7.1 Introduction

Examples of the zero-truncated negative binomial distribution have been given by Sampford \cite{39} and Brass \cite{9}. The estimation of the parameters of this distribution was discussed by David and Johnson \cite{15}, Rider \cite{36}, Sampford \cite{39}, Brass \cite{9} and Khatri \cite{24}. David and Johnson \cite{15} have considered the maximum likelihood estimation of one parameter and its three-moments estimator and observed that three-moments estimator was inefficient and so recommended the use of maximum likelihood estimator. Sampford \cite{39} has discussed the estimation of the parameters using the first two moments which was also a trial-and-error method but less laborious than the method of maximum likelihood. Brass \cite{9} has given simplified methods of estimating the parameters and also their asymptotic
efficiency. Khatri \cite{24} has extended the simplified method to the case of doubling truncated negative binomial distribution.

Rider \cite{36} obtained the estimators of the parameters of the zero-truncated negative binomial distribution in terms of the first three sample moments, but their asymptotic variances and covariance were not given by him. In this chapter, Rider's method has been extended to obtain the method of moments estimators of the parameters of the truncated negative binomial distribution in which \( k \) classes from the lower end are truncated. Further, the asymptotic variances and covariance of the three-moments estimators are derived and the three-moments estimators are compared with the maximum likelihood estimators in the case when one class from the lower end is truncated.

7.2 The truncated negative binomial distribution

The probability law of the negative binomial distribution in which \( k \) classes from the lower end of the distribution are truncated is

\[
(7.2.1) \quad p_x = \frac{L_x}{S}, \quad (x = k, k+1, \ldots, \infty)
\]

where \( L_x = (m+x-1)\frac{x}{x}p^x(1+p)^{-(m+x)} \), \( S = \sum_{x=k}^{\infty} L_x \) and \( p > 0 \), \( m > 0 \). The probabilities \( p_x \) satisfy the following
Let \( \alpha_r = \sum_{x=k}^{\infty} x^r p_x \) denote the \( r \)th order raw moment of the truncated negative binomial distribution (7.2.1). Multiplying (7.2.2) by \((x + 1)^r\) and taking the sum over \( x = k, k+1, \ldots, \infty \), we derive the following recurrence relation for the moments.

\[
(7.2.3) \quad \alpha_{r+1} = k^{r+1} p_k (1+p) + \left[ m \sum_{j=0}^{r} \binom{r}{j} \alpha_j + \sum_{j=0}^{r-1} \binom{r}{j} \alpha_{j+1} \right] p,
\]

\((r = 0, 1, 2, \ldots)\).

Putting \( r = 0, 1, 2 \) in (7.2.3), we get

\[
(7.2.4) \quad \alpha_1 = k p_k (1+p) + pm,
\]

\[
(7.2.5) \quad \alpha_2 = k^2 p_k (1+p) + pm(1+\alpha_1) + p\alpha_1,
\]

\[
(7.2.6) \quad \alpha_3 = k^3 p_k (1+p) + pm(1+2\alpha_1+\alpha_2) + p(\alpha_1+2\alpha_2).
\]

Multiplying (7.2.4) by \( k \) and \( k^2 \) and subtracting from (7.2.5) and (7.2.6) respectively, we get

\[
(7.2.7) \quad G = p(mH + \alpha_1),
\]

\[
(7.2.8) \quad L = p(mM + T),
\]
where \( G = \alpha_2 - k\alpha_1 \), \( H = \alpha_1 - k + 1 \), \( L = \alpha_3 - k^2\alpha_1 \), \( M = \alpha_2 + 2\alpha_1 - k^2 + 1 \) and \( T = \alpha_1 + 2\alpha_2 \). From (7.2.7) and (7.2.8), we obtain

\[
(7.2.9) \quad p = \frac{(HL - GM)}{(HT - \alpha_1 M)},
\]

\[
(7.2.10) \quad m = \frac{(GT - \alpha_1 L)}{(HL - GM)}.
\]

7.3 The three-moments estimators of the parameters of the singly truncated negative binomial distribution

Consider a random sample of size \( N \) from the truncated negative binomial distribution (7.2.1). Let \( a_r = \sum_{x=k}^{\infty} x^r n_x / N \) denote the sample rth order raw moment \( (r = 1, 2, \ldots) \). Let \( \hat{G}, \hat{H}, \hat{L}, \hat{M} \) and \( \hat{T} \) denote the values of \( G, H, L, M \) and \( T \) when the population moments \( \alpha_r \) are replaced by \( a_r \). The equations (7.2.9) and (7.2.10) suggest that we can estimate \( p \) and \( m \) by

\[
(7.3.1) \quad \hat{p} = \frac{(\hat{H}L - \hat{G}M)}{(\hat{HT} - \alpha_1 \hat{M})},
\]

\[
(7.3.2) \quad \hat{m} = \frac{(\hat{GT} - \alpha_1 \hat{L})}{(\hat{HL} - \hat{GM})}.
\]

The estimators \( \hat{p} \) and \( \hat{m} \) are rational functions of the sample moments and hence are asymptotically normally distributed. The asymptotic variances of \( \hat{p} \) and \( \hat{m} \) and covariance between them can be worked out by the \( \delta \)-method (Kendall and Stuart, §10.6) and are found
to be as

(7.3.3) \[ V(\hat{p}) = C'AC/NQ^2, \]

(7.3.4) \[ V(\hat{m}) = D'AD/Np^2q^2, \]

(7.3.5) \[ \text{Gov}(\hat{p}, \hat{m}) = C'AD/NpQ^2, \]

where

\[
C = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}, \quad D = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \quad \text{and}
\]

\[
P_1 = g\alpha_1 + (k-1)(2b\alpha_1 - mp - k),
\]

\[
P_2 = -h\alpha_1 + (k-1)(2b + k - \alpha_1 - 1),
\]

\[
P_3 = \alpha_1 - k + 1,
\]

\[
R_1 = -\alpha_1(m+1)g' - \alpha_1(k-1)[2(m+1)(b+p) + k]
\]

\[
+ m(k-1)[g' + (k-1)(2b+p-1) - b],
\]

\[
R_2 = \alpha_1[(m+1)h' + (k-1)(m+2)]
\]

\[
- m(k-1)(2mp + 4p + k + 1),
\]

\[
R_3 = -\alpha_1(m+1) + m(k-1),
\]

\[
Q = \alpha_1^2[p(m+1) + (k-1)] - \alpha_1(k-1)(k+3mp+2p)
\]

\[
+ (k-1)^2 2mp,
\]
\[ b = mp+p+1, \quad g = b^2+p+1, \quad g^* = g+bp, \]
\[ h = 2b+p+1, \quad h^* = h+p, \quad \text{and} \]
\[ A = \begin{bmatrix} \alpha_{ij} \end{bmatrix}, \quad i,j = 1,2,3; \quad \alpha_{ij} = \alpha_{i+j} - \alpha_i \alpha_j. \]

7.4 Comparison with the maximum likelihood estimators

For sake of illustration, we consider the case when one class from the lower end of the negative binomial distribution is truncated. The maximum likelihood estimation of the parameters of the truncated negative binomial distribution, in this case, was discussed by David and Johnson (1957) and Sampford (1939).

The maximum likelihood estimators (denoted by putting asterisks over the parameters) of \( \omega = 1/(1+p) \) and \( m \) are given by (Sampford (1939))

\[ (7.4.1) \quad \frac{Nm^*}{\omega^* (1 - \omega^m)} - \frac{Na_1}{(1 - \omega^*)} = 0, \]

\[ (7.4.2) \quad \frac{N \log \omega^*}{1 - (\omega^*)^m} + \frac{R}{\sum_{j=1}^R (m^* + j - 1) \sum_{x=j}^R n_x} = 0, \]

where \( R \) is the highest observed value of \( x \) in the sample. The determinant of the variance-covariance matrix of the estimators is given by (Sampford (1939) and Brass (1939)).
$I_0 = m^2 \gamma^3 \omega^2 / N^2 (\gamma m + \beta)^2$,

where $\gamma = (1 - \omega)$, $\beta = m \gamma / (1 - \omega^m)$, $F = (1 - \beta)W - \gamma P (1 + \log \omega / \gamma)^2$ and

$W = \sum_{r=2}^{\infty} \gamma^r / r^{(m+r-1)}$.

The three-moments estimators of $p$ and $m$ are obtained from (7.3.1) and (7.3.2) by putting $k = 1$ and are found to be

$\hat{p} = (a_3 a_1 - a_1 a_2 + a_1^2 - a_2^2) / a_1 (a_2 - a_1)$,

$\hat{m} = (2a_2^2 - a_2 a_1 - a_3 a_1) / (a_3 a_1 - a_1 a_2 + a_1^2 - a_2^2)$,

which are same as obtained by Rider \[\text{36}\]. The three-moments estimator of $\omega = 1 / (1 + p)$ is given by

$\hat{\omega} = 1 / (1 + \hat{p})$. The asymptotic variances and covariance of the three-moments estimators are obtained from (7.3.3), (7.3.4) and (7.3.5) by taking $k = 1$ and are found to be as

$V(\hat{p}) = C_1 A_1 / \alpha_1 \beta^2 (m+1)^2$,

$V(\hat{m}) = D_1 A_1 / \alpha_1^2 \beta^4$, 
(7.4.8) \[ \text{Cov}(\hat{p}, \hat{m}) = C_1^D/\alpha_1^2p^3(m+1), \]

where

\[ C_1 = \begin{bmatrix} \mathbf{g} \\ -\mathbf{h} \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -\mathbf{g}' \\ \mathbf{h}' \\ -1 \end{bmatrix}, \]

and \( A, \mathbf{g}, \mathbf{g}', \mathbf{h}, \mathbf{h}' \) have the same meanings as in (7.3.3), (7.3.4) and (7.3.5). The asymptotic variance of \( \hat{\omega} \) and covariance between \( \hat{\omega} \) and \( \hat{m} \) are given by

(7.4.9) \[ V(\hat{\omega}) = \omega^4V(\hat{p}), \]

(7.4.10) \[ \text{Cov}(\hat{\omega}, \hat{m}) = -\omega^2\text{Cov}(\hat{p}, \hat{m}). \]

Hence, the determinant of the variance-covariance matrix of the three-moments estimators is given by

(7.4.11) \[ I = \begin{bmatrix} \omega^4V(\hat{p}) & -\omega^2\text{Cov}(\hat{p}, \hat{m}) \\ -\omega^2\text{Cov}(\hat{p}, \hat{m}) & V(\hat{m}) \end{bmatrix}, \]

where \( V(\hat{p}), V(\hat{m}) \) and \( \text{Cov}(\hat{p}, \hat{m}) \) are as given by (7.4.6), (7.4.7) and (7.4.8). The joint asymptotic efficiency \( E \), of the three-moments estimators is then given by
In Table 7.4.1, the percentage asymptotic efficiency of the three-moments estimators is tabulated for selected values of \( m \) and \( U = mp \).

**TABLE 7.4.1**  
PERCENTAGE ASYMPTOTIC EFFICIENCY OF THE THREE-MOMENTS ESTIMATORS \((k = 1)\)

<table>
<thead>
<tr>
<th>( m )</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>24.3</td>
<td>32.9</td>
<td>65.5</td>
<td>84.2</td>
</tr>
<tr>
<td>1</td>
<td>15.6</td>
<td>26.1</td>
<td>37.3</td>
<td>65.2</td>
</tr>
<tr>
<td>2</td>
<td>7.4</td>
<td>15.2</td>
<td>25.6</td>
<td>36.9</td>
</tr>
<tr>
<td>5</td>
<td>3.6</td>
<td>7.1</td>
<td>14.1</td>
<td>29.2</td>
</tr>
</tbody>
</table>

From Table 7.4.1 it is observed that for large values of \( m \) and small values of \( U \) the three-moments estimators are reasonably efficient.

7.5 An illustrative example

The data of Table 7.5.1 are taken from the paper of Brass \(^9\) and were collected by the East African Medical Survey in the Kwimba district of Tanganyika territory. The observations are of the number of children
ever born to a sample of mothers over 40 years of age.

TABLE 7.5.1

<table>
<thead>
<tr>
<th>No. of children per mother</th>
<th>No. of mothers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>49</td>
</tr>
<tr>
<td>2</td>
<td>56</td>
</tr>
<tr>
<td>3</td>
<td>73</td>
</tr>
<tr>
<td>4</td>
<td>41</td>
</tr>
<tr>
<td>5</td>
<td>43</td>
</tr>
<tr>
<td>6</td>
<td>23</td>
</tr>
<tr>
<td>7</td>
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<tr>
<td>8</td>
<td>18</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>340</strong></td>
</tr>
</tbody>
</table>

(a) The three-moments estimators:

We find that $N = 340$, $a_1 = 3.9912$, $a_2 = 21.8853$, $a_3 = 148.1676$. Using (7.4.4) and (7.4.5), we find that $\hat{p} = 0.5738$ and $\hat{m} = 6.8133$. Hence $\hat{\omega} = 0.6354$ and $\hat{\tau} = 0.3646$.

Using (7.2.3) for $r = 3, 4, 5$ and substituting the values of $a_1, a_2, a_3$ for $\alpha_1, \alpha_2, \alpha_3$ and the values of $\hat{p}$ and $\hat{m}$ for $p$ and $m$, we find that the estimates of $\alpha_4, \alpha_5$ and $\alpha_6$ are given by $\alpha_4 = 1181.7578$, ...
\[ \alpha_5 = 10791.8446, \quad \alpha_6 = 110675.6518. \] Using the estimates for the population quantities, we find that
\[ A = \begin{bmatrix} 5.9556 & 60.8190 & 590.3913 \\ 60.8190 & 702.7914 & 7549.1522 \\ 590.3913 & 7549.1522 & 88718.339 \end{bmatrix}. \]

Further we find that \( b = 5.4833, \quad g = 31.6404, \quad g' = 34.7867, \quad h = 12.5404, \quad h' = 13.1142. \) Using (7.4.6), (7.4.7) and (7.4.8), we find that
\[ V(\hat{p}) = \frac{15.4925}{N}, \]
\[ V(\hat{m}) = \frac{2533.5507}{N}, \]
\[ \text{Cov}(\hat{p}, \hat{m}) = \frac{197.0041}{N}. \]

Applying (7.4.11), we find that \( I = \frac{71.7837}{N^2}. \)

(b) Maximum likelihood estimators:

The maximum likelihood estimators are given by (Brass 97), \( \omega^* = 0.565 \) and \( m^* = 4.86. \) Applying (7.4.3), we find that \( P = 0.140605, \quad W = 0.017713 \) and \( F = 0.0092489. \) Hence \( I_0 = \frac{13.1995}{N^2}. \) Thus the estimate of the asymptotic efficiency of the three-moments estimators is \( I_0/I = 0.184, \) i.e. 18.4%.

Remark. As a general case, the three-moments
estimators of the parameters of the singly truncated negative binomial distribution are inefficient. However, in the case of zero-truncated negative binomial distribution, when \( m \) is large and \( U = mp \) is very small, the three-moments estimators are reasonably efficient. They are easy to compute and hence are useful (i) for exploratory work when it is not clear which type of the distribution should be fitted, and (ii) to provide first-stage values in the iterative solution of the maximum likelihood equations.