CHAPTER 5

The Laplace-Stieltjes transformation


\[ \hat{f}(x) = \int_0^\infty \frac{f(t)}{(x^m + t^m)^\rho} dt, \quad m, \rho > 0 \]

and applied it the elements of the dual space \( (M_{a,b,c}^m)' \) of a testing function space \( M_{a,b,c}^m \). The application of the combination transform was done in the conventional way based on the methods adopted by Zemanian [50]. In this chapter, we apply the same combination transform to \( (M_{a,b,c}^m)' \), the dual of the testing function space \( M_{a,b,c}^m \).

The difference from the earlier method and the present one are the following

1. We treat the testing function space \( M_{a,b,c}^m \) as a strict countable union space.
2. \( M_{a,b,c}^m \) and its dual \( (M_{a,b,c}^m)' \) are ordered topological vector spaces.
3. The pointwise convergence topology on \( (M_{a,b,c}^m)' \) is replaced by the topology of bounded convergence.

We observe that without losing any of the original properties of the combination transform some additional features like order proper-
ties of the transform can be studied. The features of operational calculus and solution of initial value problems are retained. Comparison of initial value problems is possible in the present situation.

5.1. The testing function space $M_{a,b,c}^m$ and its dual $(M_{a,b,c}^m)'$ as ordered vector spaces

Let $(K_m)$ be a sequence of compact subsets of $\mathbb{R}_+ \times \mathbb{R}_+$ such that $K_1 \subseteq K_2 \subseteq \ldots$ and such that each compact subset of $\mathbb{R}_+ \times \mathbb{R}_+$ is contained in one $K_j, j = 1, 2, \ldots$. Let $M_{a,b,c,K_j}^m$ denote the linear space of all smooth complex-valued functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$ whose support is contained in $K_j$ on which is defined

$$
\mu_{a,b,c,K_j}^m(\phi) = \sup_{(u,t) \in K_j} \left| e^{u^{q+1}} D_u^{q+1} t^{m(1-a+l)} (1 + t^m)^{a-b} (t^{1-m} D_t^l) \phi(u, t) \right|
$$

where $a, b, c \in \mathbb{R}, q, l = 0, 1, 2, \ldots, m \in (0, \infty), D_u \equiv \frac{\partial}{\partial u}, D_t \equiv \frac{\partial}{\partial t}$. 

$\{\mu_{a,b,c,K_j,q,l}^m\}_{q,l=0}^{\infty}$ is a multinorm on $M_{a,b,c,K_j}^m$ and generates topology $\tau_{a,b,c,K_j}^m$ on $M_{a,b,c,K_j}^m$. $M_{a,b,c,K_j}^m$ is complete with respect to $\tau_{a,b,c,K_j}^m$.

$M_{a,b,c}^m = \bigcup_{j=1}^{\infty} M_{a,b,c,K_j}^m$ is a (strict) countable union space. Since each $M_{a,b,c,K_j}^m$ is complete with respect with respect to $\tau_{a,b,c,K_j}^m$, it follows that $M_{a,b,c}^m$ is complete. On each $M_{a,b,c,K_j}^m$ an equivalent multinorm is given by

$$
\overline{\mu}_{a,b,c,K_j,q,l}^m(\phi) = \sup_{0 \leq q' \leq q, 0 \leq l' \leq l} \mu_{a,b,c,K_j,q',l'}^m(\phi).
$$

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We define an order relation on $M_{a,b,c}^m$ by identifying a positive cone in it.

**Definition 5.1.1.** The positive cone of $M_{a,b,c}^m$ when $M_{a,b,c}^m$ is restricted to real valued functions is the set of all non-negative functions in $M_{a,b,c}^m$. When the field of scalars is $\mathbb{C}$, the complex numbers, the positive cone in $M_{a,b,c}^m$ is $\mathbb{C} + i\mathbb{C}$ which is also denoted as $\mathbb{C}$.

**Note.** We say that $\phi \leq \psi$ in $M_{a,b,c}^m$ when $\psi - \phi \in \mathbb{C}$ in $M_{a,b,c}^m$.

As in the previous cases it can be proved that the positive cone in $M_{a,b,c}^m$ is not normal but is a strict $b$-cone.

**Order and topology on the dual of $M_{a,b,c}^m$.** An order relation is defined on the dual $(M_{a,b,c}^m)'$, the linear space of all continuous linear functionals on $M_{a,b,c}^m$, by identifying the positive cone in $(M_{a,b,c}^m)'$ to be the dual cone $C'$ of the cone $C$ in $M_{a,b,c}^m$. The class of all $B^0$, the polars of $B$ as $B$ varies over all $\sigma(M_{a,b,c}^m; (M_{a,b,c}^m)')$-bounded subsets of $M_{a,b,c}^m$ is a neighbourhood basis of $0$ in $(M_{a,b,c}^m)'$ for the locally convex topology $\beta((M_{a,b,c}^m)', M_{a,b,c}^m)$. When $(M_{a,b,c}^m)'$ is ordered by the dual cone $C'$ and is equipped with the topology of bounded convergence $\beta((M_{a,b,c}^m)', M_{a,b,c}^m)$ it follows that $C'$ is a normal cone since $C$ is a strict $b$-cone by Corollary 1.2.6, Chapter 2, [29].

As in the case of the previous examples we observe that when the topology on $(M_{a,b,c}^m)'$ is changed to the topology of bounded convergence, $(M_{a,b,c}^m)'$ is order complete and topologically complete, the
order dual and the topological dual of $M_{a,b,c}^m$ coincide and the order topology and the topology of bounded convergence on $(M_{a,b,c}^m)'$ coincide. Also the cones of $M_{a,b,c}^m$ and $(M_{a,b,c}^m)'$ are generating.

5.2. The Laplace-Stieltjes transformation

For $\phi(u,t) \in M_{\alpha,\beta,\gamma}^m$ the Laplace-Stieltjes transformation is defined as

\[ \text{SL}_{\rho}^m \phi(u,t) = \hat{\phi}(y,x) = \int_0^\infty \int_0^\infty e^{-yu}(x^m + t^m)^{-\rho} \phi(u,t) dudt \]

for a fixed $m > 0$, $\rho \geq 1$. With suitable integrability conditions the multiple integral

\[ \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-yu}(x^m + t^m)^{-\rho} f(x,y) \phi(u,t) dudtdxdy \]

for $f \in (M_{\alpha,\beta,\gamma}^m)'$, $\phi \in M_{\alpha,\beta,\gamma}^m$ can be evaluated in two different ways so that

\[ \langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle. \]

Geetha K. V. and John J. K. [11] has proved that for $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, the Laplace-Stieltjes transform maps $M_{\alpha,\beta,\gamma}^m$ continuously into $M_{a,b,c}^m$ if

- $a \leq 1$, $a \leq \frac{1}{m} + \alpha - \rho$ and $a < 1$ if $\alpha = \rho + 1 - \frac{1}{m}$
- $b \geq 1 - \rho$, $b \geq \frac{1}{m} + \beta - \rho$ and $b > 1 - \rho$ if $\beta = 1 - 1/m$.

Now, let $f \in (M_{a,b,c}^m)'$. For each $\phi \in M_{\alpha,\beta,\gamma}^m$ we have $\text{SL}_{\rho}^m(\phi) \in M_{a,b,c}^m$. Then the adjoint mapping
\[ \langle \text{SL}_\rho^m(f), \phi \rangle = \langle f, \text{SL}_\rho^m(\phi) \rangle \]
defines the Laplace-Stieltjes transform
\[ \text{SL}_\rho^m(f) \in (M_{\alpha,\beta,\gamma}^m)' \text{ of } f \in (M_{a,b,c}^m)' \].

**Theorem 5.2.1.** *The Laplace-Stieltjes transform is strictly positive and orderbounded.*

**Proof.** \( (M_{a,b,c}^m)' \), \( (M_{\alpha,\beta,\gamma}^m)' \), \( M_{a,b,c}^m \), \( M_{\alpha,\beta,\gamma}^m \) are ordered vector spaces.

Let \( f > 0, f \in (M_{a,b,c}^m)' \). For \( \phi \in M_{\alpha,\beta,\gamma}^m, \phi > 0, \text{SL}_\rho^m(\phi) > 0, \text{SL}_\rho^m(f) \in M_{a,b,c}^m \). Thus \( \phi \to \text{SL}_\rho^m(\phi) \) is a strictly positive map. Being the adjoint of this map, \( f \to \text{SL}_\rho^m(f) \) is a strictly positive map from \( (M_{a,b,c}^m)' \) to \( (M_{\alpha,\beta,\gamma}^m)' \). Since every strictly positive map is order bounded, the theorem follows. \( \square \)

**5.3. Inversion**

For a non-negative integer \( n \) for \( \rho + n > \frac{1}{m} \) a differential operator can be defined by

\[
L_{n,y,x} \phi(y, x) = M y^{2n+1} (D_y D_x)(D_y x^{1-m} D_x)^{n-1} x^{2mn+m-1} (D_y x^{1-m} D_x)^n \phi(y, x)
\]

where
\[
M = \frac{m^{1-2n} \Gamma(\rho)}{\Gamma(n + \frac{1}{m}) \Gamma(\rho + n - 1/m) \Gamma(2n + 1)}.
\]

The Laplace-Stieltjes transform can be inverted by the application of this differential operator. The formal adjoint of this operator is itself.
Geetha K. V. and John J. K. [11] have proved the following results which are true in the present situation also.

**RESULT 5.3.1.** If $\rho + n > \frac{1}{m}$,

\[ \int_0^\infty \int_0^\infty L_{n,y,x} e^{-y x} (x^m + t^m)^{-\rho} dudt = 1. \]

**RESULT 5.3.2.** $L_{n,u,t}$ maps $M_{a,b,c}^m$ continuously into $M_{\alpha,\beta,\gamma}^m$ provided $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$.

**RESULT 5.3.3.** If $L_n$ is the differential operator and $SL_{\rho}^m$ is the Laplace-Stieltjes transform operator then either $y^{-2n} x^{1-\rho} L_n$ and $SL_{\rho}^m x^{m \rho - 1} y^{-2n}$ or $L_n x^{1-\rho} y^{-2n}$ and $y^{-2n} x^{m \rho - 1} SL_{\rho}^m$ commute on $M_{a,b,c}^m$ where $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$.

i.e., $y^{-2n} x^{m \rho - 1} SL_{\rho}^m (L_{n,u,t}(\phi)) = y^{-2n} x^{m \rho - 1} (L_{n,u,t}(\phi))$,

\[ = L_{n,y,x} \int_0^\infty \int_0^\infty \hat{e}^{-y u} (x^m + t^m)^{-\rho} u^{-2n} t^{m \rho - 1} \phi(u, t) dudt \]

\[ = L_{n,y,x} SL_{\rho}^m (u^{-2n} t^{m \rho - 1} \phi) \text{ for } \phi \in M_{a,b,c}^m. \]

**RESULT 5.3.4.** Let $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, then the sequence $\{L_{n,y,x} \hat{\phi}(y, x)\}$ converges in $M_{\alpha,\beta,\gamma}^m$ to $\phi(y, x)$.

**RESULT 5.3.5.** Let $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$ then $(L_n(\hat{\phi}))$ converges to $\phi$ in $M_{a,b,c}^m$ as $n \to \infty$.

The following result proved in [11] has been suitably modified to suit the present situation.

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RESULT 5.3.6. Let \( f \in (M_{a,b,c}^m)' \). Then \( f \in C' \) if and only if for every non-negative integer \( n \), \( L_{n,y,x}SL_{\rho}^m(f) \in C' \) where \( C' \) is the positive cone in \((M_{a,b,c}^m)'\). It follows that \( L_{n,y,x} \) is strictly positive and hence is orderbounded.

We summarize the above results as follows:

For \( f \in (M_{\alpha,\beta,\gamma}^m)' \), \( \phi \in M_{\alpha,\beta,\gamma}^m \)

\[
\langle SL_{\rho}^m L_{n,y,x}(f), \phi \rangle = \langle f, L_{n,y,x}SL_{\rho}^m(\phi) \rangle \to \langle f, \phi \rangle \text{ as } n \to \infty.
\]

For \( f \in (M_{a,b,c}^m)' \), \( \phi \in M_{a,b,c}^m \)

\[
\langle L_{n,y,x}SL_{\rho}^m(f), \phi \rangle = \langle f, SL_{\rho}^m L_{n,y,x}(\phi) \rangle \to \langle f, \phi \rangle.
\]

5.4. Operational calculus

\[
SL_{\rho}^m [DuDt(\phi)] = (m\rho)ySL_{\rho+1}^m[t^{m-1}\phi(u,t)]
\]

provided

\[
\lim_{t \to \infty} Du(\phi(u,t)) = 0 = \lim_{t \to 0} Du(\phi(u,t))
\]

\[
\lim_{u \to \infty} \phi(u,t) = 0 = \lim_{u \to 0} \phi(u,t).
\]

Consider the differential equation

\[
(DuDt)\phi(u,t) = f(u,t), \ u > 0, \ t > 0
\]

where \( f(u,t) \) is a generalized function upon which the
Laplace-Stieltjes transform can be applied.

Let \( F_1(u, t) = \int f(u, t)dt \) be such that

\[
\lim_{t \to \infty} F_1(u, t) = 0 = \lim_{t \to 0} F_1(u, t) \quad \text{and}
\]

\[
F_2(u, t) = \int F_1(u, t)du \quad \text{be such that}
\]

\[
\lim_{u \to \infty} F_2(u, t) = 0 = \lim_{u \to 0} F_2(u, t)
\]

Applying the operational calculus

\[
(m \rho)ySL^m_{\rho+1}[t^{m-1}\phi(u, t)] = (m \rho)ySL^m_{\rho+1}[t^{m-1}F_2(u, t)].
\]

Inverting using the differential operator \( L_{n,y,x} \) for \( \rho + 1 + n > \frac{1}{m} \),

\[
t^{m-1}\phi(u, t) = t^{m-1}F_2(u, t)
\]

so that \( \phi(u, t) = F_2(u, t) \) where \( F_2(u, t) = \int \int f(u, t)dtdu \). Comparison between different solutions arising out of different initial value conditions is possible since they belong to an ordered vector space.