Coker [38] introduced the concept of intuitionistic fuzzy compact sets. In this chapter, the concepts of intuitionistic fuzzy compact rings, intuitionistic fuzzy ring basic compact spaces, intuitionistic fuzzy $RB$-irresolute functions, intuitionistic fuzzy $RB$-open functions and intuitionistic fuzzy ring extremal compact spaces are introduced. In this connection, some interesting properties are established.

### 7.1 CHARACTERIZATIONS OF INTUITIONISTIC FUZZY RING BASIC COMPACT SPACES

In this section, the concepts of intuitionistic fuzzy compact rings, intuitionistic fuzzy $RB$-interiors, intuitionistic fuzzy $RB$-closures and intuitionistic fuzzy ring basic compact spaces are introduced. Some interesting properties are established.

**Definition 7.1.1.** Let $(R, \mathcal{S})$ be any intuitionistic fuzzy structure ring space and $A$ be an intuitionistic fuzzy ring in $R$. Then $A$ is said to be an intuitionistic fuzzy compact ring in $(R, \mathcal{S})$ if for every family of $\{A_i \mid i \in J\}$ of
intuitionistic fuzzy open rings in \((R, \mathcal{S})\) satisfies the condition \(A \subseteq \bigcup_{i \in J} A_i\),
there exists a finite subfamily \(J_0 = \{1, ..., n\} \subseteq J\) such that \(A \subseteq \bigcup_{i=1}^{n} A_i\).

The complement of an intuitionistic fuzzy compact ring in \((R, \mathcal{S})\) is an
intuitionistic fuzzy co-compact ring in \((R, \mathcal{S})\).

**Definition 7.1.2.** Let \((R, \mathcal{S})\) be any intuitionistic fuzzy structure ring space
and \(A\) be an intuitionistic fuzzy ring in \(R\). Then \(A\) is said to be an intuition-
istic fuzzy compact \(F_\sigma\) ring in \((R, \mathcal{S})\) if it is both intuitionistic fuzzy compact
and intuitionistic fuzzy \(F_\sigma\).

The complement of an intuitionistic fuzzy compact \(F_\sigma\) ring in \((R, \mathcal{S})\) is an
intuitionistic fuzzy co-compact \(G_\delta\) ring in \((R, \mathcal{S})\).

**Notation 7.1.1.** Let \((R, \mathcal{S})\) be any intuitionistic fuzzy structure ring space.
Then

(i) \(RB(C)\) denotes the collection of all intuitionistic fuzzy compact \(F_\sigma\) rings
    in \((R, \mathcal{S})\).

(ii) \(RB(Co)\) denotes the collection of all intuitionistic fuzzy co-compact \(G_\delta\)
rings in \((R, \mathcal{S})\).

**Definition 7.1.3.** Let \((R, \mathcal{S})\) be any intuitionistic fuzzy structure ring space.
Let \(A = \langle x, \mu_A, \gamma_A \rangle\) be an intuitionistic fuzzy ring in \(R\). Then

(i) the intuitionistic fuzzy \(RB\)-interior of \(A\) is defined and denoted as

\[
IF_{RBint}(A) = \cup\{B = \langle x, \mu_B, \gamma_B \rangle \mid B \in RB(C) \text{ and } B \subseteq A\}.
\]
(ii) the intuitionistic fuzzy RB-closure of \( A \) is defined and denoted as

\[
IF_{RB\text{cl}}(A) = \cap \{ B = \langle x, \mu_B, \gamma_B \rangle \mid B \in RB(Co) \text{ and } A \subseteq B \}. 
\]

**Remark 7.1.1.** Let \((R, \mathcal{S})\) be any intuitionistic fuzzy structure ring space. Let \( A = \langle x, \mu_A, \gamma_A \rangle \) be an intuitionistic fuzzy ring in \( R \). Then the following statements hold:

(i) \( IF_{RB\text{cl}}(A) = A \) if and only if \( A \) is an intuitionistic fuzzy co-compact \( G_δ \) ring.

(ii) \( IF_{RB\text{int}}(A) = A \) if and only if \( A \) is an intuitionistic fuzzy compact \( F_σ \) ring.

(iii) \( IF_{RB\text{int}}(A) \subseteq A \subseteq IF_{RB\text{cl}}(A) \).

(vi) \( IF_{RB\text{cl}}(\overline{A}) = \overline{IF_{RB\text{int}}(A)} \) and \( IF_{RB\text{int}}(\overline{A}) = \overline{IF_{RB\text{cl}}(A)} \).

(v) \( IF_{RB\text{cl}}(0_\sim) = 0_\sim \) and \( IF_{RB\text{int}}(1_\sim) = 1_\sim \).

(vi) \( IF_{RB\text{cl}}(1_\sim) = 1_\sim \) and \( IF_{RB\text{int}}(0_\sim) = 0_\sim \).

**Proof:** The proof is simple.

**Definition 7.1.4.** Let \((R, \mathcal{S})\) be any intuitionistic fuzzy structure ring space. Then \((R, \mathcal{S})\) is called an intuitionistic fuzzy ring basic compact space if the intuitionistic fuzzy RB-closure of every intuitionistic fuzzy compact \( F_σ \) ring is an intuitionistic fuzzy compact \( F_σ \) ring.
Example 7.1.1. Let $R = \{0, 1\}$ be a set of integers of module 2 with two binary operations as follows:

\[
\begin{array}{ccc}
+ & 0 & 1 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\cdot & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{array}
\]

Then $(R, +, \cdot)$ is a ring. Define intuitionistic fuzzy rings $A, B$ and $C$ on $R$ as follows:

\[
\mu_A(0) = 0.5, \mu_A(1) = 0.5 \quad \text{and} \quad \gamma_A(0) = 0.5, \gamma_A(1) = 0.5
\]

\[
\mu_B(0) = 0.5, \mu_B(1) = 0.4 \quad \text{and} \quad \gamma_B(0) = 0.3, \gamma_B(1) = 0.4
\]

\[
\mu_C(0) = 0.5, \mu_C(1) = 0.5 \quad \text{and} \quad \gamma_C(0) = 0.3, \gamma_C(1) = 0.4.
\]

Then $\mathcal{S} = \{0_\infty, A, B, C, 1_\infty\}$ is an intuitionistic fuzzy structure ring on $R$. Thus the pair $(R, \mathcal{S})$ is an intuitionistic fuzzy structure ring space. The intuitionistic fuzzy compact $F_\sigma$ rings in $(R, \mathcal{S})$ are $0_\infty, \overline{A}, \overline{B}, \overline{C}$ and $1_\infty$.

Now, the intuitionistic fuzzy $RB$-closure of every intuitionistic fuzzy compact $F_\sigma$ ring is an intuitionistic fuzzy compact $F_\sigma$ ring in $(R, \mathcal{S})$. Therefore, $(R, \mathcal{S})$ is an intuitionistic fuzzy ring basic compact space.

Proposition 7.1.1. Let $(R, \mathcal{S})$ be any intuitionistic fuzzy structure ring space. Then the following statements are equivalent:

(i) $(R, \mathcal{S})$ is an intuitionistic fuzzy ring basic compact space.

(ii) For each intuitionistic fuzzy co-compact $G_\delta$ ring $A$, $IF_{RB\text{int}}(A)$ is intuitionistic fuzzy co-compact $G_\delta$ ring.
(iii) For each intuitionistic fuzzy compact \( F_\sigma \) ring \( A \), we have
\[
IF_{RBcl}(IF_{RBcl}(A)) = IF_{RBcl}(A).
\]

iv) For every pair of intuitionistic fuzzy compact \( F_\sigma \) rings \( A \) and \( B \) with
\[
IF_{RBcl}(A) = \overline{B},
\]
we have \( IF_{RBcl}(B) = \overline{IF_{RBcl}(A)} \).

**Proof:**

(i) \( \Rightarrow \) (ii)

Let \( A \) be an intuitionistic fuzzy co-compact \( G_\delta \) ring in \( (R, \mathcal{S}) \). Then, \( \overline{A} \) is an intuitionistic fuzzy compact \( F_\sigma \) ring in \( (R, \mathcal{S}) \). Then by assumption, \( IF_{RBcl}(\overline{A}) \) is an intuitionistic fuzzy compact \( F_\sigma \) ring in \( (R, \mathcal{S}) \). Now, \( IF_{RBcl}(\overline{A}) = \overline{IF_{RBint}(A)} \). Therefore, \( IF_{RBint}(A) \) is an an intuitionistic fuzzy co-compact \( G_\delta \) ring in \( (R, \mathcal{S}) \). Hence, (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (iii)

Let \( A \) be an intuitionistic fuzzy compact \( F_\sigma \) ring in \( (R, \mathcal{S}) \). Then, \( \overline{A} \) is an intuitionistic fuzzy co-compact \( G_\delta \) ring in \( (R, \mathcal{S}) \). By assumption, \( IF_{RBint}(\overline{A}) = \overline{IF_{RBcl}(A)} \) is an intuitionistic fuzzy co-compact \( G_\delta \) ring in \( (R, \mathcal{S}) \). Now, \( IF_{RBcl}(\overline{IF_{RBcl}(A)}) = \overline{IF_{RBcl}(A)} \). Hence, (ii) \( \Rightarrow \) (iii).

(iii) \( \Rightarrow \) (iv)

Let \( A \) and \( B \) be any two intuitionistic fuzzy compact \( F_\sigma \) rings in \( (R, \mathcal{S}) \) such that \( IF_{RBcl}(A) = \overline{B} \). By (iii), \( IF_{RBcl}(\overline{IF_{RBcl}(A)}) = \overline{IF_{RBcl}(A)} \). This implies that \( IF_{RBcl}(B) = \overline{IF_{RBcl}(A)} \). Hence, (iii) \( \Rightarrow \) (iv).
(iv)⇒(i)

Let A and B be any two intuitionistic fuzzy compact $F_\sigma$ rings in $(R, \mathcal{S})$ such that $IFVcl(A) = B$. By (iv), it follow that $IF_{RBcl}(B) = IF_{RBcl}(A)$. That is, $IF_{RBcl}(A)$ is an intuitionistic fuzzy co-compact $G_\delta$ ring in $(R, \mathcal{S})$.

This implies that $IF_{RBcl}(A)$ is an intuitionistic fuzzy compact $F_\sigma$ ring in $(R, \mathcal{S})$. Thus, $(R, \mathcal{S})$ is an intuitionistic fuzzy ring basic compact space. Hence, (iv)⇒(i). Hence the proof.

\begin{prop}
Let $(R, \mathcal{S})$ be any intuitionistic fuzzy structure ring space. Then $(R, \mathcal{S})$ is an intuitionistic fuzzy ring basic compact space if and only if for each intuitionistic fuzzy compact $F_\sigma$ ring A and intuitionistic fuzzy co-compact $G_\delta$ ring B such that $A \subseteq B$, $IF_{RBcl}(A) \subseteq IF_{RBint}(B)$.

\textbf{Proof:}

Let A be an intuitionistic fuzzy compact $F_\sigma$ ring and B be an intuitionistic fuzzy co-compact $G_\delta$ ring in $(R, \mathcal{S})$ such that $A \subseteq B$. Then by (ii) of Proposition 7.1.1., $IF_{RBint}(B)$ is an intuitionistic fuzzy co-compact $G_\delta$ ring in $(R, \mathcal{S})$. Therefore, $IF_{RBcl}(IF_{RBint}(B)) = IF_{RBint}(B)$. Since A is an intuitionistic fuzzy compact $F_\sigma$ ring and $A \subseteq B$, $A \subseteq IF_{RBint}(B)$. Now,

$$IF_{RBcl}(A) \subseteq IF_{RBcl}(IF_{RBint}(B)) = IF_{RBint}(B).$$

Conversely, let B be an intuitionistic fuzzy co-compact $G_\delta$ ring in $(R, \mathcal{S})$. Then, $IF_{RBint}(B)$ is an intuitionistic fuzzy compact $F_\sigma$ ring in $(R, \mathcal{S})$ and
IF_{RB\text{int}}(B) \subseteq B. By assumption, IF_{RB\text{cl}}(IF_{RB\text{int}}(B)) \subseteq IF_{RB\text{int}}(B).

Also, IF_{RBB\text{int}}(B) \subseteq IF_{RB\text{cl}}(IF_{RB\text{int}}(B)). This implies that

IF_{RB\text{cl}}(IF_{RB\text{int}}(B)) = IF_{RB\text{int}}(B).

Therefore, IF_{RB\text{int}}(B) is an intuitionistic fuzzy co-compact \(G_\delta\) ring in \((R, \mathcal{S})\).

Then by (ii) of Proposition 7.1.1., \((R, \mathcal{S})\) is an intuitionistic fuzzy ring basic compact space.

\[\text{Definition 7.1.5.}\] Let \((R, \mathcal{S})\) be any intuitionistic fuzzy structure ring space. An intuitionistic fuzzy ring \(A\) in \((R, \mathcal{S})\) is said to be an intuitionistic fuzzy \(Cmpt-G_\delta F_\sigma\) ring in \((R, \mathcal{S})\) if it is both intuitionistic fuzzy compact \(F_\sigma\) and intuitionistic fuzzy co-compact \(G_\delta\).

\[\text{Remark 7.1.2.}\] Let \((R, \mathcal{S})\) be any intuitionistic fuzzy ring basic compact space. Let \(\{A_i, B_i / i \in N\}\) be a collection such that \(A_i\)'s are intuitionistic fuzzy compact \(F_\sigma\) rings and \(B_i\)'s are intuitionistic fuzzy co-compact \(G_\delta\) rings and let \(A\) and \(B\) be any two intuitionistic fuzzy \(Cmpt-G_\delta F_\sigma\) rings. If \(A_i \subseteq A \subseteq B_j\) and \(A_i \subseteq B \subseteq B_j\) for all \(i, j \in N\), then there exists an intuitionistic fuzzy \(Cmpt-G_\delta F_\sigma\) ring \(C\) such that \(IF_{RB\text{cl}}(A_i) \subseteq C \subseteq IF_{RB\text{int}}(B_j)\) for all \(i, j \in N\).

\[\text{Proof:}\]

By Proposition 7.1.2., \(IF_{RB\text{cl}}(A_i) \subseteq IF_{RB\text{cl}}(A) \cap IF_{RB\text{int}}(B) \subseteq IF_{RB\text{int}}(B_j)\) for all \(i, j \in N\). Therefore, \(C = IF_{RB\text{cl}}(A) \cap IF_{RB\text{int}}(B)\) is an intuitionistic fuzzy \(Cmpt-G_\delta F_\sigma\) ring in \((R, \mathcal{S})\) satisfying the required conditions.

\[\text{Note 7.1.1.}\] \(IF(R)\) denotes the collection of all intuitionistic fuzzy rings in \(R\).
Proposition 7.1.3. Let \((R, \mathcal{S})\) be any intuitionistic fuzzy ring basic compact space. Let \(\{A_q\}_{q \in Q}\) and \(\{B_q\}_{q \in Q}\) be monotone increasing collections of intuitionistic fuzzy compact \(F_\delta\) rings and intuitionistic fuzzy co-compact \(G_\delta\) rings of \((R, \mathcal{S})\) respectively and suppose that \(A_{q_1} \subseteq B_{q_2}\) whenever \(q_1 < q_2\) (\(Q\) is the set of all rational numbers). Then there exists a monotone increasing collection \(\{C_q\}_{q \in Q}\) of intuitionistic fuzzy \(Cmpt\)-\(G_\delta\) rings of \((R, \mathcal{S})\) such that \(IF_{RBcl}(A_{q_1}) \subseteq C_{q_2}\) and \(C_{q_1} \subseteq IF_{RBint}(B_{q_2})\) whenever \(q_1 < q_2\).

**Proof:**

Let us arrange all rational numbers into a sequence \(\{q_n\}\) (without repetitions). For every \(n \geq 2\), we shall define inductively a collection \(\{C_{q_i}/1 \leq i < n\}\) of intuitionistic fuzzy \(Cmpt\)-\(G_\delta\) rings such that

\[
IF_{RBcl}(A_{q_1}) \subseteq C_{q_2}\text{ if } q < q_1, C_{q_1} \subseteq IF_{RBint}(B_{q_2})\text{ if } q_1 < q, \text{ for all } i < n \quad (S_n)
\]

By Proposition 7.1.2., the countable collections \(\{IF_{RBcl}(A_{q_1})\}\) and \(\{IF_{RBint}(B_{q_2})\}\) satisfy \(IF_{RBcl}(A_{q_1}) \subseteq IF_{RBint}(B_{q_2})\) if \(q_1 < q_2\). By Remark 7.1.2., there exists an intuitionistic fuzzy \(Cmpt\)-\(G_\delta\) ring \(D_1\) such that

\[
IF_{RBcl}(A_{q_1}) \subseteq D_1 \subseteq IF_{RBint}(B_{q_2}).
\]

Letting \(C_{q_1} = D_1\), we get \((S_2)\). Assume that intuitionistic fuzzy rings \(C_{q_i}\) are already defined for \(i < n\) and satisfy \((S_n)\).

Define \(E = \bigcup\{C_{q_i} \mid i < n, q_i < q_n\} \cup A_{q_n}\) and \(F = \bigcap\{C_{q_j} \mid j < n, q_j > q_n\} \cap B_{q_n}\) . Then we have,

\[
IF_{RBcl}(C_{q_1}) \subseteq IF_{RBcl}(E) \subseteq IF_{RBint}(C_{q_1})
\]
and
\[ IF_{RB\text{cl}}(C_{q_i}) \subseteq IF_{RB\text{int}}(F) \subseteq IF_{RB\text{int}}(C_{q_j}) \]
whenever \( q_i < q_n < q_j \) \((i, j < n)\), as well as
\[ A_q \subseteq IF_{RB\text{cl}}(E) \subseteq B_q \]
and \( A_q \subseteq IF_{RB\text{int}}(F) \subseteq B_q' \)
whenever \( q < q_n < q' \). This shows that the countable collections
\[ \{C_{q_i}/i < n, q_i < q_n\} \cup \{A_q/q < q_n\} \text{ and } \{C_{q_j}/j < n, q_j > q_n\} \cup \{B_q/q > q_n\} \]
together with \( E \) and \( F \) fulfill the conditions of Remark 7.1.2. Hence, there exists an intuitionistic fuzzy \( C\text{mpt}-G_{\delta}F_\sigma \) ring \( D_n \) such that
\[ IF_{RB\text{cl}}(D_n) \subseteq B_q \text{ if } q_n < q, A_q \subseteq IF_{RB\text{int}}(D_n) \text{ if } q < q_n, \]
\[ IF_{RB\text{cl}}(C_{q_i}) \subseteq IF_{RB\text{int}}(D_n) \text{ if } q_i < q_n \text{ and } IF_{RB\text{cl}}(D_n) \subseteq IF_{RB\text{int}}(C_{q_j}) \text{ if } q_n < q_j \]
where \( 1 \leq i, j \leq n - 1 \).

Now, setting \( C_{q_n} = D_n \) we obtain an intuitionistic fuzzy rings \( C_{q_1}, C_{q_2}, ..., C_{q_n} \)
that satisfy \( (S_{n+1}) \). Therefore the collection \( \{C_{q_i} \mid i = 1, 2, ...\} \) has the required
property. ♦

**Definition 7.1.6.** Let \((R_1, \mathcal{J}_1)\) and \((R_2, \mathcal{J}_2)\) be any two intuitionistic fuzzy
structure ring spaces. A function \( f : (R_1, \mathcal{J}_1) \to (R_2, \mathcal{J}_2) \) is called an intuitionistic fuzzy \( RB \)-irresolute function if \( f^{-1}(A) \) is an intuitionistic fuzzy compact \( F_\sigma \) ring in \((R_1, \mathcal{J}_1)\), for each intuitionistic fuzzy compact \( F_\sigma \) ring \( A \) in \((R_2, \mathcal{J}_2)\).

**Proposition 7.1.4.** Let \((R_1, \mathcal{J}_1)\) and \((R_2, \mathcal{J}_2)\) be any two intuitionistic fuzzy
structure ring spaces. Then \( f : (R_1, \mathcal{J}_1) \to (R_2, \mathcal{J}_2) \) is an intuitionistic fuzzy
RB-irresolute function if and only if \( f(IF_{RBcl}(A)) \subseteq IF_{RBcl}(f(A)) \), for each intuitionistic fuzzy ring \( A \) in \( (R_1, \mathcal{S}_1) \).

**Proof:**

Suppose that \( f \) is an intuitionistic fuzzy RB-irresolute function and let \( A \) be an intuitionistic fuzzy ring in \( (R_1, \mathcal{S}_1) \). Then, \( IF_{RBcl}(f(A)) \) is an intuitionistic fuzzy co-compact \( G_\delta \) ring in \( (R_2, \mathcal{S}_2) \). By assumption, \( f^{-1}(IF_{RBcl}(f(A))) \) intuitionistic fuzzy co-compact \( G_\delta \) ring in \( (R_1, \mathcal{S}_1) \). Now,

\[
A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(IF_{RBcl}(f(A))).
\]

Hence by Definition 7.1.3., \( IF_{RBcl}(A) \subseteq IF_{RBcl}(f^{-1}(IF_{RBcl}(f(A)))) \). This implies that \( f(IF_{RBcl}(A)) \subseteq IF_{RBcl}(f(A)) \).

Conversely, suppose that \( A \) is an intuitionistic fuzzy co-compact \( G_\delta \) ring in \( (R_2, \mathcal{S}_2) \). Then, \( IF_{RBcl}(A) = A \). Now, by assumption,

\[
f(IF_{RBcl}(f^{-1}(A))) \subseteq IF_{RBcl}(f(f^{-1}(A)))
\]

\[
= IF_{RBcl}(A)
\]

\[
= A.
\]

This implies that \( IF_{RBcl}(f^{-1}(A)) \subseteq f^{-1}(A) \). But, \( IF_{RBcl}(f^{-1}(A)) \supseteq f^{-1}(A) \). Hence, \( IF_{RBcl}(f^{-1}(A)) = f^{-1}(A) \). That is, \( f^{-1}(A) \) is an intuitionistic fuzzy co-compact \( G_\delta \) ring in \( (R_1, \mathcal{S}_1) \). Therefore, \( f \) is an intuitionistic fuzzy RB-irresolute function.

\( \diamond \)

**Definition 7.1.7.** Let \( (R_1, \mathcal{S}_1) \) and \( (R_2, \mathcal{S}_2) \) be any two intuitionistic fuzzy structure ring spaces. A function \( f : (R_1, \mathcal{S}_1) \to (R_2, \mathcal{S}_2) \) is called an intuition-
istic fuzzy $RB$-open function if $f(A)$ is an intuitionistic fuzzy compact $F_{\sigma}$ ring in $(R_2, \mathcal{S}_2)$, for each intuitionistic fuzzy compact $F_{\sigma}$ ring $A$ in $(R_1, \mathcal{S}_1)$.

**Proposition 7.1.5.** Let $(R_1, \mathcal{S}_1)$ and $(R_2, \mathcal{S}_2)$ be any two intuitionistic fuzzy structure ring spaces. Let $f : (R_1, \mathcal{S}_1) \to (R_2, \mathcal{S}_2)$ be an intuitionistic fuzzy $RB$-open, surjective function. Then $f^{-1}(IF_{RBcl}(A)) \subseteq IF_{RBcl}(f^{-1}(A))$, for each intuitionistic fuzzy ring $A$ in $(R_2, \mathcal{S}_2)$.

**Proof:**

Let $A$ be an intuitionistic fuzzy ring in $(R_2, \mathcal{S}_2)$ and let $B = f^{-1}(\overline{A})$. Then, $IF_{RBint}(f^{-1}(\overline{A})) = IF_{RBint}(B)$ is an intuitionistic fuzzy compact $F_{\sigma}$ ring in $(R_1, \mathcal{S}_1)$. Now, $IF_{RBint}(B) \subseteq B$. Hence, $f(\overline{IF_{RBint}(B)}) \subseteq \overline{f(B)}$. That is, $IF_{RBint}(f(\overline{IF_{RBint}(B)})) \subseteq IF_{RBint}(f(B))$. Since $f$ is an intuitionistic fuzzy $RB$-open function, $f(\overline{IF_{RBint}(B)})$ is an intuitionistic fuzzy compact $F_{\sigma}$ ring in $(R_2, \mathcal{S}_2)$. Therefore, $f(\overline{IF_{RBint}(B)}) \subseteq IF_{RBint}(\overline{B}) = IF_{RBint}(\overline{A})$. Hence,

$$IF_{RBint}(f^{-1}(\overline{A})) \subseteq f^{-1}(IF_{RBint}(\overline{A})).$$

(7.1.1)

Now, (7.1.1) becomes,

$$\overline{f^{-1}(IF_{RBint}(\overline{A}))} \supseteq f^{-1}(\overline{IF_{RBint}(\overline{A})}).$$

$$\Rightarrow f^{-1}(IF_{RBint}(\overline{f^{-1}(A)})) \supseteq IF_{RBcl}(f^{-1}(A)).$$

Therefore, $f^{-1}(IF_{REC}(f^{-1}(A)) \supseteq IF_{REC}(f^{-1}(A))$. ♦

**Proposition 7.1.6.** Let $(R_1, \mathcal{S}_1)$ be any intuitionistic fuzzy ring basic compact space and $(R_2, \mathcal{S}_2)$ be any intuitionistic fuzzy structure ring space. Let $f : (R_1, \mathcal{S}_1) \to (R_2, \mathcal{S}_2)$ be an intuitionistic fuzzy $RB$-irresolute, intuitionistic
fuzzy RB-open, surjective function. Then \((R_2, \mathcal{J}_2)\) is an intuitionistic fuzzy ring basic compact space.

**Proof:**

Let \(A\) be an intuitionistic fuzzy compact \(F_\sigma\) ring in \((R_2, \mathcal{J}_2)\). Since \(f\) is an intuitionistic fuzzy RB-irresolute function, \(f^{-1}(A)\) is an intuitionistic fuzzy compact \(F_\sigma\) ring in \((R_1, \mathcal{J}_1)\). Since \((R_1, \mathcal{J}_1)\) is an intuitionistic fuzzy ring basic compact space, \(IF_{RBcl}(f^{-1}(A))\) is an intuitionistic fuzzy compact \(F_\sigma\) ring in \((R_1, \mathcal{J}_1)\). As \(f\) is an intuitionistic fuzzy RB-open function, \(f(IF_{RBcl}(f^{-1}(A)))\) is an intuitionistic fuzzy compact \(F_\sigma\) ring in \((R_2, \mathcal{J}_2)\). Then by Proposition 7.1.5.,

\[
f^{-1}(IF_{RBcl}(A)) \subseteq IF_{RBcl}(f^{-1}(A)).
\]

This implies that \(f(f^{-1}(IF_{RBcl}(A))) = IF_{RBcl}(A) \subseteq f(IF_{RBcl}(f^{-1}(A)))\). Then by Proposition 7.1.4.,

\[
IF_{RBcl}(A) \subseteq f(IF_{RBcl}(f^{-1}(A))) \subseteq IF_{RBcl}(f(f^{-1}(A))) = IF_{RBcl}(A).
\]

This implies that \(f(IF_{RBcl}(f^{-1}(A))) = IF_{RBcl}(A)\). Therefore, \(IF_{RBcl}(A)\) is an intuitionistic fuzzy compact \(F_\sigma\) ring in \((R_2, \mathcal{J}_2)\). Hence, \((R_2, \mathcal{J}_2)\) is an intuitionistic fuzzy ring basic compact space.

\[\Box\]

### 7.2 INTUITIONISTIC FUZZY RING EXTREMAL COMPACT SPACES

In this section, the concepts of intuitionistic fuzzy ring extremal compact spaces, intuitionistic fuzzy \(RE\)-irresolute functions and intuitionis-
tic fuzzy $RE$-open functions are introduced. In this connection, some interesting properties are established.

**Definition 7.2.1.** Let $(R, \mathcal{I})$ be any intuitionistic fuzzy structure ring space and $A$ be an intuitionistic fuzzy ring in $R$. Then $A$ is said to be an intuitionistic fuzzy open compact ring in $(R, \mathcal{I})$ if it is both intuitionistic fuzzy open and intuitionistic fuzzy compact.

The complement of an intuitionistic fuzzy open compact ring in $(R, \mathcal{I})$ is an intuitionistic fuzzy closed co-compact ring in $(R, \mathcal{I})$.

**Notation 7.2.1.** Let $(R, \mathcal{I})$ be any intuitionistic fuzzy structure ring space. Then

(i) $RE(OC)$ denotes the collection of all intuitionistic fuzzy open compact rings in $(R, \mathcal{I})$.

(ii) $RE(CCo)$ denotes the collection of all intuitionistic fuzzy closed co-compact rings in $(R, \mathcal{I})$.

**Definition 7.2.2.** Let $(R, \mathcal{I})$ be any intuitionistic fuzzy structure ring space. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy ring in $R$. Then

(i) the intuitionistic fuzzy $RE$-interior of $A$ is defined and denoted as

$$IF_{REint}(A) = \cup\{B = \langle x, \mu_B, \gamma_B \rangle \mid B \in RE(OC) \text{ and } B \subseteq A\}.$$

(ii) the intuitionistic fuzzy $RE$-closure of $A$ is defined and denoted as

$$IF_{REcl}(A) = \cap\{B = \langle x, \mu_B, \gamma_B \rangle \mid B \in RE(CCo) \text{ and } A \subseteq B\}.$$
**Remark 7.2.1.** Let \((R, \mathcal{I})\) be any intuitionistic fuzzy structure ring space. Let \(A = \langle x, \mu_A, \gamma_A \rangle\) be an intuitionistic fuzzy ring in \(R\). Then the following statements hold:

(i) \(IF_{REcl}(A) = A\) if and only if \(A\) is an intuitionistic fuzzy closed co-compact ring.

(ii) \(IF_{REint}(A) = A\) if and only if \(A\) is an intuitionistic fuzzy open compact ring.

(iii) \(IF_{REint}(A) \subseteq A \subseteq IF_{REcl}(A)\).

(iv) \(IF_{REcl}(\overline{A}) = IF_{REint}(\overline{A})\) and \(IF_{REint}(\overline{A}) = IF_{REcl}(\overline{A})\).

(v) \(IF_{REcl}(0\sim) = 0\sim\) and \(IF_{REint}(1\sim) = 1\sim\).

(vi) \(IF_{REcl}(1\sim) = 1\sim\) and \(IF_{REint}(0\sim) = 0\sim\).

**Proof:** The proof is simple. ♦

**Definition 7.2.3.** Let \((R, \mathcal{I})\) be any intuitionistic fuzzy structure ring space. Then \((R, \mathcal{I})\) is called an intuitionistic fuzzy ring extremal compact space if the intuitionistic fuzzy \(RE\)-closure of every intuitionistic fuzzy open compact ring is an intuitionistic fuzzy open compact ring.

**Example 7.2.1.** Let \(R = \{0, 1\}\) be a set of integers of module 2 with two binary operations as follows:

\[
\begin{array}{c|c|c}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\quad\text{and}\quad
\begin{array}{c|c|c}
\cdot & 0 & 1 \\
\hline
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]
Then $(R, +, \cdot)$ is a ring. Define intuitionistic fuzzy rings $A, B, C$ and $D$ on $R$ as follows:

$$
\mu_A(0) = 0.2, \mu_A(1) = 0.2 \text{ and } \gamma_A(0) = 0.8, \gamma_A(1) = 0.8
$$
$$
\mu_B(0) = 0.1, \mu_B(1) = 0.2 \text{ and } \gamma_B(0) = 0.9, \gamma_B(1) = 0.8
$$
$$
\mu_C(0) = 0.8, \mu_C(1) = 0.8 \text{ and } \gamma_C(0) = 0.2, \gamma_C(1) = 0.2
$$
$$
\mu_D(0) = 0.9, \mu_D(1) = 0.8 \text{ and } \gamma_D(0) = 0.1, \gamma_D(1) = 0.2.
$$

Then $\mathcal{S} = \{0_\sim, A, B, C, D, 1_\sim\}$ is an intuitionistic fuzzy structure rings on $R$. Thus the pair $(R, \mathcal{S})$ is an intuitionistic fuzzy structure ring space. The intuitionistic fuzzy open compact rings in $(R, \mathcal{S})$ are $0_\sim, A, B, C, D$ and $1_\sim$.

Therefore, the intuitionistic fuzzy $RE$-closure of every intuitionistic fuzzy open compact ring is an intuitionistic fuzzy open compact ring in $(R, \mathcal{S})$. Hence, $(R, \mathcal{S})$ is an intuitionistic fuzzy ring extremal compact space.

**Proposition 7.2.1.** Let $(R, \mathcal{S})$ be any intuitionistic fuzzy structure ring space. Then the following statements are equivalent:

(i) $(R, \mathcal{S})$ is an intuitionistic fuzzy ring extremal compact space.

(ii) For each intuitionistic fuzzy closed co-compact ring $A$, $IF_{REint}(A)$ is an intuitionistic fuzzy closed co-compact ring.

(iii) For each intuitionistic fuzzy open compact ring $A$, we have

$$
IF_{REcl}(IF_{REcl}(A)) = IF_{REcl}(A).
$$
iv) For every pair of intuitionistic fuzzy open compact rings $A$ and $B$ with

$$IF_{REcl}(A) = \overline{B},$$ we have $IF_{REcl}(B) = \overline{IF_{REcl}(A)}$.

**Proof:**

(i) $\Rightarrow$ (ii)

Let $A$ be an intuitionistic fuzzy closed co-compact ring in $(R, \mathcal{S})$. Then, $\overline{A}$ is an intuitionistic fuzzy open compact ring in $(R, \mathcal{S})$. Then by assumption, $IF_{REcl}(\overline{A})$ is an intuitionistic fuzzy open compact ring in $(R, \mathcal{S})$.

Now, $IF_{REcl}(\overline{A}) = \overline{IF_{REint}(A)}$. Hence, $IF_{REint}(A)$ is an intuitionistic fuzzy closed co-compact ring in $(R, \mathcal{S})$. Hence, (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii)

Let $A$ be an intuitionistic fuzzy open compact ring in $(R, \mathcal{S})$. Then, $\overline{A}$ is an intuitionistic fuzzy closed co-compact ring in $(R, \mathcal{S})$.

By assumption, $IF_{REint}(\overline{A}) = \overline{IF_{REcl}(A)}$ is an intuitionistic fuzzy closed co-compact ring in $(R, \mathcal{S})$. Now, $IF_{REcl}(\overline{IF_{REcl}(A)}) = \overline{IF_{REcl}(A)}$. Hence, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv)

Let $A$ and $B$ be any two intuitionistic fuzzy open compact rings in $(R, \mathcal{S})$ such that $IF_{REcl}(A) = \overline{B}$. By (iii), $IF_{REcl}(\overline{IF_{REcl}(A)}) = \overline{IF_{REcl}(A)}$. This implies that $IF_{REcl}(B) = \overline{IF_{REcl}(A)}$. Hence, (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i)

Let $A$ and $B$ be any two intuitionistic fuzzy open compact rings in $(R, \mathcal{S})$ such that $IF_{REcl}(A) = \overline{B}$. By (iv), it follow that $IF_{REcl}(B) = \overline{IF_{REcl}(A)}$. That is, $\overline{IF_{REcl}(A)}$ is an intuitionistic fuzzy closed co-compact ring in $(R, \mathcal{S})$.
This implies that $IF_{REcl}(A)$ is an intuitionistic fuzzy open compact ring in $(R,\mathcal{I})$. Hence, $(R,\mathcal{I})$ is an intuitionistic fuzzy ring extremal compact space. Hence, (iv)$\Rightarrow$(i). Hence the proof. ♦

**Definition 7.2.4.** Let $(R_1,\mathcal{I}_1)$ and $(R_2,\mathcal{I}_2)$ be any two intuitionistic fuzzy structure ring spaces. A function $f: (R_1,\mathcal{I}_1) \to (R_2,\mathcal{I}_2)$ is called an intuitionistic fuzzy $RE$-irresolute function if $f^{-1}(A)$ is an intuitionistic fuzzy open compact ring in $(R_1,\mathcal{I}_1)$, for each intuitionistic fuzzy open compact ring $A$ in $(R_2,\mathcal{I}_2)$.

**Proposition 7.2.2.** Let $(R_1,\mathcal{I}_1)$ and $(R_2,\mathcal{I}_2)$ be any two intuitionistic fuzzy structure ring spaces. Then $f: (R_1,\mathcal{I}_1) \to (R_2,\mathcal{I}_2)$ is an intuitionistic fuzzy $RE$-irresolute function if and only if $f(IF_{REcl}(A)) \subseteq IF_{REcl}(f(A))$, for each intuitionistic fuzzy ring $A$ in $(R_1,\mathcal{I}_1)$.

**Proof:**

Suppose that $f$ is an intuitionistic fuzzy $RE$-irresolute function and Let $A$ be an intuitionistic fuzzy ring in $(R_1,\mathcal{I}_1)$. Then, $IF_{REcl}(f(A))$ is an intuitionistic fuzzy closed co-compact ring in $(R_2,\mathcal{I}_2)$. By assumption, $f^{-1}(IF_{REcl}(f(A)))$ intuitionistic fuzzy closed co-compact ring in $(R_1,\mathcal{I}_1)$. Now,

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(IF_{REcl}(f(A))).$$

Hence by Definition 7.2.2., $IF_{REcl}(A) \subseteq IF_{REcl}(f^{-1}(IF_{REcl}(f(A))))$. This implies that $f(IF_{REcl}(A)) \subseteq IF_{REcl}(f(A))$.

Conversely, suppose that $A$ is an intuitionistic fuzzy closed co-compact
ring in \((R_2, \mathcal{S}_2)\). Then, \(IF_{REcl}(A) = A\). Now, by assumption,

\[
f(IF_{REcl}(f^{-1}(A))) \subseteq IF_{REcl}(f(f^{-1}(A)))
= IF_{REcl}(A)
= A.
\]

This implies that \(IF_{REcl}(f^{-1}(A)) \subseteq f^{-1}(A)\). But, \(IF_{REcl}(f^{-1}(A)) \supseteq f^{-1}(A)\). Hence, \(IF_{REcl}(f^{-1}(A)) = f^{-1}(A)\). That is, \(f^{-1}(A)\) is an intuitionistic fuzzy closed co-compact ring in \((R_1, \mathcal{S}_1)\). Therefore, \(f\) is an intuitionistic fuzzy \(RE\)-irresolute function.

Definition 7.2.5. Let \((R_1, \mathcal{S}_1)\) and \((R_2, \mathcal{S}_2)\) be any two intuitionistic fuzzy structure ring spaces. A function \(f : (R_1, \mathcal{S}_1) \to (R_2, \mathcal{S}_2)\) is called an intuitionistic fuzzy \(RE\)-open function if \(f(A)\) is an intuitionistic fuzzy open compact ring in \((R_2, \mathcal{S}_2)\), for each intuitionistic fuzzy open compact ring \(A\) in \((R_1, \mathcal{S}_1)\).

Proposition 7.2.3. Let \((R_1, \mathcal{S}_1)\) and \((R_2, \mathcal{S}_2)\) be any two intuitionistic fuzzy structure ring spaces. Let \(f : (R_1, \mathcal{S}_1) \to (R_2, \mathcal{S}_2)\) be an intuitionistic fuzzy \(RB\)-open, surjective function. Then \(f^{-1}(IF_{REcl}(A)) \subseteq IF_{REcl}(f^{-1}(A))\), for each intuitionistic fuzzy ring \(A\) in \((R_2, \mathcal{S}_2)\).

Proof:

Let \(A\) be an intuitionistic fuzzy ring in \((R_2, \mathcal{S}_2)\) and let \(B = f^{-1}(\overline{A})\). Then, \(IF_{REint}(f^{-1}(\overline{A})) = IF_{REint}(B)\) is an intuitionistic fuzzy open compact ring in \((R_1, \mathcal{S}_1)\). Now, \(IF_{REint}(B) \subseteq B\). Hence, \(f(IF_{REint}(B)) \subseteq f(B)\). That is, \(IF_{REint}(f(IF_{REint}(B))) \subseteq IF_{REint}(f(B))\).
Since \( f \) is an intuitionistic fuzzy \( RE \)-open function, \( f(IF_{RE\,int}(B)) \) is an intuitionistic fuzzy open compact ring in \( (R_2, \mathcal{S}_2) \). Therefore, \( f(IF_{RE\,int}(B)) \subseteq IF_{RE\,int}(f(B)) = IF_{RE\,int}(\overline{A}) \). Hence,

\[
IF_{RE\,int}(f^{-1}(\overline{A})) \subseteq f^{-1}(IF_{RE\,int}(\overline{A})). \tag{7.2.1}
\]

Now, (7.2.1) becomes,

\[
IF_{RE\,int}(f^{-1}(\overline{A})) \supseteq f^{-1}(IF_{RE\,int}(\overline{A}))
\]

\[
\Rightarrow f^{-1}(IF_{RE\,int}(f^{-1}(\overline{A}))) \supseteq IF_{RE\,cl}(f^{-1}(\overline{A})).
\]

Therefore, \( f^{-1}(IF_{RE\,cl}(f^{-1}(A))) \supseteq IF_{RE\,cl}(f^{-1}(A)) \).

\textbf{Proposition 7.2.4.} Let \( (R_1, \mathcal{S}_1) \) be any intuitionistic fuzzy ring extremal compact space and \( (R_2, \mathcal{S}_2) \) be any intuitionistic fuzzy structure ring space. Let \( f : (R_1, \mathcal{S}_1) \rightarrow (R_2, \mathcal{S}_2) \) be an intuitionistic fuzzy \( RE \)-irresolute, intuitionistic fuzzy \( RE \)-open, surjective function. Then \( (R_2, \mathcal{S}_2) \) is an intuitionistic fuzzy ring extremal compact space.

\textbf{Proof:}

Let \( A \) be an intuitionistic fuzzy open compact ring in \( (R_2, \mathcal{S}_2) \). Since \( f \) is an intuitionistic fuzzy \( RE \)-irresolute function, \( f^{-1}(A) \) is an intuitionistic fuzzy open compact ring in \( (R_1, \mathcal{S}_1) \). Since \( (R_1, \mathcal{S}_1) \) is an intuitionistic fuzzy ring extremal compact space, \( IF_{RE\,cl}(f^{-1}(A)) \) is an intuitionistic fuzzy open compact ring in \( (R_1, \mathcal{S}_1) \). As \( f \) is an intuitionistic fuzzy \( RE \)-open function, \( f(IF_{RE\,cl}(f^{-1}(A))) \) is an intuitionistic fuzzy open compact ring in \( (R_2, \mathcal{S}_2) \). Then by Proposition 7.2.3., \( f^{-1}(IF_{RE\,cl}(A)) \subseteq IF_{RE\,cl}(f^{-1}(A)) \).
This implies that \( f(f^{-1}(IF\text{REcl}(A))) = IF\text{REcl}(A) \subseteq f(IF\text{REcl}(f^{-1}(A))) \). Then by Proposition 7.2.2.,

\[
IF\text{REcl}(A) \subseteq f(IF\text{REcl}(f^{-1}(A))) \\
\subseteq IF\text{REcl}(f(f^{-1}(A))) \\
= IF\text{REcl}(A).
\]

This implies that \( f(IF\text{REcl}(f^{-1}(A))) = IF\text{REcl}(A) \). Therefore, \( IF\text{REcl}(A) \) is an intuitionistic fuzzy open compact ring in \((R_2, \mathcal{J}_2)\). Hence, \((R_2, \mathcal{J}_2)\) is an intuitionistic fuzzy ring extremal compact space.

**Definition 7.2.6.** Let \((R, \mathcal{S})\) be any intuitionistic fuzzy structure ring space. Let \(A = \langle x, \mu_A, \gamma_A \rangle\) and \(B = \langle x, \mu_B, \gamma_B \rangle\) be any two intuitionistic fuzzy rings in \(R\). Then \(A\) is an intuitionistic fuzzy ring quasi-coincident with \(B\) (\(AqB\)) if there is a \(x \in R\) such that \(\mu_A(x) + \mu_B(x) > 1\) and \(\gamma_A(x) + \gamma_B(x) < 1\). Otherwise, \(A\) is not an intuitionistic fuzzy ring quasi-coincident with \(B\) (\(A\tilde{q}B\)).

**Proposition 7.2.5.** Let \((R, \mathcal{S})\) be any intuitionistic fuzzy structure ring space. Then \((R, \mathcal{S})\) is an intuitionistic fuzzy ring extremal compact space if and only if for every intuitionistic fuzzy open compact rings \(A = \langle x, \mu_A, \gamma_A \rangle\) and \(B = \langle x, \mu_B, \gamma_B \rangle\) with \(A\tilde{q}B\), \(IF\text{REcl}(A)\tilde{q}IF\text{REcl}(B)\).

**Proof:**

Let \(A = \langle x, \mu_A, \gamma_A \rangle\) and \(B = \langle x, \mu_B, \gamma_B \rangle\) be any two intuitionistic fuzzy open compact rings with \(A\tilde{q}B\). Since \((R, \mathcal{S})\) is an intuitionistic fuzzy ring extremal
compact space, $IF_{REcl}(A)$ and $IF_{REcl}(B)$ are intuitionistic fuzzy open compact rings. Hence, $IF_{REcl}(A) \subseteq IF_{REcl}(B)$.

Conversely, Let $A$ be an intuitionistic fuzzy open compact ring and $\overline{IF_{REcl}(A)}$ be an intuitionistic fuzzy open compact ring in $(\mathcal{R}, \mathcal{S})$ such that $A \subseteq \overline{IF_{REcl}(A)}$.

Then by hypothesis, $IF_{REcl}(A) \subseteq IF_{REcl}(\overline{IF_{REcl}(A)})$. That is, for all $x \in \mathcal{R}$, $\mu_{IF_{REcl}(A)}(x) + \mu_{IF_{REcl}(\overline{IF_{REcl}(A)})}(x) \leq 1$ and $\gamma_{IF_{REcl}(A)}(x) + \gamma_{IF_{REcl}(\overline{IF_{REcl}(A)})}(x) \geq 1$.

This implies that

$$
\mu_{IF_{REcl}(A)}(x) \leq 1 - \mu_{IF_{REcl}(\overline{IF_{REcl}(A)})}(x) = 1 - \mu_{IF_{REcl}(IF_{REint}(A))}(x) = \mu_{IF_{REint}(IF_{REint}(A))}(x) = \mu_{IF_{REint}(IF_{REcl}(A))}(x)
$$

and

$$
\gamma_{IF_{REcl}(A)}(x) \geq 1 - \gamma_{IF_{REcl}(\overline{IF_{REcl}(A)})}(x) = 1 - \gamma_{IF_{REcl}(IF_{REint}(A))}(x) = \gamma_{IF_{REint}(IF_{REint}(A))}(x) = \gamma_{IF_{REint}(IF_{REcl}(A))}(x).
$$

Hence, $IF_{REcl}(A) \subseteq IF_{REint}(IF_{REcl}(A))$, for all $x \in \mathcal{R}$. But,

$$
IF_{REcl}(A) \supseteq IF_{REint}(IF_{REcl}(A)).
$$

This implies that $IF_{REcl}(A) = IF_{REint}(IF_{REcl}(A))$ is an intuitionistic fuzzy
open compact ring in \((R, \mathcal{S})\). Therefore, \((R, \mathcal{S})\) is an intuitionistic fuzzy ring extremal compact space.