CHAPTER-3

DEFORMATIONS IN VAGUE FUZZY DIGITAL STRUCTURE SPACES

The concepts of fuzzy homotopy and fuzzy contractible spaces were introduced by Erdal Gunar [41]. In this chapter, the concepts of vague fuzzy digital structure homotopy, vague fuzzy digital structure contractible spaces, vague fuzzy digital structure homotopy equivalent spaces, vague fuzzy digital structure retraction, vague fuzzy digital structure weak retraction, vague fuzzy digital structure cofibration and vague fuzzy digital structure homotopy extension property are introduced and studied. In this connection, some interesting properties are established.

3.1 VAGUE FUZZY DIGITAL STRUCTURE HOMOTOPY

In this section, $\alpha\beta$-cut of a vague fuzzy digital sets, vague fuzzy digital structure continuous functions, vague fuzzy digital homotopy, vague fuzzy digital structure contractible spaces and vague fuzzy digital structure homotopy equivalent spaces are introduced. In this connection, some interesting properties are discussed.
Definition 3.1.1. Let $\sum$ and $\Omega$ be any two rectangular array of integer-coordinate points and $g : \sum \to \Omega$ be a function.

(i) If $B = \{\langle Q, t_B(Q), f_B(Q) \rangle : Q \in \Omega \}$ is a vague fuzzy digital set in $\Omega$, then
the preimage of $B$ under $g$ is denoted and defined as

$$g^{-1}(B) = \{\langle P, g^{-1}(t_B)(P), g^{-1}(f_B)(P) \rangle : P \in \sum \}.$$

(ii) If $A = \{\langle P, t_A(P), f_A(P) \rangle : P \in \sum \}$ is a vague fuzzy digital set in $\sum$, then

the image of $A$ under $g$ is denoted and defined as

$$g(A) = \{\langle Q, g(t_A)(Q), g^{-1}(f_A)(Q) \rangle : Q \in \Omega \}$$

where

$$g(t_A)(Q) = \begin{cases} \sup_{P \in g^{-1}(Q)} \{t_A(P)\}, & \text{if } f^{-1}(Q) \neq \emptyset; \\ 0, & \text{otherwise}; \end{cases}$$

and

$$g^{-1}(f_A)(Q) = \begin{cases} \inf_{P \in g^{-1}(Q)} \{f_A(P)\}, & \text{if } f^{-1}(Q) \neq \emptyset; \\ 1, & \text{otherwise}. \end{cases}$$

Definition 3.1.2. Let $(\sum, D)$ be any vague fuzzy digital structure space. Let $A = \langle P, t_A, f_A \rangle$ be a vague fuzzy digital set in $\sum$. Then the induced vague fuzzy digital structure on $A$ is the family of vague fuzzy digital subsets of $A$ which are the intersections with $A$ of vague fuzzy digital structure open sets in $\sum$. The induced vague fuzzy digital structure is denoted by $D_A$ and the pair $(A, D_A)$ is called a vague fuzzy digital structure subspace of $(\sum, D)$.

Definition 3.1.3. Let $(\sum, D^1)$ and $(\Upsilon, D^2)$ be any two vague fuzzy digital structure spaces. Let $f : (\sum, D^1) \to (\Upsilon, D^2)$ be a function. Then $f$ is called a
vague fuzzy digital structure continuous function if for each vague fuzzy digital structure open set $D$ in $(\Upsilon, \mathcal{D}^2)$, $f^{-1}(D)$ is a vague fuzzy digital structure open set in $(\Sigma, \mathcal{D}^1)$.

Equivalently, $f$ is called a vague fuzzy digital structure continuous function if for each vague fuzzy digital structure closed set $A$ in $(\Upsilon, \mathcal{D}^2)$, $f^{-1}(D)$ is a vague fuzzy digital structure closed set in $(\Sigma, \mathcal{D}^1)$.

**Definition 3.1.4.** Let $\Sigma$ be a rectangular array of integer-coordinate points and $I$ be the closed interval $[0, 1]$. Let $A = \langle P, t_A, f_A \rangle$ be any vague fuzzy digital set in $\Sigma$. The set

$$\text{supp}A = A_{01} = \{ P \in \Sigma : t_A(P) > 0 \text{ and } f_A(P) < 1 \}$$

is called the support of a vague fuzzy digital set $A$.

**Definition 3.1.5.** Let $\Sigma$ be a rectangular array of integer-coordinate points and $I$ be the closed interval $[0, 1]$. For any vague fuzzy digital set $A = \langle P, t_A, f_A \rangle \in \zeta \Sigma$ and for every $\alpha, \beta \in I$ with $\alpha + \beta \leq 1$, the $\alpha\beta$-cut of a vague fuzzy digital set $A$ is defined as

$$A_{\alpha\beta} = \{ P \in \Sigma : t_A(P) \geq \alpha \text{ and } f_A(P) \leq \beta \}.$$ 

**Definition 3.1.6.** Let $\Sigma$ be a rectangular array of integer-coordinate points. A digital structure on $\Sigma$ is a family $\mathcal{D}$ of subsets of $\Sigma$ satisfying the following axioms:

(i) $\emptyset, \Sigma \in \mathcal{D}$. 

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(ii) \( G_1 \cap G_2 \in \mathcal{D} \) for any \( G_1, G_2 \in \mathcal{D} \).

(iii) \( \cup G_i \in \mathcal{D} \) for arbitrary family \( \{G_i \mid i \in J\} \subseteq \mathcal{D} \).

Then the ordered pair \((\Sigma, \mathcal{D})\) is called a digital structure space. Every member in \( \mathcal{D} \) is called a digital structure open set in \((\Sigma, \mathcal{D})\). The complement of a digital structure open set \( A \) is a digital structure closed set in \((\Sigma, \mathcal{D})\).

**Definition 3.1.7.** Let \((\Sigma, \mathcal{D}^1)\) and \((\Upsilon, \mathcal{D}^2)\) be any two digital structure spaces. Let \( f : (\Sigma, \mathcal{D}^1) \rightarrow (\Upsilon, \mathcal{D}^2) \) be a function. Then \( f \) is called a digital structure continuous function if for each digital structure open set \( D \) in \((\Upsilon, \mathcal{D}^2)\), \( f^{-1}(D) \) is a digital structure open set in \((\Sigma, \mathcal{D}^1)\).

Equivalently, \( f \) is called a digital structure continuous function if for each digital structure closed set \( A \) in \((\Upsilon, \mathcal{D}^2)\), \( f^{-1}(D) \) is a digital structure closed set in \((\Sigma, \mathcal{D}^1)\).

**Definition 3.1.8.** Let \((\Sigma, \mathcal{D})\) be any digital structure space. The collection \( \mathcal{D}^i = \{A = (P, t_A, f_A) : A \text{ is a vague fuzzy digital set in } \Sigma \text{ and } A_{01} \in \mathcal{D}\} \) is a vague fuzzy digital structure on \( \Sigma \), called the vague fuzzy digital structure on \( \Sigma \) introduced by \( \mathcal{D} \). The ordered pair \((\Sigma, \mathcal{D}^i)\) is called a vague fuzzy digital structure space introduced by \((\Sigma, \mathcal{D})\).

**Definition 3.1.9.** Let \((\Sigma, \mathcal{D})\) be any digital structure space and \( \Omega \subset \Sigma \). Let \( A = (P, t_p, f_A) \) be a vague fuzzy digital set in \( \Sigma \) and \( \mathcal{D}^* \subset \zeta \Sigma \). Let \( \mathcal{D}^* = \{C = (P, t_C, f_C), t_C : \Omega \rightarrow I \text{ and } f_C : \Omega \rightarrow I \mid \Omega \in \mathcal{D} \} \) and \( \mathcal{D}^*_{\alpha \beta} = \{C_{\alpha \beta} : C \in \mathcal{D}^*\} \), for
every $\alpha, \beta \in I$ with $\alpha + \beta \leq 1$. The pair $(A, D^*)$ is called a vague fuzzy digital structure space if and only if for every $\alpha, \beta \in I$, $(A_{\alpha\beta}, D^*_{\alpha\beta})$ is a digital structure space.

**Proposition 3.1.1.** Let $(A, D^1)$ and $(B, D^2)$ be any two vague fuzzy digital structure spaces and $g : (A, D^1) \to (B, D^2)$ be a function. Then $g$ is a vague fuzzy digital structure continuous function if and only if for every $\alpha, \beta \in I$, $g_{\alpha\beta} : (A_{\alpha\beta}, D^1_{\alpha\beta}) \to (B_{\alpha\beta}, D^2_{\alpha\beta})$ is a structure continuous function.

**Proof:**

Let $g : (A, D^1) \to (B, D^2)$ be a vague fuzzy digital structure continuous function and let $C$ be a vague fuzzy digital structure open set in $(B, D^2)$. Since $g$ is a vague fuzzy digital structure continuous function, $g^{-1}(C)$ is a vague fuzzy digital structure open set in $(A, D^1)$. Since $C$ is a vague fuzzy digital structure open set in $(B, D^2)$ and for every $\alpha, \beta \in I$, $C_{\alpha\beta}$ is a digital structure open set in $(B_{\alpha\beta}, D^2_{\alpha\beta})$.

Since $g^{-1}(C)$ is a vague fuzzy digital structure open set in $(A, D^1)$ and for every $\alpha, \beta \in I$, $g^{-1}_{\alpha\beta}(C_{\alpha\beta})$ is a digital structure open set in $(A_{\alpha\beta}, D^1_{\alpha\beta})$. Therefore, $g_{\alpha\beta} : (A_{\alpha\beta}, D^1_{\alpha\beta}) \to (B_{\alpha\beta}, D^2_{\alpha\beta})$ is a digital structure continuous function.

**Definition 3.1.10.** Let $(A, D^1)$ and $(B, D^2)$ be any two vague fuzzy digital structure spaces and $g, h : (A, D^1) \to (B, D^2)$ be any two vague fuzzy digital structure continuous functions. Then $g$ is vague fuzzy digital structure homotopic to $h$ if there exists a vague fuzzy digital structure continuous func-
tion $F : A \times I \to B$ such that for every $\alpha, \beta \in I$,

$$F_{\alpha\beta}(P, 0) = g_{\alpha\beta}(P) \text{ and } F_{\alpha\beta}(P, 1) = h_{\alpha\beta}(P).$$

The function $F$ is called vague fuzzy digital structure homotopy between $g$ and $h$ and it is denoted by $g \simeq h$.

**Proposition 3.1.2.** The relation $\simeq$ is an equivalence relation.

**Proof:**

Let $(A, \mathcal{D}_1)$ and $(B, \mathcal{D}_2)$ be any two vague fuzzy digital structure spaces.

(i) Let $g : (A, \mathcal{D}_1) \to (B, \mathcal{D}_2)$ be any vague fuzzy digital structure continuous function. Define a function $F : A \times I \to B$ such that $F_{\alpha\beta}(P, s) = g_{\alpha\beta}(P)$, for every $\alpha, \beta \in I$ and for all $s \in I$.

Then by Proposition 3.1.1., $F$ is a vague fuzzy digital structure continuous function, $F_{\alpha\beta}(P, 0) = g_{\alpha\beta}(P)$ and $F_{\alpha\beta}(P, 1) = g_{\alpha\beta}(P)$. Hence, $g \simeq g$.

(ii) If $g \simeq h$, then for every $\alpha, \beta \in I$, $g \simeq h \Rightarrow \exists F : A \times I \to B$ such that $F$ is a vague fuzzy digital structure continuous function, $F_{\alpha\beta}(P, 0) = g_{\alpha\beta}(P)$ and $F_{\alpha\beta}(P, 1) = h_{\alpha\beta}(P)$.

Let $G : A \times I \to B$ be defined as $G_{\alpha\beta}(P, s) = F_{\alpha\beta}(P, 1 - s)$, for all $s \in I$. By Proportion 3.1.1., $G$ is a vague fuzzy digital structure continuous function.

Now, $G_{\alpha\beta}(P, 0) = F_{\alpha\beta}(P, 1) = h_{\alpha\beta}(P)$ and $G_{\alpha\beta}(P, 1) = F_{\alpha\beta}(P, 0) = g_{\alpha\beta}(P)$. Therefore, $h \simeq g$.

(iii) If $g \simeq h$ and $h \simeq w$. Then, $g \simeq h \Rightarrow \exists F : A \times I \to B$ such that $F$ is a vague fuzzy digital structure continuous function, $F_{\alpha\beta}(P, 0) = g_{\alpha\beta}(P)$.
and $F_{\alpha\beta}(P,1) = h_{\alpha\beta}(P)$. Similarly, $h \simeq w \Rightarrow \exists G : A \times I \to B$ such that $G$ is a vague fuzzy digital structure continuous function, $G_{\alpha\beta}(P,0) = h_{\alpha\beta}(P)$ and $G_{\alpha\beta}(P,1) = w_{\alpha\beta}(P)$.

Let $H : A \times I \to B$ be defined as

$$H_{\alpha\beta}(P,s) = \begin{cases} F_{\alpha\beta}(P,2s), & 0 \leq s \leq 1/2; \\ G_{\alpha\beta}(P,2s-1), & 1/2 \leq s \leq 1; \end{cases}$$

for all $s \in I$. Since $F$ and $G$ are vague fuzzy digital structure continuous functions and by Proposition 3.1.1., $H$ is a vague fuzzy digital structure continuous function. Now,

$$H_{\alpha\beta}(P,0) = F_{\alpha\beta}(P,0) = g_{\alpha\beta}(P) \text{ and } H_{\alpha\beta}(P,1) = G_{\alpha\beta}(P,1) = w_{\alpha\beta}(P).$$

Therefore, $g \simeq w$. Thus the relation $\simeq$ is an equivalence relation. ♦

**Proposition 3.1.3.** Let $(A, D^1)$, $(B, D^2)$ and $(C, D^3)$ be any three vague fuzzy digital structure spaces. Suppose that $h_1$ and $h_2$ are the vague fuzzy digital structure homotopic functions from $(A, D^1)$ and $(B, D^2)$ and that $g_1$ and $g_2$ are the vague fuzzy digital structure homotopic functions from $(B, D^2)$ to $(C, D^3)$. Then, $g_1 \circ h_1 \simeq g_2 \circ h_2$.

**Proof:** The proof is clear from the following steps:

(i) $g_1 \circ h_1 \simeq g_1 \circ h_2$.

(ii) $g_1 \circ h_2 \simeq g_2 \circ h_2$.

(iii) Transitivity of (i) and (ii). ♦
Proposition 3.1.4. Let $(A, D^1)$, $(B, D^2)$ and $(C, D^3)$ be any three vague fuzzy digital structure spaces. Let $h, g : (A, D^1) \rightarrow (B, D^2)$ be any two vague fuzzy digital structure continuous functions such that $h \simeq g$. If $w : (B, D^2) \rightarrow (C, D^3)$ is a vague fuzzy digital structure continuous function, then $w \circ h, w \circ g : (A, D^1) \rightarrow (C, D^3)$ are vague fuzzy digital structure continuous functions and $w \circ h \simeq w \circ g$.

Proof:

Since $w, h$ and $g$ are vague fuzzy digital structure continuous functions, $w \circ h$ and $w \circ g$ are vague fuzzy digital structure continuous functions. Furthermore, $h \simeq g$ implies that there exists a vague fuzzy digital structure continuous function $F : A \times I \rightarrow B$ such that $F_{\alpha\beta}(P, 0) = h_{\alpha\beta}(P)$ and $F_{\alpha\beta}(P, 1) = g_{\alpha\beta}(P)$, for every $\alpha, \beta \in I$. Now, $G : A \times I \rightarrow C$ is defined as

$$G_{\alpha\beta}(P, s) = w_{\alpha\beta}(F_{\alpha\beta}(P, s)), s \in I.$$ 

Since $w$ and $F$ are vague fuzzy digital structure continuous functions, $G = w \circ F$ is a vague fuzzy digital structure continuous function. Moreover, $G$ satisfies the following conditions:

$$G_{\alpha\beta}(P, 0) = w_{\alpha\beta}(F_{\alpha\beta}(P, 0)) = w_{\alpha\beta}(h_{\alpha\beta}(P)) = (w \circ h)_{\alpha\beta}(P),$$

$$G_{\alpha\beta}(P, 1) = w_{\alpha\beta}(F_{\alpha\beta}(P, 1)) = w_{\alpha\beta}(g_{\alpha\beta}(P)) = (w \circ g)_{\alpha\beta}(P).$$

Therefore, $w \circ h \simeq w \circ g$.  ☞
Definition 3.1.11. Let \((A, D^1)\) and \((B, D^2)\) be any two vague fuzzy digital structure spaces. Let \(g, h : (A, D^1) \to (B, D^2)\) be any two vague fuzzy digital structure continuous functions and \(g \simeq h\). If \(h\) is a constant function, then \(g\) is vague fuzzy digital structure homotopic to a constant.

Definition 3.1.12. Let \((A, D)\) be any vague fuzzy digital structure space and \(\mathcal{I}_A : (A, D) \to (A, D)\) be any identity function. If \(\mathcal{I}_A\) is vague fuzzy digital structure homotopic to a constant, then \((A, D)\) is called a vague fuzzy digital structure contractible space.

Proposition 3.1.5. Let \((A, D^1)\) be any vague fuzzy digital structure space and \((B, D^2)\) be any vague fuzzy digital structure contractible space. Then every vague fuzzy digital structure continuous function \(g : (A, D^1) \to (B, D^2)\) is vague fuzzy digital structure homotopic to a constant.

**Proof:**

Let \(\mathcal{I}_B : (B, D^2) \to (B, D^2)\) be any identity function in \(B\). Since \((B, D^2)\) is a vague fuzzy digital structure contractible space, there exists a constant function \(C : (B, D^2) \to (B, D^2)\) such that \(\mathcal{I}_B \simeq C\) and \(C_{\alpha\beta}(Q) = Q_0\), for \(Q \in B_{\alpha\beta}\). Then, \(\mathcal{I}_B \simeq C \Rightarrow \exists F : B \times I \to B\) such that \(F\) is a vague fuzzy digital structure continuous function, \(F_{\alpha\beta}(Q, 0) = \mathcal{I}_{B_{\alpha\beta}}(Q)\) and \(F_{\alpha\beta}(Q, 1) = Q_0\).

Now, let \(g : (A, D^1) \to (B, D^2)\) be any vague fuzzy digital structure continuous function. Then, \(G : A \times I \to B\) is defined as

\[
G_{\alpha\beta}(P, s) = F_{\alpha\beta}(g_{\alpha\beta}(P), s).
\]
Since $F$ is a vague fuzzy digital structure continuous function and by Proposition 3.1.1., $G$ is a vague fuzzy digital structure continuous function. and has the following properties:

$$G_{\alpha\beta}(P, 0) = F_{\alpha\beta}(g_{\alpha\beta}(P), 0) = g_{\alpha\beta}(P),$$

$$G_{\alpha\beta}(P, 1) = F_{\alpha\beta}(g_{\alpha\beta}(P), 1) = Q_0 = C_{\alpha\beta}(Q).$$

Hence, $g \simeq C$. That is, every vague fuzzy digital structure continuous function is vague fuzzy digital structure homotopic to a constant. ♦

**Definition 3.1.13.** Let $(A, \mathcal{D}^1)$ and $(B, \mathcal{D}^2)$ be any two vague fuzzy digital structure spaces. Let $g : (A, \mathcal{D}^1) \rightarrow (B, \mathcal{D}^2)$ be any bijective function. Then $g$ is called a vague fuzzy digital structure homeomorphism if and only if $g$ and $g^{-1}$ are vague fuzzy digital structure continuous functions.

**Definition 3.1.14.** Let $(A, \mathcal{D}^1)$ and $(B, \mathcal{D}^2)$ be any two vague fuzzy digital structure spaces. Let $g : (A, \mathcal{D}^1) \rightarrow (B, \mathcal{D}^2)$ be any function. Then $(A, \mathcal{D}^1)$ and $(B, \mathcal{D}^2)$ are vague fuzzy digital structure homeomorphic spaces (or) vague fuzzy digital structure equivalent spaces if and only if $g$ is a vague fuzzy digital structure homeomorphism.

Equivalently, $(A, \mathcal{D}^1)$ and $(B, \mathcal{D}^2)$ are vague fuzzy digital structure equivalent spaces if and only if there exist vague fuzzy digital structure continuous functions $g : (A, \mathcal{D}^1) \rightarrow (B, \mathcal{D}^2)$ and $g^{-1} : (B, \mathcal{D}^2) \rightarrow (A, \mathcal{D}^1)$ such that $g \circ g^{-1} = \mathcal{I}_B$ and $g^{-1} \circ g = \mathcal{I}_A$ where $\mathcal{I}_A : (A, \mathcal{D}^1) \rightarrow (A, \mathcal{D}^1)$ and $\mathcal{I}_B : (B, \mathcal{D}^2) \rightarrow (B, \mathcal{D}^2)$ are identity functions of $A$ and $B$ respectively.

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**Definition 3.1.15.** Let \((A, \mathcal{D}^1)\) and \((B, \mathcal{D}^2)\) be any two vague fuzzy digital structure spaces. Let \(g : (A, \mathcal{D}^1) \to (B, \mathcal{D}^2)\) be any vague fuzzy digital structure continuous function. If there is a vague fuzzy digital structure continuous function \(g' : (B, \mathcal{D}^2) \to (A, \mathcal{D}^1)\) satisfies the following conditions:

(i) \(g \circ g' \simeq I_B\).

(ii) \(g' \circ g \simeq I_A\).

Then \(g\) is called a vague fuzzy digital structure homotopy equivalence. Further, vague fuzzy digital structure spaces \((A, \mathcal{D}^1)\) and \((B, \mathcal{D}^2)\) are called vague fuzzy digital structure homotopy equivalent spaces and denoted by \(A \equiv B\).

**Remark 3.1.1.** The relation “\(\equiv\)” is an equivalence relation.

**Proof:** The proof is simple.

**Proposition 3.1.6.** Let \((A, \mathcal{D}^1)\) and \((B, \mathcal{D}^2)\) be any two vague fuzzy digital structure spaces. If \((A, \mathcal{D}^1)\) and \((B, \mathcal{D}^2)\) are vague fuzzy digital structure equivalent spaces, then \((A, \mathcal{D}^1)\) and \((B, \mathcal{D}^2)\) are vague fuzzy digital structure homotopy equivalent spaces.

**Proof:**

Since \((A, \mathcal{D}^1)\) and \((B, \mathcal{D}^2)\) are vague fuzzy digital structure equivalent spaces, there exists a vague fuzzy digital structure continuous function \(g : (A, \mathcal{D}^1) \to (B, \mathcal{D}^2)\) such that \(g\) is one to one and surjective.
Moreover, \( g : (A, \mathcal{D}^1) \to (B, \mathcal{D}^2) \) and \( g^{-1} : (B, \mathcal{D}^2) \to (A, \mathcal{D}^1) \) are vague fuzzy digital structure continuous functions. Therefore, \( g \circ g^{-1} = \mathcal{I}_B \) and \( g^{-1} \circ g = \mathcal{I}_A \). By Proposition 3.1.2.,

\[
g \circ g^{-1} \simeq \mathcal{I}_B \text{ and } g^{-1} \circ g \simeq \mathcal{I}_A.
\]

Therefore, \((A, \mathcal{D}^1)\) and \((B, \mathcal{D}^2)\) are vague fuzzy digital structure homotopy equivalent spaces, that is, \( A \equiv B \).

\[\Box\]

### 3.2 Properties of Cofibration in Vague Fuzzy Digital Structure Spaces

In this section, the concepts of vague fuzzy digital structure retraction functions, vague fuzzy digital structure weak retraction functions, vague fuzzy digital structure cofibration, vague fuzzy digital structure homotopy extension property and vague fuzzy digital structure deformable are introduced. In this connection, some interesting properties are discussed.

Let \((A, \mathcal{D})\) be a vague fuzzy digital structure space and \( B \) be a vague fuzzy digital subset of \( A \), that is, \( B \subseteq A \). Since \( B_{\alpha\beta} \) is a subset of a digital structure space \( A_{\alpha\beta} \) for every \( \alpha, \beta \in I \), it can be given as a digital structure subspace on the set \( B_{\alpha\beta} \).

By using this structure, we can define a vague fuzzy digital structure on the vague fuzzy digital set \( B \). The vague fuzzy digital structure is denoted as \( \mathcal{D}|_B \). A pair \((B, \mathcal{D}|_B)\) is said to be a vague fuzzy digital structure subspace of \((A, \mathcal{D})\). An inclusion function \( \mathcal{I} : (B, \mathcal{D}|_B) \to (A, \mathcal{D}) \) is a vague
fuzzy digital structure continuous function on this structure.

**Definition 3.2.1.** The vague fuzzy digital structure subspace $B$ is said to be a vague fuzzy digital structure retract of vague fuzzy digital structure space $A$ if and only if there is a vague fuzzy digital structure continuous function $\mathcal{R} : (A, \mathcal{D}) \to (B, \mathcal{D}|_B)$ such that $\mathcal{R} \circ \mathcal{J} = \mathcal{I}_B$ where $\mathcal{J} : (B, \mathcal{D}|_B) \to (A, \mathcal{D})$ and $\mathcal{I}_B : (B, \mathcal{D}|_B) \to (B, \mathcal{D}|_B)$ are inclusion function and identity function respectively. The function $\mathcal{R}$ is said to be a vague fuzzy digital structure retraction function.

**Definition 3.2.2.** The vague fuzzy digital structure subspace $B$ is said to be a vague fuzzy digital structure weak retract of vague fuzzy digital structure space $A$ if and only if there is a vague fuzzy digital structure continuous function $\mathcal{R} : (A, \mathcal{D}) \to (B, \mathcal{D}|_B)$ such that $\mathcal{R} \circ \mathcal{J} \simeq \mathcal{I}_B$ where $\mathcal{J} : (B, \mathcal{D}|_B) \to (A, \mathcal{D})$ and $\mathcal{I}_B : (B, \mathcal{D}|_B) \to (B, \mathcal{D}|_B)$ are inclusion function and identity function respectively. The function $\mathcal{R}$ is said to be a vague fuzzy digital structure weak retraction function.

**Remark 3.2.1.** Every vague fuzzy digital structure retract is a vague fuzzy digital structure weak retract.

**Proof:** The proof is simple. ♦

**Remark 3.2.2.** Let $B$ be a vague fuzzy digital structure subspace of a vague fuzzy digital structure space $A$. Then for every $\alpha, \beta \in I$,
(i) the vague fuzzy digital structure subspace $B$ is a vague fuzzy digital structure retract of vague fuzzy digital structure space $A$ if and only if the digital structure subspace $B_{a\beta}$ is a digital structure retract of digital structure space $A_{a\beta}$.

(ii) the vague fuzzy digital structure subspace $B$ is a vague fuzzy digital structure weak retract of vague fuzzy digital structure space $A$ if and only if the digital structure subspace $B_{a\beta}$ is a digital structure weak retract of digital structure space $A_{a\beta}$.

**Proof:**

(i) Suppose that $B$ is a vague fuzzy digital structure retract of $A$. Then there is a vague fuzzy digital structure continuous function $R : (A, \mathcal{D}) \to (B, \mathcal{D}|_B)$ such that $R \circ J = I_B$ where $J : (B, \mathcal{D}|_B) \to (A, \mathcal{D})$ and $I_B : (B, \mathcal{D}|_B) \to (B, \mathcal{D}|_B)$ are inclusion function and identity function respectively.

By proposition 3.1.1., for any $\alpha, \beta \in I$, there is a digital structure continuous function $R_{a\beta} : (A_{a\beta}, \mathcal{D}_{a\beta}) \to (B_{a\beta}, \mathcal{D}|_{B_{a\beta}})$ such that $R_{a\beta} \circ J_{a\beta} = I_{B_{a\beta}}$ where $J_{a\beta} : (B_{a\beta}, \mathcal{D}|_{B_{a\beta}}) \to (A_{a\beta}, \mathcal{D}_{a\beta})$ and $I_{B_{a\beta}} : (B_{a\beta}, \mathcal{D}|_{B_{a\beta}}) \to (B_{a\beta}, \mathcal{D}|_{B_{a\beta}})$ are inclusion function and identity function respectively. This implies that $B$ is a digital structure retract of $A$.

Converse part can also be proved in a similar manner.

(ii) The proof is similar to that of (i).
**Notation 3.2.1.** Let $I = [0, 1]$ be the closed interval. Then

(i) For any $s \in I$, a vague fuzzy digital pair is defined and denoted as $\widehat{s} = \langle s, 1 - s \rangle$ such that $s + (1 - s) \leq 1$.

(ii) $\widehat{0} = \langle 0, 1 \rangle$ and $\widehat{1} = \langle 1, 0 \rangle$, for $0, 1 \in I$.

**Definition 3.2.3.** Let $(A, D^1)$, $(B, D^2)$ and $(C, D^3)$ be any three vague fuzzy digital structure spaces. Let $g : (A, D^1) \to (B, D^2)$ and $h : (B, D^2) \to (C, D^3)$ be any two vague fuzzy digital structure continuous functions and $G : A \times I \to C$ which satisfies the condition $G(P_{m,n}, \widehat{0}) = h(g(P_{m,n}))$ be a vague fuzzy digital structure homotopy where $P_{m,n}$ is a vague fuzzy digital point in $(A, D^1)$.

If there is a vague fuzzy digital structure homotopy $F : B \times I \to C$ such that $F(Q_{m,n}, \widehat{0}) = h(Q_{m,n})$, for every $Q_{m,n}$ is a vague fuzzy digital point in $(B, D^2)$ and $F(g(P_{m,n}), \widehat{s}) = G(P_{m,n}, \widehat{s})$, then the function $g$ is said to be a vague fuzzy digital structure cofibration.

**Proposition 3.2.1.** Let $(A, D^1)$, $(B, D^2)$ and $(C, D^3)$ be any three vague fuzzy digital structure spaces. Let $g : (A, D^1) \to (B, D^2)$ be any vague fuzzy digital structure continuous function. Then $g$ is a vague fuzzy digital structure cofibration if and only if for every $\alpha, \beta \in I$, $g_{\alpha\beta} : (A_{\alpha\beta}, D^1_{\alpha\beta}) \to (B_{\alpha\beta}, D^2_{\alpha\beta})$ is a digital structure cofibration.

**Proof:**

Let $g : (A, D^1) \to (B, D^2)$ be a vague fuzzy digital structure cofibration. For every $\alpha, \beta \in I$, the functions $h : (B, D^2) \to (C, D^3)$ and $G : A \times I \to C$
induces the functions $g_{\alpha \beta} : (B_{\alpha \beta}, D^{2}_{\alpha \beta}) \to (C_{\alpha \beta}, D^{3}_{\alpha \beta})$ and $G_{\alpha \beta} : A_{\alpha \beta} \times I \to C_{\alpha \beta}$ such that $G_{\alpha \beta}(P, 0) = h_{\alpha \beta}(g_{\alpha \beta}(P))$, for every $P \in A_{\alpha \beta}$. The vague fuzzy digital structure homotopy $F : B \times I \to C$ induces the digital structure homotopy $F_{\alpha \beta} : B_{\alpha \beta} \times I \to C_{\alpha \beta}$ such that $F_{\alpha \beta}(g_{\alpha \beta}(P), s) = G_{\alpha \beta}(P, s)$ and $F_{\alpha \beta}(Q, 0) = h_{\alpha \beta}(Q)$, for every $\alpha, \beta, s \in I$ and $Q \in B_{\alpha \beta}$. Hence, $g_{\alpha \beta} : (A_{\alpha \beta}, D^{1}_{\alpha \beta}) \to (B_{\alpha \beta}, D^{2}_{\alpha \beta})$ is a digital structure cofibration.

**Definition 3.2.4.** If the inclusion function $\mathcal{J} : (B, D|_{B}) \to (A, D)$ is a vague fuzzy digital structure cofibration, then the pair $(A, B)$ is said to have a vague fuzzy digital structure homotopy extension property.

**Proposition 3.2.2.** If the pair $(A, B)$ has a vague fuzzy digital structure homotopy extension property, then the vague fuzzy digital structure subspace $B$ is a vague fuzzy digital structure weak retract of a vague fuzzy digital structure space $A$ if and only if $B$ is a vague fuzzy digital structure retract of a vague fuzzy digital structure space $A$.

**Proof:**

It is obvious that every vague fuzzy digital structure retract is a vague fuzzy digital structure weak retract.

Conversely, let $B$ be a vague fuzzy digital structure weak retract of $A$. Then, there is a vague fuzzy digital structure continuous function $\mathcal{R} : (A, D) \to (B, D|_{B})$ such that $\mathcal{R} \circ \mathcal{J} \simeq I_B$ where $\mathcal{J} : (B, D|_{B}) \to (A, D)$ and $I_B : (B, D|_{B}) \to (B, D|_{B})$ are inclusion function and identity function respectively.
Let $G : A \times I \to A$ be a vague fuzzy digital structure homotopy between the functions $\mathcal{R} \circ \mathcal{I}$ and $\mathcal{I}_B$. Since $B_{\alpha\beta}$ is a vague fuzzy digital structure weak retract of $A_{\alpha\beta}$ and for every $\alpha, \beta \in I$, $G_{\alpha\beta} : A_{\alpha\beta} \times I \to A_{\alpha\beta}$ is a digital structure homotopy between the functions $\mathcal{R}_{\alpha\beta} \circ \mathcal{I}_{\alpha\beta}$ and $\mathcal{I}_{B_{\alpha\beta}}$ where $\mathcal{I}_{\alpha\beta} : (B_{\alpha\beta}, \mathcal{D}|_{B_{\alpha\beta}}) \to (A_{\alpha\beta}, \mathcal{D}_{\alpha\beta})$ and $\mathcal{I}_{B_{\alpha\beta}} : (B_{\alpha\beta}, \mathcal{D}|_{B_{\alpha\beta}}) \to (B_{\alpha\beta}, \mathcal{D}|_{B_{\alpha\beta}})$ are inclusion function and identity function respectively.

Since the pair $(A_{\alpha\beta}, B_{\alpha\beta})$ has a digital structure homotopy extension property, there is a digital structure homotopy $F_{\alpha\beta} : A_{\alpha\beta} \times I \to B_{\alpha\beta}$ such that $F_{\alpha\beta}|_{B_{\alpha\beta} \times I} = G_{\alpha\beta}, F_{\alpha\beta}(P, 0) = \mathcal{R}_{\alpha\beta} \circ \mathcal{I}_{\alpha\beta}(P) = \mathcal{R}_{\alpha\beta}(P)$.

If $\mathcal{R}'_{\alpha\beta} : (A_{\alpha\beta}, \mathcal{D}_{\alpha\beta}) \to (B_{\alpha\beta}, \mathcal{D}|_{B_{\alpha\beta}})$ is defined as $\mathcal{R}'_{\alpha\beta}(P) = F_{\alpha\beta}(P, 1)$, then $\mathcal{R}'_{\alpha\beta}$ is digital structure retraction function and $\mathcal{R}'_{\alpha\beta}$ is digital structure homotopic to $\mathcal{R}_{\alpha\beta}$. Thus, the vague fuzzy digital structure retraction function $\mathcal{R}' : (A, \mathcal{D}) \to (B, \mathcal{D}|_B)$ is obtained. This function $\mathcal{R}'$ is vague fuzzy digital structure homotopic to the function $\mathcal{R}$. Therefore, $B$ is a vague fuzzy digital structure retract of $A$.

**Definition 3.2.5.** If for the pair $(A, B)$, there is a vague fuzzy digital structure homotopy $G : A \times I \to A$ such that $G_{\alpha\beta}(P, 0) = P, G_{\alpha\beta}(P, 1) \in B_{\alpha\beta}(P)$, for every $\alpha, \beta \in I$. Then the vague fuzzy digital structure space $A$ is said to be vague fuzzy digital structure deformable into a vague fuzzy digital structure subspace $B$. 

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Proposition 3.2.3. The vague fuzzy digital structure space \( A \) is vague fuzzy digital structure deformable into a vague fuzzy digital structure subspace \( B \) if and only if the inclusion function \( \mathcal{I} : (B, \mathcal{D}|_B) \to (A, \mathcal{D}) \) has a right vague fuzzy digital structure homotopic inverse.

Proof:

Let the vague fuzzy digital structure space \( A \) be vague fuzzy digital structure deformable into a vague fuzzy digital structure subspace \( B \). Then, there is vague fuzzy digital structure homotopy \( G : A \times I \to A \) such that \( G_{\alpha\beta}(P, 0) = P, G_{\alpha\beta}(P, 1) \in B_{\alpha\beta}(P) \), for every \( \alpha, \beta \in I \). Let \( f_{\alpha\beta} : (A_{\alpha\beta}, \mathcal{D}_{\alpha\beta}) \to (B_{\alpha\beta}, \mathcal{D}|_{B_{\alpha\beta}}) \) be defined by the equation \( \mathcal{I}_{\alpha\beta} \circ f_{\alpha\beta}(P) = G_{\alpha\beta}(P, 1) \).

Thus, the function \( f : (A, \mathcal{D}) \to (B, \mathcal{D}|_B) \) is obtained and it is clear that \( G \) is vague fuzzy digital structure homotopy between the functions \( \mathcal{I}_A \) and \( \mathcal{I} \circ f \). Hence, \( f \) is a right vague fuzzy digital structure homotopic to the inverse function of \( \mathcal{I} \).

Conversely, suppose that the inclusion function \( \mathcal{I} : (B, \mathcal{D}|_B) \to (A, \mathcal{D}) \) has a vague fuzzy digital structure homotopic to the inverse function \( f : (A, \mathcal{D}) \to (B, \mathcal{D}|_B) \). Then, \( \mathcal{I} \circ f \simeq \mathcal{I}_A \) where \( \mathcal{I}_A : (A, \mathcal{D}) \to (A, \mathcal{D}) \) is an identity function of \( A \). Let \( F : A \times I \to A \) be vague fuzzy digital structure homotopy between the functions \( \mathcal{I}_A \) and \( \mathcal{I} \circ f \). Then,

\[
F(P, 0) = P \text{ and } F(A \times I) = \mathcal{I} \circ f(A) \subset B.
\]
Hence, \( A \) is vague fuzzy digital structure deformable into a vague fuzzy digital structure subspace \( B \).

**Definition 3.2.6.** Let \( B \) be vague fuzzy digital structure subspace of a vague fuzzy digital structure space \( A \).

(i) If the inclusion function \( I : (B, \mathcal{D}|_B) \to (A, \mathcal{D}) \) is a vague fuzzy digital structure homotopy equivalence, then the vague fuzzy digital structure subspace \( B \) is said to be a vague fuzzy digital structure weak deformation retract of vague fuzzy digital structure space \( A \).

(ii) If the vague fuzzy digital structure subspace \( B \) is a vague fuzzy digital structure retract of vague fuzzy digital structure space \( A \) and \( A \) is deformable into \( B \), then the vague fuzzy digital structure subspace \( B \) is said to be a vague fuzzy digital structure deformation retract of vague fuzzy digital structure space \( A \).

**Proposition 3.2.4.** If the pair \( (A, B) \) has the vague fuzzy digital structure homotopy extension property, then the vague fuzzy digital structure subspace \( B \) is a vague fuzzy digital structure weak deformation retract if and only if the vague fuzzy digital structure subspace \( B \) is a vague fuzzy digital structure deformation retract.

**Proof:** The proof follows from Proposition 3.2.2.