Chapter 2

Oscillation of solutions of impulsive vector hyperbolic differential equations with delays

2.1 Introduction

The theory of impulsive partial differential systems has become more important in some mathematical simulation in theoretical physics, chemistry, biotechnology, medicine, population dynamics, optimal control, economics, and in other processes and phenomena in science and technology [4, 6, 8–10, 74]. The study of the oscillatory behavior of solutions of impulsive partial differential equation has been an increasing interest in the literature [19, 21, 27–30, 32, 53, 85]. Minchev and Yoshida [61, 62] established the $H$-oscillation for vector parabolic and hyperbolic differential equation with functional arguments. Li et al. [45] discussed the $H$-oscillation of vector hyperbolic differential equations with deviating arguments. Further, Li and Han [44] proved the $H$-oscillation of solutions of impulsive vector parabolic differential equations with delays.
In this chapter, we investigate the \( H \)-oscillation of the solutions of a class of impulsive vector hyperbolic differential equations with delays of the form

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} U(x,t) &= a(t) \Delta U(x,t) + \sum_{k=1}^{m} b_k(t) \Delta U(x,t - \tau_k) \\
&\quad - \sum_{h=1}^{l} q_h(x,t) U(x,t - \sigma_h) + F(x,t),
t \neq t_j, \\
U(x,t_j^+) - U(x,t_j^-) &= p(x,t_j) U(x,t_j), \\
U_t(x,t_j^+) - U_t(x,t_j^-) &= q(x,t_j) U_t(x,t_j),
\end{aligned}
\]  

(2.1.1)

where \( I_\infty = \{1, 2, \ldots\} \), \( \mathbb{R}_+ = [0, \infty) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a piecewise smooth boundary \( \partial \Omega \), \( \Delta \) is the Laplacian in the Euclidean n-space \( \mathbb{R}^n \), \( 0 < t_1 < t_2 < \ldots < t_j < \ldots \) and \( \lim_{j \to \infty} t_j = \infty \).

Consider the following boundary condition:

\[
\frac{\partial}{\partial N} U(x,t) = \Psi(x,t), (x,t) \in \partial \Omega \times [0, \infty), t \neq t_j, j \in I_\infty,
\]  

(2.1.2)

where \( N \) is the unit exterior normal vector to \( \partial \Omega \) and \( \Psi \in PC[\partial \Omega \times \mathbb{R}_+, \mathbb{R}^M] \), \( PC \) denotes the class of functions, which are piecewise continuous in \( t \) with discontinuities of first kind only at \( t = t_j \) and left continuous at \( t = t_j, j \in I_\infty \).

To establish the results, we shall assume the following conditions:

1. \( a, b_k \in PC[\mathbb{R}_+, \mathbb{R}_+], k \in I_m; \)

2. \( \tau_k \geq 0, \sigma_h \geq 0 \) are constants, \( k \in I_m, h \in I_l; \)

3. \( q_h \in PC[\overline{\Omega}, \mathbb{R}_+], q_h(t) = \min_{x \in \Omega} q_h(x,t), h \in I_l \) and \( Q(t) = \sum_{h=1}^{l} q_h(t); \)

4. \( F \in PC[\overline{\Omega}, \mathbb{R}^M], p, r : \overline{\Omega} \to \mathbb{R}, p(x,t_j) \leq \alpha_j, \) and \( r(x,t_j) \leq \beta_j, \) \( \alpha_j, \beta_j \) are positive constants, \( j \in I_\infty, \overline{\Omega} = \Omega \times \mathbb{R}_+. \)
We establish several oscillation criteria for such systems subject to Neumann boundary condition by employing certain impulsive differential inequalities with delays.

2.2 Preliminaries

In this section, we give some definitions and notations. Also, we obtain some results for certain impulsive partial differential inequalities.

Definition 2.2.1. The vector function $U(x,t)$ is said to be a solution of the problem (2.1.1) and (2.1.2) if the following conditions are satisfied:

(i) $U(x,t)$ is a second order differentiable function for $t, t \neq t_j, j \in I_\infty$;

(ii) $U(x,t)$ is a piecewise continuous function with points of discontinuity of the first kind at $t = t_j, j \in I_\infty$ and at the moments of impulse the following relations are satisfied:

$$U(x,t_j^-) = U(x,t_j); \quad U(x,t_j^+) = U(x,t_j) + p(x,t_j)U(x,t_j),$$

$$\frac{\partial}{\partial t}U(x,t_j^-) = \frac{\partial}{\partial t}U(x,t_j); \quad \frac{\partial}{\partial t}U(x,t_j^+) = \frac{\partial}{\partial t}U(x,t_j) + r(x,t_j)\frac{\partial}{\partial t}U(x,t_j), \quad j \in I_\infty;$$

(iii) $U(x,t)$ is a second order differentiable function for $x$;

(iv) $U(x,t)$ satisfies (2.1.1) in the domain $G$ and the boundary condition (2.1.2).

Definition 2.2.2. Let $H$ be a fixed unit vector in $\mathbb{R}^M$. The solution $U(x,t)$ of the problem (2.1.1), (2.1.2) is said to be $H$-oscillatory in the domain $G$ if the inner product $<U(x,t), H>$ has a zero in $\Omega \times [\mu, \infty)$ for any $\mu > 0$. Otherwise, the solution $U(x,t)$ is said to be $H$-nonoscillatory.
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Definition 2.2.3. Let $H$ be an arbitrary unit vector in $\mathbb{R}^M$. The solution $U(x,t)$ of the problem (2.1.1), (2.1.2) is said to be strongly $H$-oscillatory in the domain $G$ if the inner product $<U(x,t), H>$ has a zero in $\Omega \times [\mu, \infty)$ for any $\mu > 0$.

Definition 2.2.4. The function $v(x,t)$, $(x,t) \in G$ is called eventually positive (negative), if there exists a number $\mu \geq 0$ such that $v(x,t) > 0$ ($v(x,t) < 0$) for $(x,t) \in \Omega \times [\mu, \infty)$. The solution $V(t)$ of an impulsive differential inequality is called eventually positive (negative), if there exists a number $\mu \geq 0$ such that $V(t) > 0$ ($V(t) < 0$) for $t \geq \mu$.

For convenience, we use the following notations:

$$u_H(x,t) = <U(x,t), H>, \quad f_H(x,t) = <F(x,t), H>, \quad \psi_H(x,t) = <\Psi(x,t), H>,$$

$$V(t) = \int_{\Omega} u_H(x,t) dx, \quad \Psi_H(t) = \int_{\partial\Omega} \psi_H(x,t) dS, \quad F_H(t) = \int_{\Omega} f_H(x,t) dx,$$

$$G_H(t) = F_H(t) + a(t) \Psi_H(t) + \sum_{k=1}^{m} b_k(t) \Psi_H(t - \tau_k), \quad t \in \mathbb{R}^+,$$

where $dS$ is the surface element on $\partial\Omega$.

Lemma 2.2.1. Let $H$ be a fixed unit vector in $\mathbb{R}^M$ and $U(x,t)$ be a solution of (2.1.1).

(i) If $u_H(x,t)$ is eventually positive, then $u_H(x,t)$ satisfies the scalar impulsive hyperbolic differential inequality

$$\begin{cases}
\frac{\partial^2}{\partial t^2} u_H(x,t) - a(t) \Delta u_H(x,t) - \sum_{k=1}^{m} b_k(t) \Delta u_H(x,t - \tau_k) \\
\quad + \sum_{h=1}^{l} q_h(t) u_H(x,t - \sigma_h) \leq f_H(x,t), \quad t \neq t_j
\end{cases}$$

$$\begin{align*}
u_H(x,t_j^+) & \leq (1 + \alpha_j) u_H(x,t_j) \\
\frac{\partial}{\partial t} u_H(x,t_j^+) & \leq (1 + \beta_j) \frac{\partial}{\partial t} u_H(x,t_j), \quad j \in I_{\infty}.
\end{align*}$$

(2.2.1)
(ii) If \( u_H(x, t) \) is eventually negative, then \( u_H(x, t) \) satisfies the scalar impulsive hyperbolic differential inequality

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u_H(x, t) - a(t) \Delta u_H(x, t) - \sum_{k=1}^m b_k(t) \Delta u_H(x, t - \tau_k) & \\
+ \sum_{h=1}^l q_h(t) u_H(x, t - \sigma_h) & \geq f_H(x, t), \ t \neq t_j
\end{align*}
\]

(2.2.2)

Proof. (i) Let \( u_H(x, t) \) be eventually positive. We calculate the inner product of (2.1.1) and \( H \).

Case 1: \( t \neq t_j \). From the first equation of (2.1.1),

\[
\frac{\partial^2}{\partial t^2} < U(x, t), H > = a(t) \Delta < U(x, t), H > + \sum_{k=1}^m b_k(t) \Delta < U(x, t - \tau_k), H > - \sum_{h=1}^l q_h(x, t) < U(x, t - \sigma_h), H > + < F(x, t), H >
\]

\[
\frac{\partial^2}{\partial t^2} u_H(x, t) = a(t) \Delta u_H(x, t) + \sum_{k=1}^m b_k(t) \Delta u_H(x, t - \tau_k) - \sum_{h=1}^l q_h(x, t) u_H(x, t - \sigma_h) + f_H(x, t)
\]

By using condition \((C_3)\), we have

\[
\frac{\partial^2}{\partial t^2} u_H(x, t) - a(t) \Delta u_H(x, t) - \sum_{k=1}^m b_k(t) \Delta u_H(x, t - \tau_k) + \sum_{h=1}^l q_h(x, t) u_H(x, t - \sigma_h) = f_H(x, t).
\]

(2.2.3)
Case 2: $t = t_j$. From the second equation in (2.1.1),

$$u_H(x, t_j^+) \leq u_H(x, t_j^-) + \alpha_j u_H(x, t_j) \leq (1 + \alpha_j) u_H(x, t_j), \ j \in I_\infty. \quad (2.2.4)$$

From the third equation in (2.1.1),

$$\frac{\partial}{\partial t} <U(x, t_j^+), H> = \frac{\partial}{\partial t} <U(x, t_j^-), H> + r(x, t_j) \frac{\partial}{\partial t} <U(x, t_j), H>$$

$$\frac{\partial}{\partial t} u_H(x, t_j^+) \leq \frac{\partial}{\partial t} u_H(x, t_j^-) + \beta_j \frac{\partial}{\partial t} u_H(x, t_j)$$

$$\frac{\partial}{\partial t} u_H(x, t_j^+) \leq (1 + \beta_j) \frac{\partial}{\partial t} u_H(x, t_j), \ j \in I_\infty. \quad (2.2.5)$$

Therefore combining (2.2.3), (2.2.4) and (2.2.5) we immediately obtain (2.2.1), which shows that $u_H(x, t)$ satisfies the scalar impulsive hyperbolic differential inequality (2.2.1).

(ii) The proof is similar to that of (i). \qed

Lemma 2.2.2. Let $H$ be a fixed unit vector in $\mathbb{R}^M$, $f_H \equiv 0$ and $U(x, t)$ be a solution of (2.1.1).

(i) If $u_H(x, t)$ is eventually positive, then $u_H(x, t)$ satisfies the scalar impulsive hyperbolic differential inequality

$$\begin{cases}
\frac{\partial^2}{\partial t^2} u_H(x, t) - a(t) \Delta u_H(x, t) - \sum_{k=1}^{m} b_k(t) \Delta u_H(x, t - \tau_k) \\
+ \sum_{k=1}^{l} q_h(t) u_H(x, t - \sigma_h) \leq 0, \ t \neq t_j \\
u_H(x, t_j^+) \leq (1 + \alpha_j) u_H(x, t_j) \\
\frac{\partial}{\partial t} u_H(x, t_j^+) \leq (1 + \beta_j) \frac{\partial}{\partial t} u_H(x, t_j), \ j \in I_\infty.
\end{cases} \quad (2.2.6)$$
(ii) If $u_H(x, t)$ is eventually negative, then $u_H(x, t)$ satisfies the scalar impulsive hyperbolic differential inequality

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} u_H(x, t) - a(t) \Delta u_H(x, t) - \sum_{k=1}^{m} b_k(t) \Delta u_H(x, t - \tau_k) \\
+ \sum_{h=1}^{l} q_h(t) u_H(x, t - \sigma_h) \geq 0, \quad t \neq t_j
\end{aligned}
\]

(2.2.7)

Proof. The proof of this Lemma is similar to that of Lemma 2.2.1. \hfill \Box

**Lemma 2.2.3.** [41] Assume that $0 \leq t_0 < t_1 < t_2 < \ldots < t_j < \ldots, \lim_{j \to \infty} t_j = \infty, w \in PC^1[\mathbb{R}_+, \mathbb{R}], h \in PC[\mathbb{R}_+, \mathbb{R}],$ and $\beta_j > 0$ is a constant, $j \in I_\infty$. If

\[
\begin{aligned}
w'(t) &\leq h(t), \quad t \geq t_0, t \neq t_j \\
w(t_j^+) &\leq (1 + \beta_j)w(t_j), \quad j \in I_\infty,
\end{aligned}
\]

then

\[
w(t) \leq \prod_{t_0 < t_j < t} (1 + \beta_j)w(t_0) + \int_{t_0}^{t} \prod_{s < t_j < t} (1 + \beta_j)h(s)ds, \quad t \geq t_0.
\]

### 2.3 Main results

Let $H$ be a fixed unit vector in $\mathbb{R}^M$. From the inner product of boundary condition (2.1.2) and $H$, we consider the following boundary condition.

\[
\frac{\partial}{\partial N} u_H(x, t) = \psi_H(x, t), \quad (x, t) \in \partial \Omega \times [0, \infty), \quad j \in I_\infty.
\]

(2.3.1)

**Theorem 2.3.1.** Let $H$ be a fixed unit vector in $\mathbb{R}^M$. If the scalar impulsive hyperbolic differential inequality (2.2.1) has no eventually positive solutions and the scalar impulsive hyperbolic differential inequality (2.2.2) has no eventually negative solutions satisfying the boundary condition (2.3.1), then every solution $U(x, t)$ of the problem (2.1.1), (2.1.2) is $H$-oscillatory in $G$. 
Proof. Suppose to the contrary that there is a $H$-nonoscillatory solution $U(x,t)$ of the problem (2.1.1) and (2.1.2), then by definition, $u_H(x,t)$ is eventually positive or eventually negative.

Let $u_H(x,t)$ is eventually positive. Then by Lemma 2.2.1, $u_H(x,t)$ satisfies the scalar impulsive hyperbolic differential inequality (2.2.1). On the other hand, it is easy to see that $u_H(x,t)$ satisfies the boundary condition (2.3.1). This is a contradiction to our hypothesis.

Similarly, if $u_H(x,t)$ is eventually negative, using Lemma 2.2.1, we easily obtain that $u_H(x,t)$ satisfies the scalar impulsive hyperbolic differential inequality (2.2.2). It is obvious that $u_H(x,t)$ satisfies the boundary condition (2.3.1). This is also a contradiction. \qed

Theorem 2.3.2. Let $H$ be a fixed unit vector in $\mathbb{R}^M$. If the impulsive differential inequality

$$\begin{cases} V''(t) + \sum_{h=1}^{l} q_h(t)V(t - \sigma_h) \leq G_H(t), & t \neq t_j \\ V(t_j^+) \leq (1 + \alpha_j)V(t_j), \\ V'(t_j^+) \leq (1 + \beta_j)V'(t_j), & j \in I_{\infty} \end{cases}$$

(2.3.2)

has no eventually positive solutions and the impulsive differential inequality

$$\begin{cases} V''(t) + \sum_{h=1}^{l} q_h(t)V(t - \sigma_h) \geq G_H(t), & t \neq t_j \\ V(t_j^+) \geq (1 + \alpha_j)V(t_j), \\ V'(t_j^+) \geq (1 + \beta_j)V'(t_j), & j \in I_{\infty} \end{cases}$$

(2.3.3)

has no eventually negative solutions then every solution $U(x,t)$ of the problem (2.1.1), (2.1.2) is $H$-oscillatory in $G$. 


Proof. To prove this theorem, it suffices to prove that the scalar impulsive partial differential inequality (2.2.1) has no eventually positive solutions and the scalar impulsive hyperbolic differential inequality (2.2.2) has no eventually negative solutions satisfying the boundary condition (2.3.1).

First, suppose that $u_H(x,t) > 0$ is a solution of the inequality (2.2.1) satisfying the boundary condition (2.3.1), $(x,t) \in \Omega \times [t_0, \infty)$, $t_0 \geq 0$.

Case 1: $t \neq t_j$. Integrating the first inequality in (2.2.1) with respect to $x$ over the domain $\Omega$, we have

$$\frac{d^2}{dt^2} \int_{\Omega} u_H(x,t)dx - a(t) \int_{\Omega} \Delta u_H(x,t)dx - \sum_{k=1}^{m} b_k(t) \int_{\Omega} \Delta u_H(x,t - \tau_k)dx$$

$$+ \sum_{h=1}^{l} q_h(t) \int_{\Omega} u_H(x,t - \sigma_h)dx \leq \int_{\Omega} f_H(x,t)dx.$$  \hfill (2.3.4)

Green’s formula and the boundary condition (2.3.1) yield

$$\int_{\Omega} \Delta u_H(x,t)dx = \int_{\partial \Omega} \frac{\partial u_H(x,t)}{\partial N}dS = \int_{\partial \Omega} \psi_H(x,t)dS = \Psi_H(t)$$  \hfill (2.3.5)

and

$$\int_{\Omega} \Delta u_H(x,t - \tau_k)dx = \int_{\partial \Omega} \frac{\partial u_H(x,t - \tau_k)}{\partial N}dS = \Psi_H(t - \tau_k).$$  \hfill (2.3.6)

Combining (2.3.4), (2.3.5) and (2.3.6), we have

$$V''(t) + \sum_{h=1}^{l} q_h(t)V(t - \sigma_h) \leq F_H(t) + a(t)\Psi_H(t) + \sum_{k=1}^{m} b_k(t)\Psi_H(t - \tau_k),$$

$$V''(t) + \sum_{h=1}^{l} q_h(t)V(t - \sigma_h) \leq G_H(t), \quad t \geq t_0.$$  \hfill (2.3.7)

Case 2: $t = t_j$. Integrating the second inequality in (2.2.1) with respect to $x$ over the domain $\Omega$, we have

$$\int_{\Omega} u_H(x,t_j^+)dx \leq (1 + \alpha_j) \int_{\Omega} u_H(x,t_j)dx$$
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\[ V(t^+_j) \leq (1 + \alpha_j)V(t_j), \quad t \geq t_0, \quad j \in I_\infty. \tag{2.3.8} \]

Similarly, integrating the third inequality in (2.2.1) with respect to \( x \) over the domain \( \Omega \), we have

\[
\frac{d}{dt} \int_{\Omega} u_H(x, t^+_j)dx \leq (1 + \beta_j) \frac{d}{dt} \int_{\Omega} u_H(x, t_j)dx
\]
\[
V'(t^+_j) \leq (1 + \beta_j)V'(t_j), \quad t \geq t_0, \quad j \in I_\infty. \tag{2.3.9}
\]

Therefore, (2.3.7), (2.3.8) and (2.3.9) show that \( V(t) > 0 \) is a positive solution of the impulsive differential inequality (2.3.2). This is a contradiction.

Similarly, suppose that \( u_H(x, t) < 0 \) is a solution of the impulsive hyperbolic differential inequality (2.2.2) satisfying the boundary condition (2.3.1), \( (x, t) \in \Omega \times [t_0, \infty), \ t_0 \geq 0 \). Using the above procedure, we also easily obtain a contradiction. \( \square \)

**Theorem 2.3.3.** Let \( H \) be a fixed unit vector in \( \mathbb{R}^M \). Assume that the following conditions hold:

\[
\sum_{j=0}^{\infty} \alpha_j < +\infty, \quad \sum_{j=0}^{\infty} \beta_j < +\infty; \tag{2.3.10}
\]

\[
\liminf_{t \to \infty} \frac{\int_{T}^{t} \prod_{T < t_j < t} (1 + \alpha_j) \int_{T}^{\xi} \prod_{s < t_j < \xi} (1 + \beta_j) G_H(s)dsd\xi}{\int_{T}^{t} \prod_{T < t_j < t} (1 + \beta_j) \prod_{\xi < t_j < t} (1 + \alpha_j)d\xi} = -\infty \quad \tag{2.3.11}
\]

\[
\limsup_{t \to \infty} \frac{\int_{T}^{t} \prod_{T < t_j < t} (1 + \alpha_j) \int_{T}^{\xi} \prod_{s < t_j < \xi} (1 + \beta_j) G_H(s)dsd\xi}{\int_{T}^{t} \prod_{T < t_j < t} (1 + \beta_j) \prod_{\xi < t_j < t} (1 + \alpha_j)d\xi} = +\infty \quad \tag{2.3.12}
\]

for every sufficiently large \( T \), then every solution \( U(x, t) \) of problems (2.1.1), (2.1.2) is \( H \)-oscillatory in the domain \( G \).
Proof. It suffices to prove that the impulsive differential inequality (2.3.2) has no eventually positive solutions and the impulsive differential inequality (2.3.3) has no eventually negative solutions.

First, assume to the contrary that the impulsive differential inequality (2.3.2) has an eventually positive solution \( V(t) > 0 \), then there exists \( T > 0 \) such that \( V(t) > 0, \ V(t - \sigma_h) > 0, \ t \geq T, \ h \in I_l \). Thus from (2.3.2) we have

\[
\begin{align*}
V''(t) & \leq G_H(t), t \geq T, t \neq t_j \\
V(t^+_j) & \leq (1 + \alpha_j)V(t_j) \\
V'(t^+_j) & \leq (1 + \beta_j)V'(t_j), j \in I_\infty.
\end{align*}
\] (2.3.13)

From first and third inequality of (2.3.13) and by using Lemma 2.2.3, we have

\[
V'(t) \leq \prod_{T < t_j < t} (1 + \beta_j)V'(T) + \int_t^T \prod_{T < t_j < s} (1 + \beta_j)G_H(s)ds, t \geq T.
\] (2.3.14)

From (2.3.14), second inequality of (2.3.13) and by using Lemma 2.2.3, we have

\[
V(t) \leq \prod_{T < t_j < t} (1 + \alpha_j)V(T) + \int_T^t \prod_{T < t_j < \xi} (1 + \alpha_j) \left[ \prod_{T < t_j < \xi} (1 + \beta_j)V'(T) \right.
\]
\[
+ \left. \int_T^\xi \prod_{T < t_j < \tau} (1 + \beta_j)G_H(\tau)ds \right] d\xi,
\]

which implies

\[
V(t) \leq \prod_{T < t_j < t} (1 + \alpha_j)V(T) + \int_T^t \prod_{T < t_j < \xi} (1 + \alpha_j) \prod_{T < t_j < \xi} (1 + \beta_j)V'(T)d\xi
\]
\[
+ \int_T^t \prod_{T < t_j < \xi} (1 + \alpha_j) \int_T^\xi \prod_{T < t_j < \tau} (1 + \beta_j)G_H(\tau)d\tau d\xi.
\] (2.3.15)
Let $t > \tau$, then there exists some integer $m$ such that $t_m < t < t_{m+1}$. Thus,

$$
\int_T^t \prod_{T < t_j < \xi} (1 + \beta_j) \prod_{\xi < t_j < t} (1 + \alpha_j) \, d\xi \leq \int_T^{t_m} \prod_{j=1}^{m} (1 + \alpha_j) \, d\xi + \sum_{k=1}^{m} \int_{t_k}^{t_{k+1}} \prod_{j=1}^{k} (1 + \beta_j) \times \prod_{j=k+1}^{m} (1 + \alpha_j) \, d\xi + \int_{t_m}^{t_{m+1}} \prod_{j=1}^{m} (1 + \beta_j) \, d\xi < +\infty.
$$

Therefore from (2.3.15), we have

$$
\frac{V(t)}{\int_T^t \prod_{T < t_j < \xi} (1 + \beta_j) \prod_{\xi < t_j < t} (1 + \alpha_j) \, d\xi} \leq \frac{\prod_{T < t_j < \xi} (1 + \alpha_j) V(T)}{\int_T^{t_m} \prod_{j=1}^{m} (1 + \alpha_j) \, d\xi} + V'(T) + \frac{\int_T^{t_m} \prod_{j=1}^{m} (1 + \alpha_j) \int_S^{t} \prod_{s < t_j < \xi} (1 + \beta_j) G_H(s) \, ds \, d\xi}{\int_T^{t_m} \prod_{j=1}^{m} (1 + \alpha_j) \prod_{\xi < t_j < t} (1 + \alpha_j) \, d\xi}.
$$

It is easy to see that

$$
\int_T^t \prod_{T < t_j < \xi} (1 + \beta_j) \prod_{\xi < t_j < t} (1 + \alpha_j) \, d\xi \to +\infty \quad \text{as} \quad t \to +\infty,
$$

then using (2.3.10) and (2.3.11) we have

$$
\liminf_{t \to +\infty} \frac{V(t)}{\int_T^t \prod_{T < t_j < \xi} (1 + \beta_j) \prod_{\xi < t_j < t} (1 + \alpha_j) \, d\xi} = -\infty,
$$

which is a contradiction to the assumption $V(t) > 0$.

Next, assume to the contrary that the impulsive differential inequality (2.3.3) has an eventually negative solution $\tilde{V}(t)$. Then there exists $T > 0$ such that $\tilde{V}(t) < 0$, $\tilde{V}(t - \sigma_k) < 0$, $t \geq T$, $h \in I_t$. Thus from (2.3.3) we have

$$
\begin{cases}
\tilde{V}''(t) \geq G_H(t), t \geq T, t \neq t_j \\
\tilde{V}(t_j^+) \geq (1 + \alpha_j) \tilde{V}(t_j) \\
\tilde{V}'(t_j^+) \geq (1 + \beta_j) \tilde{V}'(t_j), j \in I_\infty.
\end{cases}
$$

(2.3.16)

Letting $W(t) = -\tilde{V}(t)$, we have $W(t) > 0$. It follows from (2.3.16) that

$$
\begin{cases}
W''(t) \leq -G_H(t), t \geq T, t \neq t_j \\
W(t_j^+) \leq (1 + \alpha_j) W(t_j) \\
W'(t_j^+) \leq (1 + \beta_j) W'(t_j), j \in I_\infty.
\end{cases}
$$

(2.3.17)
From first and third inequality of (2.3.17) and by using Lemma 2.2.3, we have

\[
W'(t) \leq \prod_{T < t_j < t} (1 + \beta_j)W'(T) - \int_T^t \prod_{T < t_j < s} (1 + \beta_j)G_H(s)ds, t \geq T. \tag{2.3.18}
\]

From (2.3.18) and second inequality of (2.3.17) and by using Lemma 2.2.3, we have

\[
W(t) \leq \prod_{T < t_j < t} (1 + \alpha_j)W(T) + \int_T^t \prod_{T < t_j < \xi} (1 + \alpha_j) \left[ \prod_{T < t_j < \xi} (1 + \beta_j)W'(T) 
- \int_T^\xi \prod_{s < t_j < \xi} (1 + \beta_j)G_H(s)ds \right] d\xi,
\]

\[
W(t) \leq \prod_{T < t_j < t} (1 + \alpha_j)W(T) + \int_T^t \prod_{T < t_j < \xi} (1 + \alpha_j) \prod_{T < t_j < \xi} (1 + \beta_j)W'(T) d\xi
- \int_T^t \prod_{T < t_j < \xi} (1 + \alpha_j) \int_T^\xi \prod_{s < t_j < \xi} (1 + \beta_j)G_H(s)ds d\xi.
\]

Therefore, we obtain

\[
\int_T^t \prod_{T < t_j < \xi} (1 + \beta_j) \prod_{\xi < t_j < \xi}(1 + \alpha_j) d\xi \
\geq \int_T^t \prod_{T < t_j < \xi} (1 + \alpha_j) \prod_{\xi < t_j < \xi}(1 + \alpha_j) d\xi,
\]

\[
\int_T^t \prod_{T < t_j < \xi} (1 + \beta_j) \prod_{\xi < t_j < \xi}(1 + \alpha_j) d\xi \
- \int_T^t \prod_{T < t_j < \xi} (1 + \alpha_j) \int_T^\xi \prod_{s < t_j < \xi} (1 + \beta_j)G_H(s)ds d\xi.
\]

Noting the condition (2.3.12) and taking \( t \to \infty \), from (2.3.10) we have

\[
\limsup_{t \to +\infty} \frac{\tilde{V}(t)}{\int_T^t \prod_{T < t_j < \xi}(1 + \beta_j) d\xi} = +\infty,
\]

which is a contradiction to the assumption \( \tilde{V}(t) < 0 \).

Consider the following boundary condition

\[
\frac{\partial}{\partial N} U(x, t) = 0. \tag{2.3.19}
\]
Theorem 2.3.4. Let $H$ be fixed unit vector in $\mathbb{R}^M$. If the impulsive differential inequality
\[
\begin{aligned}
V''(t) + \sum_{h=1}^{l} q_h(t) V(t - \sigma_h) &\leq 0, \quad t \neq t_j \\
V(t_j^+) &\leq (1 + \alpha_j) V(t_j), \\
V'(t_j^+) &\leq (1 + \beta_j) V'(t_j), \quad j \in I_\infty
\end{aligned}
\tag{2.3.20}
\]
has no eventually positive solutions and the impulsive differential inequality
\[
\begin{aligned}
V''(t) + \sum_{h=1}^{l} q_h(t) V(t - \sigma_h) &\geq 0, \quad t \neq t_j \\
V(t_j^+) &\geq (1 + \alpha_j) V(t_j), \\
V'(t_j^+) &\geq (1 + \beta_j) V'(t_j), \quad j \in I_\infty
\end{aligned}
\tag{2.3.21}
\]
has no eventually negative solutions then every solution $U(x, t)$ of the problem (2.1.1), (2.3.19) is $H$-oscillatory in $G$.

Proof. We can prove this theorem by similar procedure to those used in the proof of Theorem 2.3.1. \qed

Theorem 2.3.5. Let $H$ be a fixed unit vector in $\mathbb{R}^M$. Assume that the condition (2.3.10) and the following condition hold:
\[
\limsup_{t \to \infty} \int_T^t \prod_{s < t_k < t} (1 + \beta_k) Q(s) ds = \infty,
\tag{2.3.22}
\]
for every sufficiently large $T$, then every solution $U(x, t)$ of the problem (2.1.1) and (2.3.19) is $H$-oscillatory in the domain $G$.

Proof. Suppose to the contrary assume that $U(x, t)$ is $H$-nonoscillatory solution of problem (2.1.1) and (2.3.19). Then $u_H(x, t)$ is eventually positive or eventually negative. Assume that $u_H(x, t) > 0$ for $(x, t) \in \Omega \times [t_0, \infty)$. It is clear that the
solution of the inequality (2.3.20), \( V(t) > 0 \) for \( t \geq t_1 \geq t_0 \) and also \( V(t - \sigma_h) > 0 \) for \( t \geq t_1 \geq t_0, \ h \in I_t \).

Therefore, from (2.3.20), we have

\[
\begin{cases}
V''(t) \leq 0, & t \neq t_k, t \geq t_1 \\
V'(t_k^+) \leq (1 + \beta_k) V'(t_k), & k = 1, 2, \cdots
\end{cases}
\]  

(2.3.23)

By Lemma 2.2.3, we have

\[ V'(t) \leq (1 + \beta_k) V'(t_1). \]

We have to show that

\[ V'(t) \geq 0, \quad t \geq t_1. \]  

(2.3.24)

Suppose that this is not true, then there exists a number \( t^* \geq t_1 \) such that \( V'(t^*) \leq 0 \). By Lemma 2.2.3 we obtain

\[ V'(t) \leq (1 + \beta_k) V'(t^*). \]  

(2.3.25)

By (2.3.25), second inequality of (2.3.20) and using Lemma 2.2.3, we get

\[
V(t) \leq \prod_{t^* < t_k < t} (1 + \alpha_k) V(t^*) + \int_{t^*}^t \prod_{s < t_k < t} (1 + \alpha_k) \left[ \prod_{t^* < t_k < s} (1 + \beta_k) V'(t_k) \right] ds
\]

\[
= \prod_{t^* < t_k < t} (1 + \alpha_k) V(t^*) + V'(t^*) \int_{t^*}^t \prod_{s < t_k < t} (1 + \beta_k) \prod_{s < t_k < t} (1 + \alpha_k) ds
\]

Letting \( t \to \infty \) in above inequality, using (2.3.10) and

\[
\int_{t^*}^t \prod_{s < t_k < t} (1 + \beta_k) \prod_{s < t_k < t} (1 + \alpha_k) ds \to \infty \quad \text{as} \quad t \to \infty,
\]

we get \( \limsup_{t \to \infty} V(t) = -\infty \), which is a contradiction. Thus (2.3.24) holds.
2.3. Main results

Set

\[ W(t) = \frac{V'(t)}{V(t - \sigma)}, \quad t \geq t_1, \]

where \( \sigma = \max \{\sigma_h : h \in I_t\} \). Then \( W(t) \geq 0 \) for \( t \geq t_1 \) and

\[ W''(t) = \frac{V''(t)}{V(t - \sigma)} - \frac{V'(t)V'(t - \sigma)}{V^2(t - \sigma)}, \quad t \geq T, \quad t \neq t_k. \]  \hfill (2.3.26)

We have by (2.3.23),

\[ V'(t) \leq V'(t - \sigma), \quad t \geq T, \quad t \neq t_k \]  \hfill (2.3.27)

and from (2.3.20) it follows that

\[ V''(t) \leq - \sum_{h=1}^{l} q_h(t)V(t - \sigma) = -Q(t)V(t - \sigma), \quad t \geq T, \quad t \neq t_k. \]  \hfill (2.3.28)

Combining (2.3.26)-(2.3.29) yields

\[ W'(t) \leq -Q(t) - \left[ \frac{V''(t)}{V(t - \sigma)} \right]^2 \leq -Q(t), \quad t \geq T, \quad t \neq t_k. \]  \hfill (2.3.29)

For \( t = t_k \), we consider two cases:

**Case 1:** \( t_i - \sigma \in (t_j, t_{j+1}) \) for some \( i, j \in \{1, 2, \cdots, k, \cdots\} \).

It is easy to see that

\[ W(t_i^+) = \frac{V'(t_i^+)}{V(t_i^+ - \sigma)} \leq \frac{(1 + \beta_i)V'(t_i)}{V(t_i - \sigma)} = (1 + \beta_i)W(t_i). \]

**Case 2:** \( t_i - \sigma = t_j \) for some \( i, j \in \{1, 2, \cdots, k, \cdots\} \).

We obtain \( V(t_j^+) \geq V(t_j) \).

Thus, we have

\[ W(t_i^+) = \frac{V'(t_i^+)}{V(t_j^+)} \leq \frac{V'(t_i^+)}{V(t_j)} \leq \frac{(1 + \beta_i)V'(t_i)}{V(t_i - \sigma)} = (1 + \beta_i)W(t_i). \]
In view of the cases 1 and 2, we get

\[ W(t^+_k) \leq (1 + \beta_i)W(t_i), \quad k = 1, 2, \cdots. \] (2.3.30)

By Lemma 2.2.3 and from (2.3.29), (2.3.30), we get

\[ W(t) \leq \prod_{T < t_k < t} (1 + \beta_k)W(T) - \int_{T}^{t} \prod_{s < t_k < t} (1 + \beta_k)Q(s)ds. \]

Letting \( t \to \infty \) in above inequality, using (2.3.10) and (2.3.22), we get

\[ \limsup_{t \to \infty} W(t) = -\infty, \]

which is a contradiction to \( W(t) \geq 0 \) for \( t \geq t_1 \).

Similarly, suppose that \( u_H(x, t) < 0 \) for \( (x, t) \in \Omega \times [t_0, \infty) \). Using the above procedure, we also easily obtain a contradiction. \( \square \)

**Remark 2.3.1.** In the above results, if we take \( H \) to be an arbitrary unit vector in \( \mathbb{R}^M \) instead of a fixed unit vector in \( \mathbb{R}^M \), then we can easily obtain the results for strongly \( H \)-oscillatory in \( G \).

**2.4 Examples**

In this section, we give two examples to illustrate our results.

**Example 2.4.1.** Consider the impulsive vector hyperbolic differential equations of the form

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} U(x, t) &= e^t \Delta U(x, t) + 2\Delta U(x, t - \pi/2) - e^t U(x, t - \pi) \\
&\quad - 2U(x, t - 3\pi/2) + F(x, t), t \neq 2^j, \\
U(x, t^+_j) - U(x, t^-_j) &= t_j^{-3}U(x, t_j), \\
U_t(x, t^+_j) - U_t(x, t^-_j) &= t_j^{-5}U_t(x, t_j),
\end{aligned}
\] (2.4.1)
where
\[ F(x, t) = \begin{pmatrix} -e^t \cos t - 2 \sin t - (1 + \cos x) \cos t \\ -2e^t \cos x - e^t - 1 - \cos x \end{pmatrix}, \]
with the boundary condition
\[ U_x(0, t) = U_x(\pi, t) = 0, \quad t \neq 2^j. \]

Here \( n = 1, M = 2, \Omega = (0, \pi), m = 1, l = 2, a(t) = e^t, b(t) = 2, q_1(x, t) = e^t, q_2(x, t) = 2, \tau_1 = \pi/2, \sigma_1 = \pi, \sigma_2 = 3\pi/2, \quad p(x, t_j) = \alpha_j = t_j^{-3}, r(x, t_j) = \beta_j = t_j^{-5}, \Psi(x, t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Letting \( H = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \) we have \( \Psi_H(t) = \Psi_H(t - \tau_1) = 0, \quad f_H(x, t) = -e^t \cos t - 2 \sin t - (1 + \cos x) \cos t \) and \( G_H(t) = -\pi((1 + e^t) \cos t + 2 \sin t). \) It is easy to see that all the conditions of Theorem 2.3.3 are satisfied. Hence every solution of (2.4.1) is \( H \)-oscillatory in domain \((0, \pi) \times [0, \infty).\)

**Example 2.4.2.** Consider the impulsive vector hyperbolic differential equations of the form
\[
\begin{align*}
\frac{\partial^2}{\partial t^2} U(x, t) &= e^t \Delta U(x, t) + \Delta U(x, t - 3\pi/2) + \sqrt{2} e^t \Delta U(x, t - \pi/4) \\
-e^t U(x, t - 3\pi/2) - \sqrt{2} U(x, t - \pi/4) - 2e^t U(x, t - \pi) + F(x, t), \quad t \neq 2^j, \\
U(x, t_j^+) - U(x, t_j^-) &= j^{-4} U(x, t_j), \\
U_t(x, t_j^+) - U_t(x, t_j^-) &= j^{-3} U_t(x, t_j),
\end{align*}
\]
where
\[
F(x, t) = \begin{pmatrix} 0 \\ (1 + \sqrt{2}e^{\pi/4} + e^{3\pi/2} + 2e^{\pi} + e^{-(t-3\pi/2)} + \sqrt{2}e^{-(t-\pi/4)} - e^{-t}) \sin x \end{pmatrix},
\]
with the boundary condition
\[ U_x(0, t) = U_x(\pi, t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \neq 2^j. \]
Here $n = 1$, $M = 2$, $\Omega = (0, \pi)$, $m = 2$, $l = 3$, $a(t) = e^t$, $b_1(t) = 1$, $b_2(t) = \sqrt{2}e^t$, $q_1(x, t) = e^t$, $q_2(x, t) = \sqrt{2}$, $q_3(x, t) = 2e^t$, $\tau_1 = 3\pi/2$, $\tau_2 = \pi/4$, $\sigma_1 = 3\pi/2$, $\sigma_2 = \pi/4$, $\sigma_3 = \pi$, $p(x, t_j) = \alpha_j = j^{-2}$, $r(x, t_j) = \beta_j = j^{-3}$, $\Psi(x, t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Letting $H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have $f_H(t) = 0$ and $Q(s) = \sum_{h=1}^{3} q(s) = e^s + \sqrt{2} + 2e^s = 3e^s + \sqrt{2}$.

Note that

$$\sum_{j=1}^{\infty} j^{-2} < \infty, \quad \sum_{j=1}^{\infty} j^{-2} < \infty.$$

It is easy to see that

$$\limsup_{t \to \infty} \int_{T}^{t} \prod_{s < t_j < t} (1 + \beta_j)Q(s)ds = \infty.$$

Clearly, all the conditions of Theorem 2.3.5 are satisfied. Hence every solution of (2.4.2) is $H$-oscillatory in domain $(0, \pi) \times [0, \infty)$. 