right basin boundary of \( x_r^* \). The basins of attraction are well defined and are not intermingled, unlike in the case of the logistic map under certain parametric perturbations[83,84]. Such well defined basin structures have already been reported for other discontinuous maps[85].

Within the parameter range for the two attractors to co-exist, it is observed that the width of the basin of attraction of \( x_i^* \) increases with \( \mu \) and reaches its maximum value at \( \mu = 0.5 \). If \( \mu < (1/2 - C) \), the fixed point is \( x_i^* \) or zero. For \( 0.5 < \mu < \mu_i \), the fixed point is \( x_r^* \). Table 4.2 shows the width of the basin of attraction \( (R) \) at \( \mu = 0.5 \) for different values of \( C \), obtained numerically and analytically. There is excellent agreement between the two, which verifies our theoretical analysis.
Fig 4.12 The basins of attraction of the discontinuous logistic map with $C = 0.1$. The asymptotic state is plotted against the seed values. The value of $\mu$ is taken as 0.5.
Fig 4.13 The basins of attraction of the discontinuous logistic map with $C = 0.2$. The asymptotic state is plotted against the seed values. The value of $\mu$ is taken as 0.5.
Table 4.2

Width of the basin of attraction of the right fixed point for various values of $C$. The control parameter $\mu$ is kept at 0.5.

<table>
<thead>
<tr>
<th>Discontinuity parameter $(C)$</th>
<th>Width of the basin of attraction of $x^*_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1581139</td>
</tr>
<tr>
<td>0.10</td>
<td>0.2236068</td>
</tr>
<tr>
<td>0.15</td>
<td>0.2738613</td>
</tr>
<tr>
<td>0.20</td>
<td>0.3162278</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3535534</td>
</tr>
<tr>
<td>0.30</td>
<td>0.3872984</td>
</tr>
<tr>
<td>0.35</td>
<td>0.4183300</td>
</tr>
</tbody>
</table>
4.4 Conclusion

The work presented above gives a theoretical understanding of the route to chaos in discontinuous systems. A detailed analysis of the dynamics of a discontinuous logistic map is carried out, both analytically and numerically, to understand the route it follows to chaos. We have shown analytically that the discontinuous logistic map has got multiple attractors with different basins of attraction. We also give expressions for the basin boundaries and a theory for the bifurcation phenomenon of the map is developed. In contrast to the standard pitchfork bifurcation in which the slope of a cycle becomes -1, the border collision bifurcation occurs whenever a cycle element touches the discontinuity of the map; For the map with a discontinuity at the extremum, the slope of the cycle becomes 0 at this point. Our results are verified by numerical investigations of the map. We have presented a parameter space plot for the map so that one can have a desired dynamics for the system, by suitable choice of the discontinuity parameter \( C \) for every value of the control parameter \( \mu \). However, the present analysis deals only with the case of the discontinuity parameter applied to the right half of the interval of mapping. A detailed analysis for the \( n \)-furcations of various periodicities can be made and a more general theory for the map with discontinuities applied at different positions can be formulated on a similar footing.